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Variable population welfare and poverty orderings satisfying replication properties*

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Abstract

We discuss and compare the variable population axioms of Critical Level (CL) and Population Replication Invariance (PRI) introduced in the economic and philosophical literature for evaluating distributions with different population size. We provide a common framework for analyzing these competing views considering a strengthening of the Population Replication Principle (PRP) based on Dalton's (1920) "principle of proportionate additions to persons" that requires an ordering defined over populations of the same size to be invariant w.r.t. replication of the distributions, not necessarily imposing indifference between the original distribution and the replica. The strong version of PRP extends the invariance condition to hold also when distributions of different population size are compared. We suggest ethically meaningful general specifications of the invariance requirement underlying the Strong PRP and characterize the associated classes of parameterized evaluation functions that include CL principles and PRI properties. Moreover, we identify a general class of evaluation functions satisfying the Strong PRP: the social evaluation ordering will be represented by the simple formula considering the product of the population size times a strictly monotonic function of the Equally Distributed Equivalent Income (EDEI). Interesting ethical properties are shown to be associated with the shape of the function transforming the EDEI. Implications for poverty measurement are investigated.

Keywords: Variable Population Social Choice, Population Replication, Welfare Measurement, Poverty Measurement.

JEL Classification Numbers: D31, D63.

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1 Introduction

Consider the problem of implementing an intertemporal welfare comparison over populations of size n_t possibly differing between periods t. Suppose information is available concerning a representative income level ζ_t that may correspond to per-capita GDP or to a more general measure that takes also into account distributive concerns. Similar problems occurs in evaluating demographic policies that might affect overall GDP as well as its distribution.

The immediate course of action will be to specify a welfare function W_t that makes use of period t information on population size and representative income. One might support ethical views arguing that either population size or per capita representative income are the only relevant aspects to take into account and therefore identify a lexicographic order between them. However, if some degree of "compensation" between population size and social indicators of "quality of life" is allowed then one of the simplest and intuitively appealing formulations for W_t might require to apply a continuous increasing transformation to the representative individual income and aggregate the obtained transformed measure of representative "quality of life" across the entire populations. That is, social evaluation can be represented by the function

$$W_t = n_t \cdot \Upsilon(\zeta_t) \tag{1}$$

where $\Upsilon(.)$ is increasing and continuous.

Despite the simple formulation of (1), through the specification of $\Upsilon(.)$ it is possible to take into account a number of ethical views on the joint social evaluation of population size and (inequality corrected) per-capita GDP. For instance if there exists an income level $\tilde{\zeta}$ [e.g. a poverty line or a critical (subsistence) level of income] such that $\Upsilon(\tilde{\zeta}) = 0$ then for all $\zeta_t > \tilde{\zeta}$ an increase in population size improves welfare while the opposite occurs for all populations whose representative income is below the threshold $\tilde{\zeta}$.

The function in (1) is among those we study in this paper within an axiomatic framework. We provide characterizations of parametric versions of (1) and identify a general class of evaluation functions leading essentially to (1) and to its dual where $W_t = \Upsilon(\zeta_t)/n_t$. The two classes of parametric indices representing the variable population social evaluation are

$$W_t = (n_t)^\alpha \cdot (\zeta_t - c) \text{ and } W_t = \beta \ln(n) + \zeta_t \quad \text{for } \alpha, \beta, c \in \mathbb{R}.$$

The results are derived following a logical perspective where social evaluations are required to be consistent when population are replicated i.e. when identical distributions are merged. Implications for variable population poverty evaluations and related indices are also explored.

The debate on variable population social evaluations. The *Population Replication Principle (PRP)*, introduced in Bossert (1990a) and based on Dalton's (1920) "principle of proportionate additions to persons" requires inequality, welfare and poverty comparisons of income distributions exhibiting the same population size to be invariant w.r.t. replications of the distributions. It is a weaker version of the *Population Replication Invariance (PRI)* property often applied in inequality, welfare and poverty measurement requiring that an ordering ranks as socially indifferent income profiles obtained by replicating the population. While the first principle usually meets the common intuition, the second stronger property is sometimes questioned and considered to lead to ethically unappealing results.[1] As pointed out by Hurka (1983) the average evaluation principles (which are obtained from population replication invariant orderings) are questionable because they favor only population increases leading to improvement in the average welfare, penalizing therefore addition of individuals at acceptable levels of income but below the (average) representative income (or welfare) level of the population.[2] Moreover Amiel and Cowell (1998, 1999) draw attention to evidence from questionnaires showing that the PRI property moderately meets people views in the context of inequality and welfare comparisons (about 66% of questionnaire answers), and its support substantially decreases when poverty comparisons are involved (50%). As Amiel and Cowell (1999) state (pp. 101-102, part within brackets added): "In this context *(poverty measurement)* the *a priori basis* for the (PRI) principle is less clear: one could plausibly argue that if the distribution is replicated...then poverty is doubled...In assessing poverty -as opposed to inequality- people think much more in absolute than relative terms". This view is consistent with Kanbur (2001)'s criticism of "classical" poverty measures that take into account the population share of poor individuals in a society but neglect the relevance of their absolute number.[3]

It is therefore not always clear why the PRI principle has to be considered the most appropriate criterion to apply in order to compare populations of different size. One possible alternative is to posit that poverty (or alternatively welfare) increases, possibly in a multiplicative way, when population replicates. This is for instance what happens if welfare is evaluated according to the Classical Utilitarian (CU) criterion. The associated social evaluation ordering is represented by the sum of the levels of well-being of each individual within the society measured by a utility function defined over the individual income. This criterion accommodates average welfare and population effects in a multiplicative way, it is immune from the critiques applied to the average principles, but is subject to a major flaw. As pointed out by Parfit (1982) the CU evaluation, as well as many other criteria that consider total welfare as the product of representative welfare and population size, suffers from the *repugnant conclusion.*[4] That is to say, for any population with positive high representative individual welfare there exists a sufficiently large alternative population with slightly

[1] See amongst many Blackorby et al. (1997a,b; 2005), Carlson (1998), Arrhenius (2000) and literature cited therein.

[2] Of course the extent of this critique depends also on the way in which inequality is accommodated within the evaluation. One may consider a symmetric utilitarian welfare function where utilities are increasing and concave and realize that the "more concave" are the utility functions, the lower is the level of income at which an additional individual will induce a social welfare improvement. For extreme inequality averse functions representing maximin positions, we get that no addition of any individual with any income level will be a substitute for increasing the income of the poorest individual in the society.

[3] See Chakravarty et al. (2006) for an exception.

[4] For a discussion of the repugnant conclusion see also Blackorby and Donaldson (1991), Blackorby et al. (1997a,b, 1998, 2001, 2005), Carlson (1998) and Arrhenius (2000).

positive representative individual welfare exhibiting a larger total welfare. Different alternatives have been suggested in order to avoid this problem within a non replication invariant framework of evaluation [see Hurka (1983), Ng (1986, 1989) and Blackorby et al. (1995, 1997a, b)]. One of the most appealing solutions to the problem of the repugnant conclusion is the Blackorby, Bossert and Donaldson criterion of *Critical Level* (generalized) *Utilitarianism* (CLU), which is a modification of the Classical Utilitarianism. CLU sets a positive critical level of utility (or income within the framework we will consider) which corresponds to the level of utility (or income) that if held by one individual makes the addition of this person indifferent to the society. Total welfare is given by the sum of the gaps between individual utilities and the critical level for each individual. Therefore in order to improve the welfare (or standard of living) of an affluent society, an alternative society has necessarily to guarantee to the members a representative welfare above the critical level. It follows that the CLU is immune from the repugnant conclusion.

Unfortunately as shown by Bossert (1990a) the critical level variable population criterion and the replication invariance property cannot coexist within an inequality averse framework. This result leaves open a fundamental question. What is the most appropriate criterion to use? Moreover, the different criteria are based on alternative perceptions of the principles allowing comparisons between distributions with different population size. The procedure followed by the replication invariance property considers hypothetical replications, thus any pair of distributions over populations of different size can be indirectly compared, because there always exists a common multiple for the number of individuals in both populations. The critical level appears more restrictive for what concerns direct comparisons since it requires to compare only distributions obtained adding one individual; however, indirectly, through sequential addition of individuals, all distributions of different population size can also be compared. The principles adopted by the two approaches focus either on the *information about the distribution* or on the *existence of a reference income* levels in order to identify the "neutral transformation" to apply to a distribution. In this respect, recalling Bossert (1990a) result, it may seem impossible to embed simultaneously the two aspects in a principle that allows to compare distributions of different population size. However, this is not the case, within the replication framework it is possible to reconcile the approaches.

Aims of the work. Here we drop the PRI view and follow the consistency perspective underlying the PRP. As also shown in Blackorby et al. (2005) many of the orderings adopted for the evaluation of variable population income prospects do indeed satisfy this principle, we argue that this is the case also for its modification: the *Strong Population Replication Principle* (SPRP) that extends the range of application of the PRP also to comparisons of distributions with different population size.[5]

We start considering the value function associated with a general ordering defined

[5] Blackorby et al. (2005) refer to this principle [as Extended Replication Invariance] quoting an earlier version of this paper and applying it in order to assess the evaluation criteria illustrated in the book.

over distributions with different population size. We use a result of Blackorby and Donaldson (1984) and Blackorby et al. (2001a) which allows to represent the welfare evaluation through a function that is separable in population size and representative income of the society. We investigate some ethically appealing specifications of the SPRP that lead to the characterization of classes of *parameterized evaluation functions accommodating representative welfare and population sizes.* The more general class provides a characterization for both the generalized versions of the average criteria derived making use of population replication invariant evaluation functions and a general version of the critical level variable population criteria. We then present a general class of variable population evaluation criteria satisfying SPRP that also includes the family of evaluation functions previously discussed. According to the result the social evaluation can be represented by the *product of population size times a monotonic transformation of the Equally Distributed Equivalent Income* (EDEI) of the society. We then identify for this class of evaluation functions the conditions under which it allows to avoid the "repugnant conclusion". The results can be appropriately modified in order to be extended to poverty comparisons over distributions with different population size, as suggested in the final part of the paper.

In next section we introduce the notation and some preliminary results, we then provide and discuss the main results on welfare evaluations and conclude with applications to poverty evaluations. Proofs can be found in the Appendix.

2 Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers $\{1, 2, 3, ...\}$, where $n \in \mathbb{N}$ denotes a population size. Let $\mathbb{R}$ ($\mathbb{R}_+$) [$\mathbb{R}_{++}$] be the set of all (non-negative) [positive] real numbers and $\mathbb{R}^n$ the n-fold Cartesian product of $\mathbb{R}$. We consider the income distributions $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{1}_n$ denotes the vector consisting of n ones, therefore $\zeta \cdot \mathbf{1}_n$ will represent an equal distribution of income over n individuals where everybody gets $\zeta \in \mathbb{R}$.[6] The set of all distributions with any possible population size is $\Omega := \cup_{n\in\mathbb{N}}\mathbb{R}^n$.

We are interested in investigating social evaluation functions that represent a variable population social ordering [i.e. a reflexive, transitive and complete binary relation] on Ω. For this purpose following Blackorby and Donaldson (1984) and Blackorby et al. (2001, 2001a) we assume from the outset that the ordering satisfies *extended continuity* and *weak monotonicity*, that is respectively a continuity property defined on Ω and a fixed population condition requiring that distribution $\mathbf{x}$ is evaluated as strictly welfare superior with respect to distribution $\mathbf{y}$ if it componentwise exhibits higher income values.[7]

The implications of these properties are the following:

(i) We can derive a fixed population cardinal indicator of welfare: the "*Equally Distributed Equivalent Income* ζ (EDEI)" that is the level of income that, if owned by all the individuals makes the distribution be socially indifferent w.r.t. $\mathbf{x}$. Thus extended continuity and weak monotonicity imply that there exists a sequence of

[6] Our results are valid for $\mathbf{x} \in \mathbb{R}^n$, but they can also be restricted to hold for positive incomes.

[7] We refer to Blackorby et al. (2001a, 2005) for formalization and discussion of the properties.

representative-income (EDEI) functions $\Xi^n : \mathbb{R}^n \to \mathbb{R}$, a function for any population size n, that represents the social ordering for all fixed-population comparisons, where $\Xi^n(.)$ is continuous, weakly increasing [i.e. $x_i > y_i$ for all $i = 1, 2, ..., n \Rightarrow \Xi^n(\mathbf{x}) > \Xi^n(\mathbf{y})$], and satisfies $\Xi^n(\zeta \cdot \mathbf{1}_n) = \zeta$.

(ii) We also obtain a separable representation of the ordering on Ω in terms of the population size and the EDEI of the distribution. The evaluation can be made according to the *Variable Population Value Function* $V(n, \Xi^n(\mathbf{x}))$ i.e. a function $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ continuous and increasing in the second argument [see Blackorby et al. (2001a) Th. 2].

Relying on the above results we take as primitive concept the notion of *Social Evaluation Function (SEF)* $W : \Omega \to \mathbb{R}$ representing the variable population welfare ordering. When appropriate we will equivalently consider directly the variable population value function $V(.,.)$. We will take into account *anonymous* and *inequality averse* SEFs but it has to be stressed that our results are independent from the adoption of these properties usually applied for inequality and welfare measurement. *Anonymity* requires that only the income distribution matters irrespective of the identity of the individuals to whom incomes are assigned thus $\Xi^n(.)$ is symmetric. While *inequality aversion* is formalized by the consistency of the SEF with the *Principle of Transfers* favoring rank-preserving transfers from rich to poor individuals, thus $\Xi^n(.)$ is S-Concave.[8]

We now introduce the main properties concerning comparisons between distributions of different population size.

The *Population Replication Principle (PRP)* (see Dalton, 1920 and Bossert, 1990a) states that a population replication does not affect the ranking of two distributions of the same population size. That is, if a distribution is considered welfare superior to another involving the same number of individuals then their ranking is not affected if the individuals in both distributions are cloned. Let $\mathbf{x}^r \in \mathbb{R}^{rn}$, for $r \in \mathbb{N}$, be the distribution obtained from $\mathbf{x} \in \mathbb{R}^n$ replicating it r times, that is $\mathbf{x}^r = (\underbrace{\mathbf{x}, ., ..\mathbf{x}, .., \mathbf{x}}_{r \text{ times}})$, then PRP requires that: for all $\mathbf{y}, \mathbf{x} \in \mathbb{R}^n$ and all $r, n \in \mathbb{N}$

$$V(n, \Xi^n(\mathbf{x})) \geq V(n, \Xi^n(\mathbf{y})) \Longleftrightarrow V(rn, \Xi^{rn}(\mathbf{x}^r)) \geq V(rn, \Xi^{rn}(\mathbf{y}^r)).$$

The *Population Replication Invariance (PRI)* criterion strengthens PRP by requiring indifference between distributions obtained through the replication process: for all $\mathbf{x} \in \mathbb{R}^n$ and all $r, n \in \mathbb{N}$

$$V(n, \Xi^n(\mathbf{x})) = V(rn, \Xi^{rn}(\mathbf{x}^r)).$$

Properties PRP and PRI impose different restrictions on the value function. As shown by Bossert (1990a): if PRI holds then $V(n, \Xi^n(\mathbf{x})) = H(\Xi^n(\mathbf{x}))$ where $H(.)$ is increasing and $\Xi^n(.)$ is *replication invariant* i.e. satisfies $\Xi^{rn}(\mathbf{x}^r) = \Xi^n(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $r, n \in \mathbb{N}$; while if PRP holds then $\Xi^n(.)$ is replication invariant

[8] It is well known that Anonymity and Principle of Transfers correspond to imposing S-Concavity on the equality (welfare) ordering, and that over distributions of same total income and population size the inequality (welfare) ordering is equivalent to the Lorenz ranking. See Kolm (1969), Dasgupta, Sen and Starrett (1973) and Fields and Fei (1978).

without any further restriction on the functional form of $V(.)$. The second part of the previous statements may come unexpected: an EDEI function may not be necessarily replication invariant (see Example 8 in Appendix).

An alternative criterion to PRI is the *Critical Level Principle (CLP),* originally suggested by Blackorby and Donaldson (1984). CLP involves the identification of a *critical level of welfare (or income)* c such that if an individual with c is added to the original population the social welfare is unaffected. Formally, there exists $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^n$:

$$V(n, \Xi^n(\mathbf{x})) = V(n+1, \Xi^{n+1}(\mathbf{x}, c)).$$

Note that in this case c is fixed, in analogy with an absolute poverty line it is independent from the original distribution and its population size.[9] Bossert (1990a) proves that CLP and PRI are not consistent for anonymous, inequality averse and weak monotonic orderings on Ω. The conflict between PRI and CLP is therefore inevitable, they are mutually exclusive. They are related to different logical structures (replication vs addition of individuals) and apparently share very little in common, leading to different orderings and welfare functions. The Average Utilitarian SEF $W_{AU}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} U(x_i)$ satisfies PRI, while the Classical Utilitarian SEF $W_{CU}(\mathbf{x}) = \sum_{i=1}^{n} U(x_i) = nW_{AU}(\mathbf{x})$ satisfies CLP for c such that $U(c) = 0$.[10] In general the Critical Level Utilitarian SEF can be represented as $W_{CLU}(\mathbf{x}) = \sum_{i=1}^{n} [U(x_i) - U(c)]$ or as $\hat{W}_{CLU}(\mathbf{x}) = \sum_{i=1}^{n} \hat{U}(x_i - c)$. Many other welfare representations can accommodate the two principles, for instance if we adopt the rank-dependent SEFs [see Weymark, 1981; Donaldson and Weymark, 1980; Yaari, 1987 and Ebert, 1988] we can represent welfare through $W_{RD}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} \left(V\left(\frac{i}{n}\right) - V\left(\frac{i-1}{n}\right)\right)$ where $x_{(i)} \leq x_{(i+1)}$, $V(1) = 1, V(0) = 0$ that satisfies PRI, while $W_{CLRD}(\mathbf{x}) = n(W_{RD}(\mathbf{x}) - c) = n\sum_{i=1}^{n}\left(x_{(i)} - c\right)\left(V\left(\frac{i}{n}\right) - V\left(\frac{i-1}{n}\right)\right)$ satisfies CLP.

In general we can express the previous SEFs through functions combining population size, EDEI and a reference level $\hat{c}$ as follows:

$$W(\mathbf{x}) = \Psi[\Xi^n(\mathbf{x})] \text{ or } W(\mathbf{x}) = n \cdot (\Psi[\Xi^n(\mathbf{x})] - \hat{c}) \tag{2}$$

where $\Xi^n(.)$ satisfies PRI and $\Psi(.)$ is continuous and increasing. The previous cases are obtained letting $\Xi^n(\mathbf{x}) = W_{RD}(\mathbf{x})$ and $\Psi(t) = t$ or $\Xi^n(\mathbf{x}) = U^{-1}[W_{AU}(\mathbf{x})]$, $\Psi(t) = U(t)$ and $\hat{c} = U(c)$. If the effect of n is ruled out we have the standard PRI SEFs otherwise we obtain evaluation functions satisfying CLP.

How can these rules be characterized within a common framework of analysis? Our starting point is the observation that all the evaluation functions we have considered satisfy the PRP (and also a strong form of it specified in next section) even if some of them are characterized without imposing any replication property.

In next section we will consider restrictions imposed on the SEF when a strong version of the PRP is adopted. Because of the general nature of the property we

[9] Blackorby and Donalson (1984) and Bossert (1990a,c) consider a variable critical level $c(\mathbf{x})$ that depends on the starting distribution, while Blackorby et al. (2002) consider the number dependent critical level $c(n)$ that depends on the population size.

[10] Note that $W_{AU}(\mathbf{x})$ satisfies the variable critical level principle for $c(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} U(x_i)$.

will focus only on some plausible restrictions allowing consistency with the principle. These restrictions are sufficient to lead to parametric specifications of the variable population value function generalizing those above introduced and suggest a common framework of analysis where evaluation functions satisfying PRI and CLP are obtained from the same generic welfare representation as the values of the parameters change.

3 Results

We require that social evaluations satisfy a strong version of the PRP obtained extending the replication consistency condition on $V(n, \Xi^n(.))$ to hold also for comparisons of distributions exhibiting different population size.

Axiom 3.1 (Strong Population Replication Principle (SPRP)) *For all* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ *and all* $r, n, m \in \mathbb{N}$

$$V(n, \Xi^n(\mathbf{x})) \geq V(m, \Xi^m(\mathbf{y})) \Longleftrightarrow V(rn, \Xi^{rn}(\mathbf{x}^r)) \geq V(rm, \Xi^{rm}(\mathbf{y}^r)).$$

Clearly SPRP implies PRP but the reverse implication does not hold, we can therefore identify value functions that violate SPRP but satisfy PRP.

Example 1 *PRP* $\not\Longrightarrow$ *SPRP. Let* $G : \mathbb{R} \to (0, 1)$ *denote an increasing function bounded in* $(0, 1)$. *Define* $\tilde{V}(n, \Xi^n(\mathbf{x})) := 2 \cdot G[\Xi^n(\mathbf{x})] + n - 1$. *Clearly* $\tilde{V}(n, \Xi^n(\mathbf{x})) \geq \tilde{V}(n, \Xi^n(\mathbf{y}))$ *implies* $\Xi^n(\mathbf{x}) \geq \Xi^n(\mathbf{y})$; *thus if* $\Xi^n(.)$ *is replication invariant then* $\tilde{V}(.)$ *satisfies PRP. Does it satisfy also the SPRP?*
Take distributions $\mathbf{x} \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^2$ *where* $\mathbf{x} = (x)$ *and* $\mathbf{y} = (y, y)$ *such that* $\mathbf{x}$ *and* $\mathbf{y}$ *are considered socially indifferent according to* $\tilde{V}(.)$, *i.e.* $\tilde{V}(1, x) = 2G(x) = 2G(y) + 1 = \tilde{V}(2, y)$. *If we replicate once both distributions, we get* $\tilde{V}(2, x) = 2G(x) + 1$ *while* $\tilde{V}(4, y) = 2G(y) + 3$. *It follows that* $\tilde{V}(4, y) > \tilde{V}(2, x)$ *given that* $2G(x) = 2G(y) + 1$. *Thus SPRP is violated.*

The classes of SEFs where population sizes are lexicographically ordered w.r.t. the EDEI do satisfy SPRP. Note that for these SEFs the socially optimal distribution is obtained either when just one individual exists or when the population size is maximized irrespective of the EDEI. This second case is an extreme example of the type of SEFs suffering from the "repugnant conclusion" where variations in population size can offset differences in EDEI. Here we assume instead that *changes in EDEI can compensate for differences in population size* and we impose some degree of "welfare comparability" between distributions with different population sizes. We first consider a minimal version of comparability ruling out the evaluations where some welfare levels can be achieved only by distributions exhibiting a given population size. Then we strengthen the comparability assumption requiring that EDEI changes can always compensate for populations variations. As we will show even in these cases we can obtain SEFs leading to the "repugnant conclusion".

Let $V(n, \mathbb{R})$ denote the image of $V(n, .)$ for a fixed $n \in \mathbb{N}$, that is $V(n, \mathbb{R}) := \{V(n, \zeta) : \zeta \in \mathbb{R}\}$, while $V(\mathbb{N}, \mathbb{R})$ denotes the image of $V(., \mathbb{R})$ that is $V(\mathbb{N}, \mathbb{R}) := \{V(n, \mathbb{R}) : n \in \mathbb{N}\}$.

Axiom 3.2 (Minimal Welfare Comparability (MWC)) *For any $w \in V(\mathbb{N}, \mathbb{R})$ there exist $n, m \in \mathbb{N}$, and $\zeta_n, \zeta_m \in \mathbb{R}$ such that $V(n, \zeta_n) = V(m, \zeta_m) = w$.*

The MWC property requires that any feasible welfare level can be obtained for appropriate EDEI values by distributions of different population size, it can be strengthened requiring that this condition holds irrespective of the population size, that is $V(n, \mathbb{R}) = V(\mathbb{N}, \mathbb{R})$ for all $n \in \mathbb{N}$ i.e. the function $V(n, .)$ is surjective.

Axiom 3.3 (Full Welfare Comparability (FWC)) *For any $n, m \in \mathbb{N}$, and $\zeta_n \in \mathbb{R}$ there exists $\zeta_m \in \mathbb{R}$ such that $V(n, \zeta_n) = V(m, \zeta_m)$.*

Note that FWC does not specify any relation between ζ_n and ζ_m. Changes in population size may induce either positive or negative effects on welfare that may depend on the value of the initial EDEI as well.

Example 1 provides evidence of a value function that does not satisfy neither FWC nor SPRP, however it has to be noticed that the FWC (and also the MWC) property is independent from PRP and from SPRP as shown in next examples.

Example 2 *$FWC \not\Longrightarrow$ PRP, and $FWC + PRP \not\Longrightarrow$ SPRP. Let*

$$\hat{V}(n, \Xi^n(\mathbf{x})) := \begin{cases} n \cdot |\Xi^n(\mathbf{x})|^n & \text{if } \Xi^n(\mathbf{x}) \geq 0 \\ -n \cdot |\Xi^n(\mathbf{x})|^n & \text{if } \Xi^n(\mathbf{x}) < 0 \end{cases}, \tag{3}$$

then FWC is satisfied. However PRP is not satisfied unless $\Xi^n(.)$ is replication invariant and even in this latter case SPRP is not satisfied. Suppose we compare, as in Example 1, distributions $\mathbf{x} = (x)$ and $\mathbf{y} = (y, y)$ where $x, y > 0$ that are considered socially indifferent according to $\hat{V}(n, \Xi^n(.))$, i.e. $\hat{V}(1, x) = x = 2y^2 = \hat{V}(2, y)$. If we replicate once both distributions, we get $\hat{V}(2, x) = 2x^2$ while $\hat{V}(4, y) = 4y^4 = [2y^2]^2 = x^2$. It follows that $\hat{V}(4, y) < \hat{V}(2, x)$ thereby violating SPRP.

Example 3 *$SPRP \not\Longrightarrow MWC$. This is the case when population size is lexicographically ordered w.r.t. the EDEI, for instance if*

$$\check{V}(n, \Xi^n(\mathbf{x})) := \frac{1}{\pi} \arctan[\Xi^n(\mathbf{x})] + n - \frac{1}{2} \tag{4}$$

where $\Xi^n(.)$ is replication invariant. Distributions exhibiting different population size are ranked according to it, thus MWC is violated but replications do not affect the social rank. While for distribution with the same population size SPRP is satisfied because $\Xi^n(.)$ is replication invariant.

According to (4) no compensation of any degree in terms of EDEI is allowed in order to obtain socially indifferent distributions involving different population sizes. However $SPRP$ does not imply FWC even when some degree of compensation is allowed and MWC holds.

Example 4 *SPRP + MWC* $\not\Longrightarrow$ *FWC. Let*

$$\bar{V}(n, \Xi^n(\mathbf{x})) := n \cdot \left[\frac{1}{\pi}\arctan[\Xi^n(\mathbf{x})] + \frac{1}{2}\right] - \frac{1}{2} \tag{5}$$

where $\Xi^n(.)$ *is replication invariant. The function* $\bar{V}(n, \Xi^n(\mathbf{x}))$ *satisfies SPRP but violates FWC. Note that* $\sup \bar{V}(n, \mathbb{R}) = n - 1/2$ *and* $\bar{V}(n, 0) = n/2 - 1/2$ *thus if* $\zeta' > 0$ *then* $\bar{V}(2n, \zeta') > n - 1/2 > \bar{V}(n, \zeta)$ *for any* $\zeta \in \mathbb{R}$*.*

Fulfillment of SPRP in conjunction with MWC corresponds to assume that a replication operation affects the social evaluation in a separable way based on the replication coefficient and on the value of the SEFs at the starting distribution. Moreover MWC also requires that the set of potentially feasible welfare values is not affected if the set of all distribution with a fixed generic population size is not taken into account, that is no population size is essential for reaching a given welfare level.

Lemma 3.1 $V(n, \Xi^n(.))$ *satisfies MWC and SPRP if and only if there exists a sequence of functions* $\Xi^n : \mathbb{R}^n \to \mathbb{R}$ *satisfying* $\Xi^{rn}(\mathbf{x}^\mathbf{r}) = \Xi^n(\mathbf{x})$ *and a function* $\Pi : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ *increasing in the second argument such that*

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = \Pi\left[r, V(n, \Xi^n(\mathbf{x}))\right] \tag{6}$$

for all $r, n \in \mathbb{N}$ *and all* $\mathbf{x} \in \mathbb{R}^n$*, where* $\Pi\left[n, V(1, \mathbb{R})\right] = V(n, \mathbb{R})$*, and*

$$V(\mathbb{N}, \mathbb{R}) = \Pi\left[\mathbb{N}, V(1, \mathbb{R})\right] = \Pi\left[\mathbb{N}\backslash n', V(1, \mathbb{R})\right] = V(\mathbb{N}\backslash n', \mathbb{R})\ \forall n' \in \mathbb{N}. \tag{7}$$

Specifying the functional form of $\Pi[r, V(.)]$ amounts to impose restrictions on $V(.)$. These restrictions are precisely those we are going to investigate by adopting for (6) a simple plausible and separable functional form. The most general results associated with the mentioned specification will turn out to be equivalent to the relevant solutions of (6) derived in the second part of the section.

We compare distributions whose population is replicated r times, and the replicated individuals are identical in all respects to the original ones. Each person therefore compares him/herself with a larger set of reference individuals: an r times replica of the individuals different from him/her and $r - 1$ individuals similar to him/her. Welfare changes are supposed to depend on the replication parameter r according to a positive *multiplicative* factor $\phi(r)$ and/or and *additive* factor $\psi(r)$.

We require that the *positive* function $\phi(r)$ and the function $\psi(r)$ are *monotonic:* they are either non-decreasing or non-increasing in terms of the replication coefficient r; given that these functions are independent this is the case also for the direction of their monotonic effect.

The *proportional effect* formalized by $\phi(r)$ is consistent with the view that changes in overall social welfare that are induced by increases in population size depend on the "average" welfare of the new population added, possibly compared with that of the original population. When the merged population is a replica of the original one then the welfare impact of the merging operation can be either positive or negative, it is related to the ratio between the new and the original population size and is proportional to the average welfare of the original population.

The idea behind the *additive effect* formalized by $\psi(r)$ is that welfare may decrease [or increase] according to whether there exists a fixed social cost [or a fixed benefit] expressed in per capita units deriving from merging identical populations. Moreover, because of the replication operation the per capita representative welfare may be affected depending on the number of clones that are associated with each individual. The impact of a replication therefore is independent from either the initial population size or the EDEI, it only depends on the number of clones of each individual. A positive or negative value for the mere existence of individuals is postulated, the more individuals of a given type are included in the society the higher/or lower is the per capita value of the mere existence of each one of them.

Next properties combine the *proportional effect* based on the per capita representative income of the distribution and the *pure population effect* that are induced by the replication of the society. We suggest two versions of the affine replication effect. The first version (GAfPR) is consistent with a cardinal concept of welfare, while the weaker second version retains the replication effects within an ordinal framework (O-GAfPR).

Axiom 3.4 (GAfPR: Generalized Affine Population Replication) *There exist monotonic functions* $\psi : \mathbb{N} \to \mathbb{R}$ *where* $\psi(1) = 0$, *and* $\phi : \mathbb{N} \to \mathbb{R}_{++}$ *where* $\phi(1) = 1$, *such that* $W(\mathbf{x}^r) = \psi(r) + \phi(r) \cdot W(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$, *and all* $r, n \in \mathbb{N}$.

Axiom 3.5 (O-GAfPR: Ordinal - GAfPR) *There exist monotonic functions* $\psi : \mathbb{N} \to \mathbb{R}$ *where* $\psi(1) = 0$, *and* $\phi : \mathbb{N} \to \mathbb{R}_{++}$ *where* $\phi(1) = 1$, *and an increasing transformation* $F : \mathbb{R} \to \mathbb{R}$ *such that* $F[W(\mathbf{x}^r)] = \psi(r) + \phi(r) \cdot F[W(\mathbf{x})]$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *and all* $r, n \in \mathbb{N}$.

These axioms are weaker than the PRI, in that the functions $\phi(.)$ and $\psi(.)$ are not specified while PRI requires to set $\phi(r) = 1$ and $\psi(r) = 0$ for all $r \in \mathbb{N}$. Both replication factors are assumed to be monotonic because there is no reason to allow for changes in trends of the evaluation related to the *pure population effect* and the *proportional effect* as the number of replications increase, however these effects can take independent directions. Replicating a distribution may improve or worsen or leave unchanged the welfare in the original distribution. All possible patterns are still feasible, but it is not the case that different replication coefficients induce changes in the pattern of evaluation.

It is worth recalling that the evaluations satisfying O-GAfPR are made in an ordinal framework, and we are not assuming from the outset that the function $V(.)$ carries a cardinal meaning. What O-GAfPR implicitly requires is that for a given ordinally invariant specification of the value function $V(.)$, and of the representative income function $\Xi^n(.)$, it is possible to find a distribution with different population size and representative income that is indifferent to the original one, and that this comparison process can be carried out according to the pure population and the proportional effect. It is the independence between $\phi(.)$, $\psi(.)$ and $V(.)$ which turns out to play a major role in imposing restrictions on the functional form of the latter and identifying invariance conditions. The restrictions imposed by GAfPR on the variable population evaluation functions are summarized in the following propositions.

Proposition 3.1 *$W(.)$ satisfies GAfPR if and only if there exist constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, an increasing continuous function $\Psi : \mathbb{R} \to \mathbb{R}$ and a sequence of functions $\Xi^n : \mathbb{R}^n \to \mathbb{R}$ satisfying $\Xi^{rn}(\mathbf{x^r}) = \Xi^n(\mathbf{x})$ such that:*
(i) if $\alpha \neq 0$,

$$W(\mathbf{x}) = W_{\alpha,\beta}(\mathbf{x}) := \beta + n^\alpha \cdot [\Psi(\Xi^n(\mathbf{x})) - \beta] \tag{8}$$

for all $\mathbf{x} \in \mathbb{R}^n$, all $n \in \mathbb{N}$, where the ordering induced by $W(.)$ is invariant w.r.t. affine transformation of $\Psi(.)$ scaling the gap w.r.t. β i.e. $T[\Psi] := \begin{cases} b_1[\Psi - \beta] + \beta & \text{if } \Psi \geq \beta \\ b_2[\Psi - \beta] + \beta & \text{if } \Psi < \beta \end{cases}$ where $b_1, b_2 > 0$.
(ii) if $\alpha = 0$

$$W(\mathbf{x}) = W_{0,\beta}(\mathbf{x}) := \beta \ln(n) + \Psi[\Xi^n(\mathbf{x})] \tag{9}$$

for all $\mathbf{x} \in \mathbb{R}^n$, all $n \in \mathbb{N}$ where, if $\beta \neq 0$ the ordering induced by $W(.)$ is invariant w.r.t. additive transformations of $\Psi(.)$, i.e. $T[\Psi] := a + \Psi$ where $a \in \mathbb{R}$. If $\beta = 0$ the ordering induced by $W(.)$ is invariant w.r.t. increasing transformations of $\Psi(.)$.

Applying GAfPR we obtain a parametrized class of variable population evaluation criteria that generalizes the Critical Level SEFs and those in (2). Following a population replication approach it is then possible to derive criteria that have been characterized following a single individual addition perspective. The parametric welfare representations in Proposition 3.1 are associated with the following specifications of the replication coefficients

$$\phi(r) = r^\alpha; \; \psi(r) = \begin{cases} \beta \cdot [1 - r^\alpha] & \text{if } \alpha \neq 0 \\ \beta \ln(r) & \text{if } \alpha = 0 \end{cases}.$$

They can include population replication invariant perceptions (if $\alpha = \beta = 0$) but mainly identify flexible weights to apply to the population effects. If $\alpha \neq 0$ an increase in individual representative income above the threshold $\Psi^{-1}(\beta)$ is weighted according to the population size and induces a larger welfare effect as α increases. On the other hand changes in the population size for a fixed EDEI increase welfare if the representative income is above the threshold $\Psi^{-1}(\beta)$ and α is positive. The value of $\Xi^n(\mathbf{x}) = \Psi^{-1}(\beta)$ thus represents a *reference point* for the evaluation playing a role analogous to the critical level (or to a poverty line). Indeed, if we set $\alpha = 1$ and $\beta > 0$ we can obtain the class of Critical Level SEFs, where $\Psi^{-1}(\beta)$ is the critical income level above which the representative income induces a positive welfare effect.[11] Note

[11] If in addition we impose the *Population Substitution Principle* (see Blackorby and Donaldson, 1984), requiring that replacing the income of each individual within a subgroup of the population by the representative income of the subgroup does not affect the aggregate welfare evaluation, we get an additively separable representation for $\Xi^n(.)$. That is, $\Xi^n(\mathbf{x}) = U^{-1}[\frac{1}{n}\sum_{i=1}^n U(x_i)]$, leading to $W^n(\mathbf{x}) = n^\alpha \cdot \left\{\Psi\left[U^{-1}\left(\frac{1}{n}\sum_{i=1}^n U(x_i)\right)\right] - \Psi(c)\right\}$ where $\Psi(c) = \beta$ and c is the critical income level. If we set $\Psi = U$ then the evaluation of the representative income is obtained adopting the utility of the representative individual. Then $W^n(\mathbf{x}) = n^{\alpha-1} \cdot \{\sum_{i=1}^n U(x_i) - nU(c)\}$ which for $\alpha = 1$ gives the Critical Level Utilitarian SEF.

that if $\alpha > 0$ then the representation in (8) suffers from the repugnant conclusion if $\inf \Psi(\mathbb{R}) \geq \beta$ implying that $\Psi(\Xi^n(\mathbf{x})) - \beta > 0$ for all $\mathbf{x} \in \mathbb{R}^n$.[12]

The other interesting family of SEFs in (9) is obtained when $\alpha = 0$, i.e. when only the pure population effect plays a role. When $n = 1$ (9) coincides with the evaluation of the individual income, as the population replicates, depending on the sign of β, welfare may increase or decrease. Within a perfectly equal society adding an extra individual with the same income induces a welfare effect quantified by $\beta\left(\ln(\frac{n+1}{n})\right) \approx \beta/n$. If $\beta > 0$ the addition of individuals whose income leaves unaffected the EDEI of the society induce a positive externality impact effect β/n which is inversely related with the original population size. Thus this class of criteria suffers from the repugnant conclusion when $\beta > 0$, while on the other hand for $\beta = 0$ it is subject to the critiques given to the average utilitarianism, as is also the case for $\beta < 0$.

As a result both criteria obtained from the GAfPR specification of SPRP may suffer from some of the critiques discussed in the introduction. But the reason does not lie in the replication perspective, it comes from the way the SPRP has been specified in GAfPR and also relates to the shape of the transformations $\Psi(.)$ in Proposition 3.1. If we apply the more general O-GAfPR retaining the ordinal content of the social evaluation then it is possible to recover the orderings induced by the class of functions in (8) and (9) as special cases of a general representation that is multiplicatively separable between the population size and a replication invariant EDEI.

Proposition 3.2 *$W(.)$ satisfies O-GAfPR if and only if there exist a sequence of functions $\Xi^n : \mathbb{R}^n \to \mathbb{R}$ satisfying $\Xi^{rn}(\mathbf{x}^\mathbf{r}) = \Xi^n(\mathbf{x})$, an increasing continuous function $\bar{\Upsilon} : \mathbb{R} \to \mathbb{R}$, and an increasing function $H : \mathbb{R} \to \mathbb{R}$ such that:*

$$W(\mathbf{x}) = H \circ [n^k \cdot \bar{\Upsilon}(\Xi^n(\mathbf{x}))] \quad \textit{where } k \in \{-1, 0, 1\}. \tag{10}$$

Thus any non replication invariant variable population evaluation satisfying O-GAfPR can be represented as the product or the ratio of an increasing transformation of the replication invariant EDEI and the population size. The representations in Propositions 3.1 and 3.2 satisfy the previous welfare comparability assumptions under some restrictions on the values of the parameters or the image of functions $\Psi(.)$ and $\bar{\Upsilon}(.)$. Next remarks identify the exhaustive list of these conditions.

Remark 3.1 *Only if at least one of the following conditions is satisfied then the welfare representations in Proposition 3.1*

- satisfy MWC: (1) $\alpha = 0 = \beta$; *(2)* $\sup \Psi(\mathbb{R}) \geq \beta \geq \inf \Psi(\mathbb{R})$; *(3)* $\alpha \neq 0$, $\inf \Psi(\mathbb{R}) > \beta$ *and* $\frac{\sup \Psi(\mathbb{R}) - \beta}{\inf \Psi(\mathbb{R}) - \beta} > 2^{|\alpha|}$; *(4)* $\alpha \neq 0$, $\sup \Psi(\mathbb{R}) < \beta$ *and* $\frac{|\inf \Psi(\mathbb{R}) - \beta|}{|\sup \Psi(\mathbb{R}) - \beta|} > 2^{|\alpha|}$; *(5):* $\alpha = 0$ *and* $\sup \Psi(\mathbb{R}) - \inf \Psi(\mathbb{R}) > |\beta| \cdot \ln(2)$ *i.e.* $\frac{\sup[\exp \circ \Psi(\mathbb{R})]}{\inf[\exp \circ \Psi(\mathbb{R})]} > 2^{|\beta|}$.

- satisfy FWC: (1') $\alpha = 0 = \beta$; *(2')* $\alpha \neq 0$ *and* $\Psi(\mathbb{R}) = \mathbb{R}$; $\mathbb{R}_{++}$; $\mathbb{R}_{--}$; *(3')* $\alpha = 0, \beta \neq 0$ *and* $\Psi(\mathbb{R}) = \mathbb{R}$.

[12] For instance (5) can be obtained from (8) letting $\alpha = 1$, $\beta = -1/2$ and setting $\Psi(t) := \arctan(t)/\pi$, and suffers from the repugnant conclusion given that $\inf[\arctan(\mathbb{R})/\pi] \geq -1/2$.

Conversely, if the welfare representation in Proposition 3.2 satisfies MWC [respectively FWC] we necessarily have that:

Remark 3.2 *Only if at least one of the following conditions is satisfied the welfare representations in Proposition 3.2*

- satisfy MWC: (1) $k = 0$*; (2)* $\bar{\Upsilon}(\mathbb{R}) = \mathbb{R}$, $\bar{\Upsilon}(\mathbb{R}) = (\underline{w}, +\infty)$ *where* $\underline{w} \leq 0$, $\bar{\Upsilon}(\mathbb{R}) = (-\infty, \bar{w})$ *where* $\bar{w} \geq 0$*; (3)* $k = 1$ *and* $\frac{\sup \bar{\Upsilon}(\mathbb{R})}{\inf \bar{\Upsilon}(\mathbb{R})} > 2$ *(4)* $k = -1$ *and* $\frac{\sup \bar{\Upsilon}(\mathbb{R})}{\inf \bar{\Upsilon}(\mathbb{R})} < 2^{-1}$.

- satisfy FWC: (1') $k = 0$*; (2')* $\bar{\Upsilon}(\mathbb{R}) = \mathbb{R}$*;* $\mathbb{R}_{++}$*;* $\mathbb{R}_{--}$.

The choice of the function $\Psi(.)$ and therefore of $\bar{\Upsilon}(.)$ transforming the EDEI thus plays a crucial role. However, when incomes are positive and, for interpretative reasons, transformations are required to map in the income domain then the orderings associated with different families of evaluation functions can no longer be always derived through modifications of the monotonic transformations applied to the EDEI. This view is summarized by the following axiom:

Axiom 3.6 (N: Normalization) $V(1, \zeta) = \zeta$ *for all* $\zeta \in \mathbb{R}$.

Note that by construction we have that $V(n, \mathbb{R}) \subseteq V(1, \mathbb{R}) = \mathbb{R}$, thus the Normalization axiom allows to represent social welfare in monetary terms as the EDEI does for fixed population comparisons. Moreover, it is independent from MWC and FWC as shown in next examples.

Example 5 *N* $\not\Longrightarrow$ *MWC. Let*

$$V^*(n, \Xi^n(\mathbf{x})) := \tan \cdot \left[n^{-1} \cdot \arctan[\Xi^n(\mathbf{x})]\right]. \tag{11}$$

The function $V^*(n, \Xi^n(\mathbf{x}))$ *satisfies N but violates MWC. Note that for* $n \geq 2$ *we get* $V^*(n, \mathbb{R}) = [\tan(-\pi/2n); \tan(\pi/2n)]$ *thus for any* $n \geq 2$ $V^*(n, \mathbb{R}) \subset V^*(1, \mathbb{R}) = \mathbb{R} = V^*(\mathbb{N}, \mathbb{R})$ *therefore some values in* $V^*(\mathbb{N}, \mathbb{R})$ *can be reached only if* $n = 1$*, thereby violating MWC.*

FWC $\not\Longrightarrow$ *N. The increasing transformation* $\exp \circ \hat{V}$ *of* $\hat{V}(n, \Xi^n(\mathbf{x}))$ *in (3) satisfies FWC however it violates N as* $\exp[\hat{V}(1, \Xi^1(\mathbf{x}))] = \exp[\Xi^1(\mathbf{x})]$.

Adopting N amounts to restrict the set of admissible transformations of the function $\Psi(.)$ in Propositions 3.1. As next corollary shows (omitting the straightforward proof) the only admissible set of functions $\Psi(.)$ are identities:

Corollary 3.1 $W(.)$ *satisfies GAfPR and N if and only if there exists a sequence of functions* $\Xi^n : \mathbb{R}^n \to \mathbb{R}$ *satisfying* $\Xi^{rn}(\mathbf{x^r}) = \Xi^n(\mathbf{x})$ *such that:*

$$W(\mathbf{x}) = \begin{cases} \tilde{W}_{\alpha,\beta}(\mathbf{x}) := \beta + n^{\alpha} \cdot [\Xi^n(\mathbf{x}) - \beta] & \text{if } \alpha \neq 0; \\ \tilde{W}_{0,\beta}(\mathbf{x}) := \beta \ln(n) + \Xi^n(\mathbf{x}) & \text{if } \alpha = 0 \end{cases} \tag{12}$$

for all $\mathbf{x} \in \mathbb{R}^n$*, all* $n \in \mathbb{N}$.

To summarize we essentially obtain two families of evaluation functions. If social evaluation is assumed to be cardinal fully measurable and therefore N holds then either

$$W^n(\mathbf{x}) = n^\alpha \cdot [\Xi^n(\mathbf{x}) - c]\,;\ \text{or } W^n(\mathbf{x}) = \beta \ln(n) + \Xi^n(\mathbf{x}).$$

In this case all the well known classes of EDEI functions satisfying PRI adopted in welfare and inequality measurement can be substituted for $\Xi^n(\mathbf{x})$. For instance if we consider the linear rank-dependent functions we obtain $W(\mathbf{x}) = n^\alpha\,(W_{RD}(\mathbf{x}) - c) = n^{\alpha-1}W_{CLRD}(\mathbf{x})$. Moreover, note that as a by-product of the result the characterization in (12) satisfy FWC, indeed we have that irrespective of the population size $V(n,\mathbb{R}) = \mathbb{R}$.

On the other hand if we retain O-GAfPR and apply N the direct implication is that according to (10) $V(1,\zeta) = H \circ [\bar{\Upsilon}(\zeta)] = \zeta$, thereby implying that $H = \bar{\Upsilon}^{-1}$, as a result $W(\mathbf{x}) = \bar{\Upsilon}^{-1} \circ [n^k \cdot \bar{\Upsilon}(\Xi^n(\mathbf{x}))]$ where $k \in \{-1, 0, 1\}$. Moreover, the normalization assumption induces further restrictions on the image of $\bar{\Upsilon}(.)$ in order to allow that $\bar{\Upsilon}^{-1}(.)$ maps from the domain given by the set of values in $\mathbb{N}^k \cdot \bar{\Upsilon}(\mathbb{R})$. Thus, it has to hold that $\bar{\Upsilon}(\mathbb{R}) = \mathbb{N}^k \cdot \bar{\Upsilon}(\mathbb{R})$.

Corollary 3.2 *$W(.)$ satisfies O-GAfPR and N if and only if there exist an increasing function $\bar{\Upsilon} : \mathbb{R} \to \mathbb{R}$ and a sequence of functions $\Xi^n : \mathbb{R}^n \to \mathbb{R}$ satisfying $\Xi^{rn}(\mathbf{x}^\mathbf{r}) = \Xi^n(\mathbf{x})$ such that either*

$$V(n, \Xi^n(\mathbf{x})) = \Xi^n(\mathbf{x}) \tag{13}$$

or

$$V(n, \Xi^n(\mathbf{x})) = \bar{\Upsilon}^{-1} \circ [n^k \cdot \bar{\Upsilon}(\Xi^n(\mathbf{x}))] \text{ where } k \in \{-1, 1\} \tag{14}$$

moreover (i) if $k = 1$ then either $\bar{\Upsilon}(\mathbb{R}) = \mathbb{R}$; $\bar{\Upsilon}(\mathbb{R}) = (\underline{w}, +\infty)$ where $\underline{w} \geq 0$, or $\bar{\Upsilon}(\mathbb{R}) = (-\infty, \bar{w})$ where $\bar{w} \leq 0$; (ii) if $k = -1$ then either $\bar{\Upsilon}(\mathbb{R}) = \mathbb{R}$; $\bar{\Upsilon}(\mathbb{R}) = (\underline{w}, +\infty)$ where $\underline{w} \leq 0$; $\bar{\Upsilon}(\mathbb{R}) = (-\infty, \bar{w})$ where $\bar{w} \geq 0$ or $\bar{\Upsilon}(\mathbb{R}) = (\underline{w}, \bar{w})$ where $\bar{w} > 0 > \underline{w}$.

As argued in next remark under the assumptions underlying the representation in (14) the properties of MWC and FWC are equivalent.

Remark 3.3 *The family of evaluation functions in (14) satisfy MWC and FWC under the same assumptions i.e. only if $\bar{\Upsilon}(\mathbb{R}) = \mathbb{R}$; $\mathbb{R}_{++}$; $\mathbb{R}_{--}$.*

The representation in (11) provides an example of a specification of (14) for $k = -1$ that violates MWC given that $\bar{\Upsilon}(\mathbb{R}) := \arctan[\mathbb{R}] = (-\pi/2; \pi/2)$. The functions in Corollary 3.2 are related to those we will derive in next section dropping O-GAfPR and focussing on the general class of evaluation functions satisfying SPRP, N and MWC. Before moving to the analysis of these general results we highlight the relevance of the results in (12) and (14) for empirical applications based on summary statistics. In particular Corollary 3.1 identifies the normative implications for adopting evaluations based on parametric aggregation of average income, population size and an inequality index.

Evaluations based on aggregations of summary statistics. Following the *normative approach to income inequality measurement* [see Kolm, 1969; Atkinson, 1970; Sen, 1973 and Blackorby and Donaldson, 1978] then a SEF can be specified consistently from a relative inequality index. In particular the EDEI can be formalized as an "Abbreviated SEF" (Lambert, 2001 Ch. 5) weighting the average income by a scale invariant (in)equality index.[13] Suppose to restrict attention to positive incomes and let $I^n : \mathbb{R}^n_{++} \to [0,1]$ denote an S-Convex, scale invariant, population replication invariant inequality index, then the replication invariant EDEI can be written as

$$\Xi^n(\mathbf{x}) := \mu(\mathbf{x}) \cdot [1 - I^n(\mathbf{x})].$$

Thus the SEFs in (14) can be specified as $W^n(\mathbf{x}) = \bar{\Upsilon}^{-1}\left\{n^k \cdot \bar{\Upsilon}[\mu(\mathbf{x}) \cdot [1 - I^n(\mathbf{x})]]\right\}$ for $k \in \{-1, 1\}$ aggregating information on population size, average income and inequality making use of the transformation function $\bar{\Upsilon}(.)$ that may implicitly identify a critical level $\bar{\Upsilon}^{-1}(0)$ for the representative income.

Example 6 *Let $c > 0$ denote the critical level. For $k = 1$ the specification $\bar{\Upsilon}(x) := \ln(x) - \ln(c)$ gives $W^n(\mathbf{x}) = c \cdot \{\mu(\mathbf{x}) \cdot [1 - I^n(\mathbf{x})]/c\}^n$, while $\bar{\Upsilon}(x) := x^\alpha - c^\alpha$ for $\alpha > 0$ gives $W^n(\mathbf{x}) = [c^\alpha + n \cdot [\mu(\mathbf{x})^\alpha \cdot [1 - I^n(\mathbf{x})]^\alpha - c^\alpha]]^{1/\alpha}$.*

If on the other hand we consider the cardinal SEFs in (12) then we will obtain two families of evaluation that can allow to represent a wide variety of ethical views depending on the choices of the parameters and making use of available summary statistics on population size, per-capita income and inequality:

$$\tilde{W}_{\alpha,\beta}(\mathbf{x}) \ : \ = \beta + n^\alpha \cdot [\mu(\mathbf{x}) \cdot [1 - I^n(\mathbf{x})] - \beta]\,; \qquad (15)$$

$$\tilde{W}_{0,\beta}(\mathbf{x}) \ : \ = \beta \ln(n) + \mu(\mathbf{x}) \cdot [1 - I^n(\mathbf{x})]. \qquad (16)$$

3.1 A general class of evaluation functions satisfying "SPRP"

In this section we highlight the analogy between imposing SPRP, MWC and N on the evaluation function $V(n, \Xi^n(.))$ and the solution of a class of functional equations applied in iteration theory: the "multiplicative translation equation". We derive a version of the "multiplicative translation equation" which is defined over the set of positive integers (population sizes or number of replications). The general solution of this functional equation is still an open problem that goes beyond the scope of this section. We consider the continuous particular solution associated with the most common specifications of the functional equation defined on the set of real numbers. This solution coincides with the class of evaluation functions derived in Corollary 3.2, it is sufficiently general to include as special cases all the results obtained in the previous section as well as most of the variable population evaluation procedures presented in the literature. Moreover, it will make possible to identify a simple condition that separates the evaluation functions that are subject to the "repugnant conclusion" from those that are immune. For this latter class of functions the critical level of income will be directly identified.

[13] See for instance Chakravarty (1990) and Blackorby et al. (1999a).

We consider variable population evaluations functions $V(n, \Xi^n(.))$ that are increasing w.r.t. $\Xi^n(.)$ and satisfy N. Under the further assumptions of MWC and SPRP, we can make use of condition (6) in Lemma 3.1; setting $n = 1$ and requiring $V(1, x) = V(1, \Xi^1(x)) = x$ we get

$$V(r, \Xi^r(x\mathbf{1}_r)) = \Pi[r, V(1, x)] = \Pi(r, x). \tag{17}$$

Therefore $\Pi(r, x) = V(r, x)$ for all $r \in \mathbb{N}, x \in \mathbb{R}$.

Substituting $V(r, x)$ for $\Pi(r, x)$ in (6) we obtain the following functional equation:

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = V[r, V(n, \Xi^n(\mathbf{x}))] \text{ for all } r, n \in \mathbb{N}, \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Noting that $\Xi^n(.)$ is necessarily replication invariant (see Bossert, 1990a), and letting $\Xi^n(\mathbf{x}) = \zeta$, the functional equation can be expressed as

$$V(rn, \zeta) = V[r, V(n, \zeta)] \text{ for all } r, n \in \mathbb{N}, \text{ for all } \zeta \in \mathbb{R}. \tag{18}$$

This is the "multiplicative translation equation" (see Aczél 1966, 1987 Sect. 6, and Moszner, 1995 and literature cited therein).[14] It is important to note that (18) is defined for natural numbers r and n as well as for real values of ζ. This specification opens the possibility of a variety of solutions generalizing the most common ones derived for domains considering real numbers (see e.g. Aczél 1966, 1987). The derivation of the general solution of (18) goes beyond the scope of this paper. Here we present particular solutions that can be obtained when $r, n, \zeta \in \mathbb{R}$. A particular solution of (18) is obtained if $V(n, \Xi^n(\mathbf{x}))$ is independent from n for all $n \in \mathbb{N}$. This happens when $V(n, \Xi^n(\mathbf{x})) = \Xi^n(\mathbf{x})$ where $\Xi^n(.)$ is replication invariant. As in Aczél 1987 (section 6) the solution of (18) that is non-constant in $n \in \mathbb{R}$ is obtained if there exists a strictly monotonic [either increasing or decreasing] continuous function $\Upsilon : \mathbb{R} \to \mathbb{R}$ (or $\Upsilon : \mathbb{R} \to \mathbb{R}_{++}$, or $\Upsilon : \mathbb{R} \to \mathbb{R}_{--}$) such that

$$V(n, \zeta) = \Upsilon^{-1}[n \cdot \Upsilon(\zeta)]$$

that is

$$V(n, \Xi^n(\mathbf{x})) = \Upsilon^{-1}[n \cdot \Upsilon(\Xi^n(\mathbf{x}))] \tag{19}$$

where $\Xi^n(.)$ is replication invariant.

Noting that if $\hat{\Upsilon}(\zeta) := -\Upsilon(\zeta)$ then $\Upsilon^{-1}[n \cdot \Upsilon(\zeta)] = \hat{\Upsilon}^{-1}[n \cdot \hat{\Upsilon}(\zeta)]$ we can then restrict attention only to functions $\Upsilon(.)$ such that either $\Upsilon(\mathbb{R}) = \mathbb{R}$ or $\Upsilon(\mathbb{R}) = \mathbb{R}_{++}$. Moreover, if $\Upsilon : \mathbb{R} \to \mathbb{R}_{++}$ is decreasing we can let $\tilde{\Upsilon}(\zeta) := \frac{1}{\Upsilon(\zeta)}$, thereby obtaining an increasing function $\tilde{\Upsilon}(.)$. We can then restate (19) requiring that there exists an increasing function $\tilde{\Upsilon} : \mathbb{R} \to \mathbb{R}_{++}$ such that $V(n, \Xi^n(\mathbf{x})) = \tilde{\Upsilon}^{-1}[n^{-1} \cdot \tilde{\Upsilon}(\Xi^n(\mathbf{x}))]$ where $\Xi^n(.)$ is replication invariant.

To summarize, according to the solution of (18) we have that $V(n, \Xi^n(\mathbf{x})) \geq V(m, \Xi^m(\mathbf{x}))$ if and only if there exist replication invariant functions $\Xi^n(.)$ and increasing function $\Upsilon(.)$ such that if $\Upsilon(\mathbb{R}) = \mathbb{R}$ then

$$n \cdot \Upsilon(\Xi^n(\mathbf{x})) \geq m \cdot \Upsilon(\Xi^m(\mathbf{y})), \tag{20}$$

[14] It has to be noticed that (18) satisfies MWC only if conditions (7) in Lemma 3.1 hold. Recalling that if N holds then $V(1, \mathbb{R}) = \mathbb{R}$ it follows that the only binding condition in (7) has to require that $V(\mathbb{N}\backslash 1, \mathbb{R}) = \mathbb{R}$. Property FWC instead requires that $V(n, \mathbb{R}) = \mathbb{R}\ \forall n \in \mathbb{N}$.

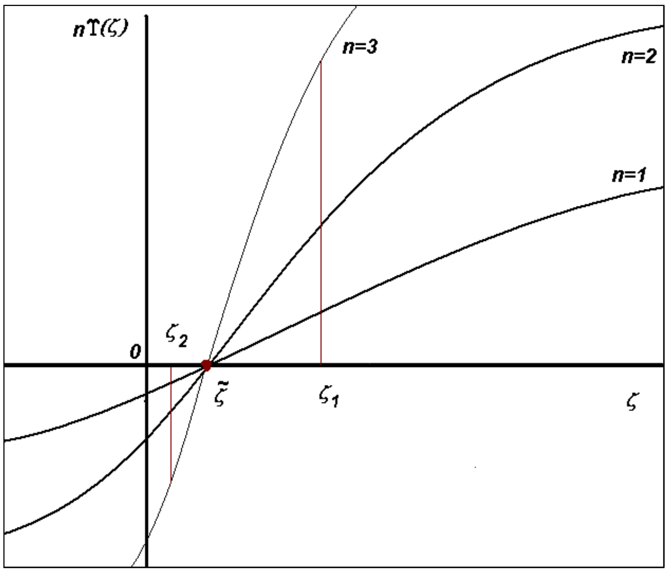
while if $\Upsilon(\mathbb{R}) = \mathbb{R}_{++}$ then either (20) holds or

$$\frac{1}{n} \cdot \Upsilon\left(\Xi^n(\mathbf{x})\right) \geq \frac{1}{m} \cdot \Upsilon\left(\Xi^m(\mathbf{y})\right). \tag{21}$$

Note that in both cases $V(.)$ is increasing w.r.t. $\Xi^n(.)$.

Of particular interest in (19) is the case where $\tilde{\zeta} = \Upsilon^{-1}(0)$ - that is the EDEI $\tilde{\zeta}$ s.t. $\Upsilon(\tilde{\zeta}) = 0$. In this case any addition of individuals to the society implemented such that the EDEI is kept at $\tilde{\zeta}$ will leave the evaluation unchanged, also any replication of the society will give this result. This is a stationary point for the solution of (18).

Is it possible to overcome the "repugnant conclusion"? The answer is affirmative. For sure this is the case when $\Upsilon(\mathbb{R}) = \mathbb{R}_{++}$ and (21) holds but it is also possible to overcome the repugnant conclusion when $\Upsilon(.)$ obtains both positive and negative values. This happens because by continuity and strict monotonicity of $\Upsilon(.)$ there exists a unique $\tilde{\zeta}$ such that $\Upsilon(\tilde{\zeta}) = 0$. Then, if Υ is increasing any population with representative income below $\tilde{\zeta}$ will experience a negative value of $\Upsilon(\zeta)$, therefore increasing the population size (keeping ζ fixed) will lead to a decrease in $n \cdot \Upsilon(\zeta)$. Irrespective of its size the population component will not be able to compensate for a low level of representative income: an increase in population size with an EDEI below $\tilde{\zeta}$ will reduce welfare. From next figure it is evident that the existence of $\tilde{\zeta} > 0$ ensures that for any $n, m \in \mathbb{N}$ for all $\zeta_1 > \tilde{\zeta}$ and all $\zeta_2 < \tilde{\zeta}$ the condition $n\Upsilon(\zeta_2) < m\Upsilon(\zeta_1)$ is always satisfied.



The solution to the repugnant conclusion for $W = n \cdot \Upsilon(\zeta)$

Therefore if $\Upsilon(.)$ is chosen such that $\Upsilon(\tilde{\zeta}) = 0$, then any $\mathbf{x} \in \mathbb{R}^n$ such that $\Xi^n(\mathbf{x}) > \tilde{\zeta}$ will be socially preferred to any $\mathbf{y} \in \mathbb{R}^m$ such that $\Xi^m(\mathbf{y}) < \tilde{\zeta}$ irrespective of $n, m \in \mathbb{N}$, limiting therefore the population size impact on welfare evaluations. Alternatively if $\Upsilon(\mathbb{R}) = \mathbb{R}_{++}$ and (20) holds then for any $\Xi^n(\mathbf{x}) > \Xi^n(\mathbf{y})$ there will always exist a number of replications of $\mathbf{y}$ such that $n \cdot \Upsilon(\Xi^n(\mathbf{x})) < rn \cdot \Upsilon(\Xi^{rn}(\mathbf{y}^r))$ i.e. $r > \frac{\Upsilon(\Xi^n(\mathbf{x}))}{\Upsilon(\Xi^n(\mathbf{y}))}$.

The most common classes of variable population evaluation functions can be obtained for a suitable specification of $\Upsilon(\zeta)$ and $\Xi^n(\mathbf{x})$. In particular if $\Upsilon(\zeta) = U(\zeta) - U(c)$, and $\Xi^n(\mathbf{x}) = U^{-1}\left(\frac{1}{n}\sum_{i=1}^n U(x_i)\right)$ then the evaluation is made according to CLU. Note that in this case the critical level is such that if $\tilde{\zeta} = c$ we get $\Upsilon(\tilde{\zeta}) = 0$. If $U(c)$ is set equal to 0 we get the standard total utilitarianism.

In general if a critical level of income, independent from the income distribution, exists then it has to be equal to $\tilde{\zeta}$.[15] However existence of such fixed level may imply restrictions on $\Upsilon(.)$ and $\Xi^n(\mathbf{x})$ as for the CLU specification.[16]

3.1.1 An example: a pure population problem

We consider an application of (19) to a pure population problem [see Blackorby and Donaldson, 1984]. A positive fixed amount of resources $Z > 0$ has to be divided between a population of $n \in \mathbb{N}$ individuals. The optimal population size n^* is chosen in order to maximize the social welfare. We consider the evaluation function in (19) where $\Upsilon(.)$ is increasing and we impose inequality aversion for the fixed population evaluation leading to S-concavity of $\Xi^n(\mathbf{x})$. It follows that Z has to be shared equally between individuals, therefore we have $\Xi^n(\frac{Z}{n}\cdot\mathbf{1}_n) = \frac{Z}{n}$. Thus the optimization problem becomes

$$Max_n \;\; n \cdot \Upsilon\left(\frac{Z}{n}\right).$$

Alternatively, letting $I = \frac{Z}{n}$ denote the per capita income, the objective function is rewritten as $\frac{Z}{I}\cdot\Upsilon(I)$ leading to

$$Max_I \left(\frac{\Upsilon(I)}{I}\right) \quad \text{s.t. } Z \geq I > 0.$$

If $\Upsilon(I)$ is positive for all $I \geq 0$ [i.e. $\lim_{I\to 0}\Upsilon(I) = \Upsilon(0) > 0$] then the solution I^* to the problem requires letting $I^* \to 0$ implying that $n^* \to \infty$. This is an example of the "repugnant conclusion", a very large population ($n^* \to \infty$) with almost null per-capita income ($I^* \to 0$) will dominate in terms of welfare all the other variable population income distributions. On the other hand, if $\Upsilon(I) < 0$ for all $I \geq 0$ then replicating the population, for a fixed EDEI, will reduce welfare. As a result $n^* \to \infty$ will not be a solution because, given that $\Upsilon(I) < 0$, $\Upsilon(Z)/Z$ will for sure dominate $\lim_{I\to 0}\Upsilon(I)/I$. Any n^* can be a solution depending on the shape of $\Upsilon(.)$, here follows an example where $n^* = 1$.

Example 7 *Let $\Upsilon(I) = -\beta\exp(-I)$ where $\beta > 0$. We get $\frac{\Upsilon(I)}{I} = \frac{-\beta\exp(-I)}{I}$ if we consider I as a continuous variable and we note that*

$$d\left(\frac{\Upsilon(I)}{I}\right)/dI = \frac{\beta\exp(-I)I + \beta\exp(-I)}{I^2} = (\beta\exp(-I))\frac{1+I}{I^2} > 0,$$

[15] Consider distribution $c\mathbf{1}_n$, if a social evaluation neutral individual is added with income c then $n\cdot\Upsilon(c) = n\cdot\Upsilon(\Xi^n(c\mathbf{1}_n)) = (n+1)\cdot\Upsilon(\Xi^n(c\mathbf{1}_{n+1})) = (n+1)\cdot\Upsilon(c)$ for all n, giving $\Upsilon(c) = 0$.

[16] The Number Dependent Critical Level (Blackorby et al., 2002) where the critical level depends on the population size cannot in general be represented for a suitable choice of $\Upsilon(.)$ and $\Xi^n(\mathbf{x})$. The reason is that $\Upsilon(.)$ is independent from the population size and therefore cannot accommodate any consideration taking into account a critical level dependent upon the population size.

then it turns out that the function $\frac{\Upsilon(I)}{I}$ *is increasing in* I*. Thus the pure population problem solution requires to maximize the per capita income, giving* $I^* = Z$*,* $n^* = 1$*.*

The remaining alternative is when the range of $\Upsilon(I)$ is the set of real numbers (or when $\lim_{I\to 0}\Upsilon(I) = 0$). Then there exists a value $\tilde{\zeta}$ such that $\Upsilon(\tilde{\zeta}) = 0$. If $\tilde{\zeta} > 0$ then $\Upsilon(I)$ will have both positive and negative values for $I > 0$. It follows that for sure $I^* > \tilde{\zeta}$ and therefore $n^* < Z/\tilde{\zeta}$, thereby avoiding the repugnant conclusion. The optimal value n^* will depend on the shape of the function $\Upsilon(.)$. The case where $\lim_{I\to 0}\Upsilon(I) = 0$ is equivalent to setting $\tilde{\zeta} = 0$. Then n^* will depend on the shape of $\Upsilon(.)$ and in principle an outcome leading to the repugnant conclusion can still be optimal.

4 Poverty evaluations

We highlight the implications of applying the SPRP for deriving ethically bases poverty indices.

Let $P^n(\mathbf{x},z) : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}_+$ denote a poverty index evaluated for the income distribution $\mathbf{x} \in \mathbb{R}^n$ given a poverty line $z > 0$. As shown by Foster and Shorrocks (1988a,b) poverty orderings are consistent with welfare orderings associated with income distributions $\mathbf{x}^*$ that are censored at the level of the poverty line, i.e. such that $x_i^* = x_i$ if $x_i \leq z$, and $x_i^* = z$ if $x_i > z$. Here we illustrate three parametric formulations of variable population poverty orderings that are consistent with this framework and the analysis developed in the previous sections.

First formulation. Poverty can be assessed as the welfare loss associated with the existence of incomes strictly below the poverty line. More precisely, inspired by Clark et al. (1981) and Chakravarty (1983) we suggest to derive ethical poverty indices measuring the gap between the welfare associated with a censored income distribution with no individuals strictly below the poverty line, i.e. the distribution $z\mathbf{1}_n$, and the welfare of the censored income distribution $\mathbf{x}^*$. Thus a continuous and monotone absolute poverty index is obtained as

$$P^n(\mathbf{x},z) := W^n(z\mathbf{1}) - W^n(\mathbf{x}^*).$$

Considering the general specification of $W^n(\mathbf{x})$ in (19) we get

$$P^n(\mathbf{x},z) = \Upsilon^{-1}\left[n \cdot \Upsilon(z)\right] - \Upsilon^{-1}\left[n \cdot \Upsilon\left(\Xi^n(\mathbf{x}^*)\right)\right].$$

Setting the representative "critical level" at the poverty line level z, i.e. setting $\Upsilon(z) = 0$, allows to avoid a "poverty line based" version of the *repugnant conclusion* where increases in population size through replication (thus also increases in the number of poor individuals) can compensate for increases in individuals' poverty levels. As a result $\Upsilon^{-1}\left[n \cdot \Upsilon(z)\right] = z$ giving

$$P^n(\mathbf{x},z) = z - \Upsilon^{-1}\left[n \cdot \Upsilon\left(\Xi^n(\mathbf{x}^*)\right)\right].$$

If $\Upsilon(.)$ is strictly increasing then $\Upsilon(\zeta) < 0$ for $\zeta < z$, any population replication increases n (and also the number of poor individuals) leading to a poverty increase. Consider for instance the specification

$$\Upsilon(\zeta) = -(z-\zeta)^{\frac{1}{\alpha}} \text{ for } \zeta \leq z \text{ and } \alpha > 0$$

thus $\Upsilon^{-1}(x) = z - (-x)^{\alpha}$ for $x \leq 0$ and $\alpha > 0$, it follows that

$$P^n(\mathbf{x},z) = n^{\alpha} \cdot [z - \Xi^n(\mathbf{x}^*)]\,. \tag{22}$$

The poverty index in (22) is obtained weighting by n^{α} the absolute version of the normative poverty index derived in Chakravarty (1983) $P^n_{W^*}(\mathbf{x},z) = [1 - \Xi^n(\mathbf{x}^*)/z]$ thus $P^n(\mathbf{x},z) = n^{\alpha} \cdot AP^n_{W^*}(\mathbf{x},z)$ where $AP^n_{W^*}(\mathbf{x},z) = z \cdot P^n_{W^*}(\mathbf{x},z)$.

Note that if we denote by $q(\mathbf{x},z)$ the number of poor individuals and let $H(\mathbf{x},z) := q(\mathbf{x},z)/n$ denote the *head count ratio,* then

$$P^n(\mathbf{x},z) = \left[\frac{q(\mathbf{x},z)}{H(\mathbf{x},z)}\right]^{\alpha} \cdot [z - \Xi^n(\mathbf{x}^*)]\,. \tag{23}$$

Therefore proportional population growth both for poor and non-poor groups of individuals affects (23) only through $q(\mathbf{x},z)$, given that $\Xi^n(\mathbf{x}^*)$ and $H(\mathbf{x},z)$ are population replication invariant. Thus, if $\alpha > 0$, poverty increases because of the increase in the number of poor individuals as suggested for instance in Amiel and Cowell (1999) and Kanbur (2001).

One straightforward poverty index, for instance, can be obtained letting $\Xi^n(\mathbf{x}^*) = \frac{1}{n}\sum_{i=1}^n x_i^*$. Then denoting by $AI(\mathbf{x},z) = \frac{1}{q(\mathbf{x},z)}\sum_{i=1}^n (z - x_i^*)$ the *Absolute Income Gap,* i.e. the average income shortfall to the poverty line evaluated only over the population of poor individuals, it is possible to rewrite (23) as

$$P^n(\mathbf{x},z) = [q(\mathbf{x},z)]^{\alpha} \cdot [H(\mathbf{x},z)]^{1-\alpha} \cdot AI(\mathbf{x},z). \tag{24}$$

combining the information on the *Absolute Incidence* of poverty [i.e. $q(\mathbf{x},z)$], on the *Relative Incidence* of poverty [i.e. $H(\mathbf{x},z)$] and on the *Absolute Intensity* of poverty [i.e. $AI(\mathbf{x},z)$].[17] Of course it is possible to specify poverty measures that take into account also the inequality aspect of poverty [which is neglected by $AI(\mathbf{x},z)$].

Second formulation. An alternative procedure allows to identify directly poverty orderings that are consistent with welfare orderings associated with censored distributions. Poverty indices are required to satisfy for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$, the consistency condition

$$P^n(\mathbf{x},z) \leq P^m(\mathbf{y},z) \Longleftrightarrow V(n, \Xi^n(\mathbf{x}^*)) \geq V(m, \Xi^m(\mathbf{y}^*)). \tag{25}$$

Recalling that $\Xi^n(.)$ is replication invariant it follows that $\Xi^n(\mathbf{x}^*)$ can be written in terms of the poverty line and a replication invariant normatively based index of poverty. We can either consider $\Xi^n(\mathbf{x}^*) = z - AP^n_{W^*}(\mathbf{x},z)$, or focus on a generic (absolute) poverty index aggregating individual deprivation measured by a function $p(.)$

[17] Note that if $\alpha \in (0,1)$ the index $P^n(\mathbf{x},z)$ in (24) is increasing in all components.

defined in terms of individuals income gaps. Taking for instance Chakravarty (1983a) specification of the absolute index of poverty deprivation $AP_d^n(\mathbf{x},z) := \frac{1}{n}\sum_{i=1}^n p(z - x_i^*)$ where $p(.)$ is continuous, increasing, convex and $p(0) = 0$ we obtain $\Xi^n(\mathbf{x}^*) = z - p^{-1}[AP_d^n(\mathbf{x},z)]$.

Thus $P^n(\mathbf{x},z) \leq P^m(\mathbf{y},z)$ will require, for this latter case, that

$$\Upsilon^{-1}\left\{n \cdot \Upsilon[z - p^{-1}[AP_d^n(\mathbf{x},z)]]\right\} \geq \Upsilon^{-1}\left\{m \cdot \Upsilon\left[z - p^{-1}[AP_d^m(\mathbf{y},z)]\right]\right\}.$$

Setting for instance the following transformation for the EDEI based on censored incomes $\Upsilon(\zeta) = \{\exp[-p(z-\zeta)]\}^{\frac{1}{\beta}}$ where $\beta \in \mathbb{R}\backslash 0$ we get

$$z + p^{-1}\{\beta\ln(n) - AP_d^n(\mathbf{x},z)\} \geq z + p^{-1}\{\beta\ln(m) - AP_d^m(\mathbf{y},z)\}.$$

Rearranging we obtain:

$$P^n(\mathbf{x},z) \leq P^m(\mathbf{y},z) \Leftrightarrow AP_d^n(\mathbf{x},z) - \beta\ln(n) \leq AP_d^m(\mathbf{y},z) - \beta\ln(m).$$

Thus the poverty ordering can be represented by the class of poverty indices

$$P^n(\mathbf{x},z) := AP_d^n(\mathbf{x},z) - \beta\ln(n), \tag{26}$$

which is in spirit related to the absolute version of the family of poverty indices investigated in Chakravarty et al. (2006) and approaches the replication invariant classical type of indices as $\beta \to 0$. Alternatively denoting by $NAP_d^n(\mathbf{x},z)$ the normalized average poverty deprivation evaluated over the population of poor individuals i.e. such that $AP_d^n(\mathbf{x},z) = H(\mathbf{x},z) \cdot NAP_d^n(\mathbf{x},z)$ we can rewrite

$$P^n(\mathbf{x},z) = H(\mathbf{x},z) \cdot NAP_d^n(\mathbf{x},z) - \beta\ln[q(\mathbf{x},z)] + \beta\ln[H(\mathbf{x},z)].$$

Note that for fixed population comparisons the relevant information is given by $AP_d^n(\mathbf{x},z)$, if q increases then [if $\beta < 0$] poverty increases. An increase in population with q and NAP_d^n fixed reduces H and induces a marginal change in $P^n(\mathbf{x},z)$ that for large populations can be approximated by $dP^n = [NAP_d^n + \beta/H] \cdot dH$. Thus $dP^n < 0$ only if $H \cdot NAP_d^n > -\beta > 0$. That is, only if the original level of poverty averaged across the entire population is sufficiently large then a marginal increase in the number of non-poor individuals reduces poverty.[18]

Third formulation. Moreover, if we set $\Upsilon(\zeta) = \alpha - p(z-\zeta)$ where $\alpha \in \mathbb{R}$ we get

$$z - p^{-1}\{\alpha - n\cdot\alpha + n \cdot AP_d^n(\mathbf{x},z)\} \geq z - p^{-1}\{\alpha - m\cdot\alpha + m\cdot AP_d^m(\mathbf{y},z)\}.$$

Rearranging we obtain:

$$P^n(\mathbf{x},z) \leq P^m(\mathbf{y},z) \Leftrightarrow n\cdot[AP_d^n(\mathbf{x},z) - \alpha] \leq m\cdot[AP_d^m(\mathbf{y},z) - \alpha]$$

[18]This finding is in line with the theorem by Kundu and Smith (1983) stating the impossibility of combining inequality aversion of poverty measures with a behavior that takes properly into account increases in the number of poor individual and increases in the size of the non-poor population. In our case in order to have a poverty index responding positively to absolute increases in the number of poor individuals we need to set $\beta < 0$, while if the poverty index has to unambiguously decrease with the number of non poor individuals it is necessary that $\beta \geq 0$.

which is precisely the absolute version of the class of poverty orderings characterized in Chakravarty et al. (2006). Note that the "critical level" $\tilde{\zeta}$ such that $\Upsilon(\tilde{\zeta}) = \alpha - p(z - \tilde{\zeta}) = 0$ is obtained when $AP_d^n(\mathbf{x},z) = \alpha$, thus population replications induce an improvement in the welfare evaluation based on censored incomes an thereby reduce poverty only if $AP_d^n(\mathbf{x},z) < \alpha$.

The procedure in (25) allows to derive families of poverty indices that are sensitive to the absolute number of poor individuals and are consistent with traditional poverty indices for fixed population comparisons.

5 Conclusions

Two major properties are applied in the literature for the evaluation of variable populations income distributions, the Population Replication Invariance (PRI) and the Critical Level Principle (CLP). Their application leads to different orderings and associated ordinal or cardinal welfare and inequality indicators. Moreover, they follow completely different perspectives in comparing distributions with different numbers of individuals. The former allows comparability between distributions obtained through replications, the latter considers comparisons obtained adding recursively one extra individual.

In order to derive a common framework of analysis we have considered weaker versions of the PRI, the Population Replication Principle (PRP) and the Strong PRP. The former requires invariance of the welfare ordering over distributions of the same population size if they are replicated, the latter extends the requirement to consider also distributions with different population size.

Imposing additional properties specifying the invariance of the ordering w.r.t. replications we have characterized within a common framework of analysis general classes of variable population evaluation functions and identified parametric evaluation functions that are based on summary statistics on population size, average income and an inequality index. The more general of these classes includes as special cases both replication invariant functions and the critical level principles that are thus derived within a population replication framework and not a population addition one.

Furthermore, we note that the SPRP induces restrictions on the class of variable population evaluation functions that are equivalent to those imposed by a version of the “multiplicative translation functional equation”. A class of particular solutions of this functional equation is investigated. These solutions are analytically very simple but generalize the results presented in the first part of the paper and provide an attractive specification to the variable population evaluation function suggested in Blackorby and Donaldson (1984a) that is still general enough to include both average (population replication) principles, critical level principles and classical utilitarian principles. Making use of this general class of evaluation functions we have derived ethically based poverty indices that may take into account also the absolute number of poor individuals and not only their population share in the overall distribution.

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A Appendix

Example 8 *Consider the symmetric and inequality averse SEF:*

$$\tilde{W}^n(\mathbf{x}) := \begin{cases} \sum_{i=1}^{n/2} \left(x_{(i)} \cdot x_{(n+1-i)}\right) & \text{if } n \text{ is even} \\ \sum_{i=1}^{(n-1)/2} \left(x_{(i)} \cdot x_{(n+1-i)}\right) + [x_{\left(\frac{n+1}{2}\right)}]^2 & \text{if } n \text{ is odd} \end{cases},$$

where $0 < x_{(i)} \leq x_{(i+1)}$. *Let* $n = 3$, *the EDEI* ζ_3 *is s.t.* $\tilde{W}^3(\zeta_3 \cdot \mathbf{1}_3) = 2\left(\zeta_3\right)^2 = \tilde{W}^3(\mathbf{x}) = x_{(1)}x_{(3)} + \left(x_{(2)}\right)^2$, *that is* $\zeta_3 = \{[x_{(1)}x_{(3)} + \left(x_{(2)}\right)^2]/2\}^{\frac{1}{2}}$. *Replicating the population we get* $\tilde{W}^6(\zeta_6 \cdot \mathbf{1}_6) = 3\left(\zeta_6\right)^2 = \tilde{W}^6(\mathbf{x}^2) = x_{(1)}x_{(3)} + x_{(1)}x_{(3)} + \left(x_{(2)}\right)^2$, *that is* $\zeta_6 = \{[2x_{(1)}x_{(3)} + \left(x_{(2)}\right)^2]/3\}^{\frac{1}{2}}$. *But* $\zeta_3 = \zeta_6$ *only if* $\left(x_{(2)}\right)^2 = x_{(1)}x_{(3)}$ *which shows that the EDEI function is not replication invariant.*

A.1 Proofs

A.1.1 Proof of Lemma 3.1

Sufficiency. If the specification in (6) holds, then SPRP is satisfied because Π is order preserving w.r.t. $V(.)$ while (7) ensures that MWC is also satisfied.

Necessity. Imposing SPRP in conjunction of MWC amounts to require that (6) holds. MWC ensures that for any value in $V(\mathbb{N}, \mathbb{R})$ there exist $\mathbf{x}, \mathbf{y} \in \Omega$ and $m, n \in \mathbb{N}$ s.t. $V(n, \Xi^n(\mathbf{x})) = V(m, \Xi^m(\mathbf{y}))$. If $V(n, \Xi^n(\mathbf{x})) = V(m, \Xi^m(\mathbf{y}))$ then SPRP requires that $V(rn, \Xi^{rn}(\mathbf{x}^r)) = V(rm, \Xi^{rm}(\mathbf{y}^r))$ irrespective of the replication parameter thus (6) holds. Condition (7) is a direct implication of MWC applied to (6). Moreover, given that SPRP implies PRP then following Bossert (1990a) the EDEI function is necessarily replication invariant. ■

In order to prove Proposition 3.1 we first need to prove the following lemma.

Lemma A.1 *A monotonic function* $\phi : \mathbb{N} \to \mathbb{R}_{++}$ *satisfies*

$$\phi(rv) = \phi(r) \cdot \phi(v)$$

for all $r, v \in \mathbb{N}$ *if and only if* $\phi(r) = r^\alpha$ *where* $\alpha \in \mathbb{R}$, *for all* $r \in \mathbb{N}$.

Note that $\phi(rt) = \phi(r)\phi(v)$ is a Cauchy functional equation, since $\phi(.)$ is defined over $\mathbb{N}$, we cannot make use of the solution valid for functions defined over dense sets (see Aczél, 1966). Aczél and Dhombres (1989) in Ch. 16.5 provide a general result for deriving solutions of the Cauchy equation defined over restricted domains, including natural numbers (as in this case). As they show the previous lemma results as a corollary of their Th. 16. Here we will follow the results in Donaldson and Weymark (1980) and prove directly our claim. The next result follows from Lemma 1 in Donaldson and Weymark (1980):

Lemma A.2 (Donaldson and Weymark (1980)) *The function* $\phi : \mathbb{N} \to \mathbb{R}_{++}$ *satisfies* $\phi(rv) = \phi(r)\phi(v)$ *for all* $r, v \in \mathbb{N}$ *if and only if*

$$\phi(t) = \prod_{i \in \mathbb{N}} c_i^{n_i},$$

where $c_i \in \mathbb{R}$ is an arbitrary constant and n_i is the number of times the i^{th} prime number ϱ_i ($\varrho_1 = 2$, $\varrho_2 = 3$, $\varrho_3 = 5$, $\varrho_4 = 7, ..$) occurs in the unique factorization of t into primes.

Proof of Lemma A.1: Since $\phi(1v) = \phi(1)\phi(v)$, it is evident that $\phi(1) = 1$. We follow now the proof of Theorem 2 in Donaldson and Weymark (1980) in order to prove that given $\phi(r)$ is *positive* and *weakly monotonic*, the solution of the functional equation is $\phi(r) = r^\alpha, \alpha \in \mathbb{R}$.

Theorem 2 in Donaldson and Weymark (1980) shows that if $\phi(r)$ is monotonically non-decreasing then $\phi(r) = r^\beta$, $\beta \geq 0$. Since we restrict $\phi(r)$ being *monotonic*, we are left with the case in which $\phi(r)$ is monotonically non-increasing. Following exactly the same logical arguments as in the proof of Theorem 2 in Donaldson and Weymark (1980), we prove the second part of our statement.

Since $\phi(r) > 0$ then $c_i > 0$ for all $i \in \mathbb{N}$ it follows that we can write $c_i = \varrho_i^{\beta_i}$, $\beta_i \in \mathbb{R}$. We need to prove that if $\phi(r)$ is non increasing then $\beta_i = \beta$ for all $i \in \mathbb{N}$. Suppose the contrary i.e. $\phi(r)$ is non increasing but $\beta_i \neq \beta_j$ for some $i, j \in \mathbb{N}$. Since $\mathbb{N}$ has a countable number of elements, then there exist two real numbers $\beta_L < \beta_H$, and a disjoint partition of $\mathbb{N} = (\mathbb{N}^L, \mathbb{N}^H)$ such that: $i \in \mathbb{N}^L \Leftrightarrow \beta_i \leq \beta_L$, and $i \in \mathbb{N}^H \Leftrightarrow \beta_i \geq \beta_H$.

Consider now a number $\bar{r} \in \mathbb{N}$, $\bar{r} > 1$, such that $n_i = 0$ for all $i \in \mathbb{N}^H$, that is $\bar{r}$ is factorized by all the prime numbers in $\mathbb{N}^L$. Therefore $\phi(\bar{r}) \leq \bar{r}^{\beta_L}$, as well as $\phi(\bar{r}^T) \leq \bar{r}^{T\beta_L}$ for all $T \in \mathbb{N}$. Consider now $\bar{r} + 1$, or more generally $\bar{r}^T + 1$, no number ϱ_i with index $i \in \mathbb{N}^L$ is a prime factor of $\left(\bar{r}^T + 1\right)$, so $\left(\bar{r}^T + 1\right)$ could be decomposed into the product of prime factors ϱ_j where $j \in \mathbb{N}^H$. It follows that $\phi\left(\bar{r}^T + 1\right) \geq \left(\bar{r}^T + 1\right)^{\beta_H}$. Combining both the obtained inequalities we get

$$\frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \frac{\left(\bar{r}^T + 1\right)^{\beta_H}}{\bar{r}^{T\beta_L}},$$

that can be rearranged as

$$\frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \left(\frac{\bar{r}^T + 1}{\bar{r}^T}\right)^{\beta_H} \left(\bar{r}^T\right)^{(\beta_H - \beta_L)}.$$

From which follows

$$\lim_{T\to\infty} \frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \lim_{T\to\infty} \left(\frac{\bar{r}^T + 1}{\bar{r}^T}\right)^{\beta_H} \left(\bar{r}^T\right)^{(\beta_H - \beta_L)} = \infty,$$

which contradicts the fact that $\phi(.)$ is non-increasing. It follows that necessarily $\beta_i = \beta$ for all $i \in \mathbb{N}$, in which case $\phi(r) = r^\beta$. In order to satisfy non-increasingness of $\phi(r)$ we need to consider $\beta \leq 0$. Monotonicity of $\phi(r)$ requires $\phi(r) = r^\beta$, where $\beta \in \mathbb{R}$. ■

A.1.2 Proof of Proposition 3.1

We consider explicitly the value function $V(n, \Xi^n(\mathbf{x}))$. The *sufficiency* part can be verified replicating $\mathbf{x}$. For the *necessity* part we first notice that the welfare evaluation of the rk replication of $\mathbf{x}$ is $V(rkn, \Xi^{rkn}(\mathbf{x}^{rk})) = \psi(kr) + \phi(kr) \cdot V(n, \Xi^n(\mathbf{x}))$ and that it can be decomposed into two replications of size r and k

$$\begin{aligned} V(rkn, \Xi^{rkn}(\mathbf{x}^{rk})) &= \psi(k) + \phi(k) \cdot V(rn, \Xi^{rn}(\mathbf{x}))) \\ &= \psi(k) + \phi(k) \left[\psi(r) + \phi(r) \cdot V(n, \Xi^n(\mathbf{x}))\right] \\ &= \psi(k) + \phi(k)\psi(r) + \phi(k)\phi(r) \cdot V(n, \Xi^n(\mathbf{x})) \end{aligned}$$

leading to the welfare consistency requirement:

$$\psi(kr) + \phi(kr)V(n, \Xi^n(\mathbf{x})) = \psi(k) + \phi(k)\psi(r) + \phi(k)\phi(r)V(n, \Xi^n(\mathbf{x})) \tag{27}$$

which has to be satisfied for all $\mathbf{x} \in \mathbb{R}^n$, all $n \in \mathbb{N}$ and all $r, k \in \mathbb{N}$. This condition implies that the following system of functional equations has to be satisfied by $\psi(.)$ and $\phi(.)$:

$$\begin{aligned} \psi(k) + \phi(k) \cdot \psi(r) &= \psi(kr) \quad \text{and} && (28) \\ \phi(kr) &= \phi(k) \cdot \phi(r) \quad \text{for all } r, k \in \mathbb{N}. && (29) \end{aligned}$$

We require that the restrictions on the scalar and level coefficients in the affine transformation have to be independently satisfied. Otherwise, we can always set appropriate values for $V(n, \Xi^n(\mathbf{x})) \in \mathbb{R}$ (or find appropriate distributions $\mathbf{x} \in \mathbb{R}^n$) such that the equation (27) is not satisfied. We substitute the solution to (29) from Lemma A.1, getting

$$\begin{aligned} \psi(k) + k^\alpha \psi(r) &= \psi(kr) \quad \text{and} && (30) \\ \phi(n) &= n^\alpha \quad \text{for all } r, k, n \in \mathbb{N}, \ \alpha \in \mathbb{R}. \end{aligned}$$

For $\alpha = 0$ we get

$$\begin{aligned} \psi(k) + \psi(r) &= \psi(kr) \quad \text{for all } r, k \in \mathbb{N} \text{ and} \\ \phi(n) &= 1. && (31) \end{aligned}$$

We let $\psi(r) := \ln \phi(r)$ where $\phi(.)$ is the function in (29). Given the logarithmic transformation both the domain restriction and the property of monotonicity of $\phi(.)$ are satisfied. We then get

$$\begin{aligned} \ln \phi(rk) &= \ln \phi(k) + \ln \phi(r) \text{ for all } r, k \in \mathbb{N} && (32) \\ &\Leftrightarrow \ \phi(rk) = \phi(k) \cdot \phi(r) \text{ for all } r, k \in \mathbb{N} \end{aligned}$$

from which we can apply the result in Lemma A.1 and get $\phi(n) = n^\beta$ where $\beta \in \mathbb{R}$, for all $n \in \mathbb{N}$, which gives $\psi(n) = \ln \phi(n) = \beta \ln(n)$.

If $\alpha \neq 0$, the first functional equation in (30) can be rewritten also noticing that $\psi(kr) = \psi(rk)$:

$$\psi(k) + k^\alpha \psi(r) = \psi(kr) = \psi(rk) = \psi(r) + r^\alpha \psi(k) \quad \text{for all } r, k \in \mathbb{N},$$

that is

$$\psi(k) \cdot [1 - r^\alpha] = \psi(r) \cdot [1 - k^\alpha] \quad \text{for all } r, k \in \mathbb{N}. \tag{33}$$

Since $\alpha \neq 0$ then $1 - k^\alpha \neq 0$ and $1 - r^\alpha \neq 0$ (if $k, r \neq 1$) we can therefore divide both sides and obtain

$$\frac{\psi(k)}{[1-k^\alpha]} = \frac{\psi(r)}{[1-r^\alpha]} \quad \text{for all } r, k \in \mathbb{N}\ k, r \neq 1.$$

It follows that $\frac{\psi(n)}{[1-n^\alpha]}$ is independent from n which gives

$$\frac{\psi(n)}{[1-n^\alpha]} = \beta \quad \text{for all } n \in \mathbb{N},\ n \neq 1 \text{ if } \alpha \neq 0,$$

where $\beta \in \mathbb{R}$, from which

$$\psi(n) = \beta\left[1-n^\alpha\right] \quad \text{for all } n \in \mathbb{N},\ n \neq 1 \text{ if } \alpha \neq 0. \tag{34}$$

If $k = 1$ then from (30) $\psi(1) = 0$, similarly if $r = 1$. This extends the solution in (34) allowing to include also $n = 1$ in the domain.

We need now to substitute into $W(\mathbf{x})$ for $\phi(n) = n^\alpha$ and $\psi(n) = \beta\left[1-n^\alpha\right]$ $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$.

We choose $n = 1$, and $x = \zeta$, then, after replicating r times we get

$$V(r, \zeta\mathbf{1}_r) = \psi(r) + \phi(r) \cdot V(1, \zeta)$$

for all $r \in \mathbb{N}$ and all $\zeta \in \mathbb{R}$. Letting $\Psi(\zeta) := V(1, \zeta)$ we have

$$V(n, \Xi^n(\mathbf{x})) = \beta\left[1-n^\alpha\right] + n^\alpha \cdot \Psi(\Xi^n(\mathbf{x}))$$

where $\Psi(.)$ is increasing and continuous given that $V(n, \Xi^n(\mathbf{x}))$ is weakly monotonic [implying that it is increasing] and continuous in $\Xi^n(\mathbf{x})$. Rearranging we get

$$V(n, \Xi^n(\mathbf{x})) = \beta + n^\alpha\left[\Psi(\Xi^n(\mathbf{x})) - \beta\right].$$

We now show that $\Xi^n(\mathbf{x})$ has to be *population replication invariant.*

According to the obtained specification for social evaluation functions consistent with GAfPR we have:

If $\alpha = 0$,

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = \beta \ln(rn) + \Psi(\Xi^{rn}(\mathbf{x}^r)),$$

and by GAfPR

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = \beta \ln(r) + V(n, \Xi^n(\mathbf{x})) = \beta \ln(r) + \left[\beta \ln(n) + \Psi(\Xi^n(\mathbf{x}))\right]$$

for all $r \in \mathbb{N}$ all $\mathbf{x} \in \mathbb{R}^n$, all $n \in \mathbb{N}$.

If $\alpha \neq 0$,

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = (nr)^\alpha\left[\Psi(\Xi^{rn}(\mathbf{x})) - \beta\right] + \beta,$$

and applying GAfPR we get

$$V(rn, \Xi^{rn}(\mathbf{x}^r)) = (r)^\alpha\left[(n)^\alpha\left[\Psi(\Xi^n(\mathbf{x})) - \beta\right]\right] + \beta.$$

Equating both sides gives

$$(nr)^\alpha\left[\Psi(\Xi^{rn}(\mathbf{x})) - \beta\right] = (r)^\alpha\left[(n)^\alpha\left[\Psi(\Xi^n(\mathbf{x})) - \beta\right]\right]$$

for all $r \in \mathbb{N}$ all $\mathbf{x} \in \mathbb{R}^n$, all $n \in \mathbb{N}$.

It follows that in both cases $\Psi(\Xi^{rn}(\mathbf{x}^r)) = \Psi(\Xi^n(\mathbf{x}))$, from which, given that $\Psi(.)$ is increasing, we necessarily have

$$\Xi^{rn}(\mathbf{x}^r) = \Xi^n(\mathbf{x}).$$

Invariance conditions. Note that welfare evaluation is no longer invariant w.r.t. any increasing transformations of $\Psi(.)$ unless $\alpha=\beta=0$.

If $\alpha \neq 0$, let $T(.)$ be a transformation of $\Psi(.)$, it (strictly) preserves the ordering if for all $\mathbf{x},\mathbf{y}\in\Omega$:

$$\begin{aligned} n^{\alpha}\left[\Psi(\Xi^{n}(\mathbf{x}))-\beta\right] &= (>)\ m^{\alpha}\left[\Psi(\Xi^{m}(\mathbf{y}))-\beta\right] \\ &\Leftrightarrow\ n^{\alpha}\left[T[\Psi(\Xi^{n}(\mathbf{x}))]-\beta\right]=(>)\ m^{\alpha}\left[T[\Psi(\Xi^{m}(\mathbf{y}))]-\beta\right] \end{aligned}$$

We consider first the case with equality. Let $\mathbf{x}=x\mathbf{1}_n$, $\mathbf{y}=y\mathbf{1}_m$, $\Psi=\Psi(x)$, and $\Psi'=\Psi(y)$ we get that whenever

$$n^{\alpha}\Psi-n^{\alpha}\beta=m^{\alpha}\Psi'-m^{\alpha}\beta \tag{35}$$

it has to be that $n^{\alpha}\left[T(\Psi)-\beta\right]=m^{\alpha}\left[T(\Psi')-\beta\right]$ for all $n,m\in\mathbb{N}$. That is $n^{\alpha}\left[T(\Psi)-\beta\right]=m^{\alpha}\left[T(\Psi')-\beta\right]$ for all $\Psi,\Psi'\in\mathbb{R}$ such that $\frac{\Psi'-\beta}{\Psi-\beta}=\left(\frac{n}{m}\right)^{\alpha}=\rho^{\alpha}$ where $\rho\in\mathbb{Q}_{++}$ is any positive rational number.

If we rearrange $n^{\alpha}\left[T(\Psi)-\beta\right]=m^{\alpha}\left[T(\Psi')-\beta\right]$ we obtain $\frac{T(\Psi')-\beta}{T(\Psi)-\beta}=\left(\frac{n}{m}\right)^{\alpha}=\rho^{\alpha}=\frac{\Psi'-\beta}{\Psi-\beta}$. That is for any $\beta\in\mathbb{R}$

$$\begin{aligned} \frac{T(\Psi')-\beta}{T(\Psi)-\beta} &= \frac{\Psi'-\beta}{\Psi-\beta}\longleftrightarrow\frac{T(\Psi')-\beta}{\Psi'-\beta}=\frac{T(\Psi)-\beta}{\Psi-\beta}, \\ \text{for all } \Psi',\Psi &\in \mathbb{R}\quad\text{such that } \frac{\Psi'-\beta}{\Psi-\beta}=\rho^{\alpha},\ \text{where } \rho\in\mathbb{Q}_{++}. \end{aligned}$$

Let $S_{\Psi_0}^{\alpha,\beta}=\{\Psi:\Psi-\beta=\rho^{\alpha}\left[\Psi_0-\beta\right] \text{ for all } \rho\in\mathbb{Q}_{++}\}$ and $\frac{T(\Psi_0)-\beta}{\Psi_0-\beta}=b_0$, it follows that for all $\Psi\in S_{\Psi_0}^{\alpha,\beta}$ we have $\frac{T(\Psi)-\beta}{\Psi-\beta}=\frac{T(\Psi_0)-\beta}{\Psi_0-\beta}=b_0$. That is, the admissible transformation is $T(\Psi)=b_0\Psi+(1-b_0)\beta$ for all $\Psi\in S_{\Psi_0}^{\alpha,\beta}$.

We first prove the result for the case $\alpha\neq 0$ and $\beta=0$, and then we will extend it to $\alpha\neq 0$ and $\beta\neq 0$.

Let $\beta=0$ then $S_{\Psi_0}^{\alpha,0}=\{\Psi:\Psi=\rho^{\alpha}\cdot\Psi_0 \text{ for all } \rho\in\mathbb{Q}_{++}\}$. Note that $S_{\Psi_0}^{\alpha,0}$ is a dense set in $\mathbb{R}_{++}$ if $\Psi_0>0$, while it is dense in $\mathbb{R}_{--}$ (the set of all negative real numbers) if $\Psi_0<0$. Therefore at least the admissible transformations are in principle different depending on whether $\Psi>0$ or $\Psi<0$.

We first note that given that

$$n^{\alpha}\Psi(\Xi^{n}(\mathbf{x}))>m^{\alpha}\Psi(\Xi^{m}(\mathbf{y}))\Leftrightarrow n^{\alpha}T\left[\Psi(\Xi^{n}(\mathbf{x}))\right]>m^{\alpha}T\left[\Psi(\Xi^{m}(\mathbf{y}))\right]$$

necessarily $b_0>0$.

We now prove that $T(\Psi)=b_1\Psi$ for all $\Psi\in\mathbb{R}_{++}$, and $T(\Psi)=b_2\Psi$ for all $\Psi\in\mathbb{R}_{--}$ where $b_1,b_2>0$. We show this result for $\Psi\in\mathbb{R}_{++}$. A similar result will also apply to all $\Psi\in\mathbb{R}_{--}$.

Suppose there exist $\Psi_0,\Psi_1\in\mathbb{R}_{++}$ such that $S_{\Psi_0}^{\alpha,0}\cap S_{\Psi_1}^{\alpha,0}=\emptyset$. Note that if there exists a value Ψ' such that $\Psi'\in S_{\Psi_0}^{\alpha,0}$, and $\Psi'\in S_{\Psi_1}^{\alpha,0}$ then $S_{\Psi_0}^{\alpha,0}=S_{\Psi_1}^{\alpha,0}$. Let $\Psi'=\rho^{\alpha}\Psi_0$ while $\Psi'=\sigma^{\alpha}\Psi_1$, for $\rho,\sigma\in\mathbb{Q}_{++}$ it follows that $\Psi_1=\left(\frac{\rho}{\sigma}\right)^{\alpha}\Psi_0=\varrho^{\alpha}\Psi_0$ for $\varrho\in\mathbb{Q}_{++}$. Therefore $\Psi''\in S_{\Psi_0}^{\alpha,0}$ if and only if $\Psi''\in S_{\Psi_1}^{\alpha,0}$.

If $S^{\alpha,0}_{\Psi_0} \cap S^{\alpha,0}_{\Psi_1} = \emptyset$, we show that the coefficients b_0 and b_1 have to coincide if $V(.)$ satisfies continuity and weak monotonicity.

Let $\mathbf{x} = x\mathbf{1}_n$, for $x \in \mathbb{R}$ then $\Psi(\Xi^n(\mathbf{x})) = \Psi(x)$, by weak monotonicity $x'' > x' \longleftrightarrow \Psi(x'') > \Psi(x')$ that is equivalent to $T(\Psi(x'')) > T(\Psi(x'))$. Denote $\Psi'' = \Psi(x'')$ and $\Psi' = \Psi(x')$ and suppose that $\Psi' \in S^{\alpha,0}_{\Psi_0}$ and $\Psi'' \in S^{\alpha,0}_{\Psi_1}$ where $S^{\alpha,0}_{\Psi_0} \cap S^{\alpha,0}_{\Psi_1} = \emptyset$. Given that both $S^{\alpha,0}_{\Psi_0}$ and $S^{\alpha,0}_{\Psi_1}$ are dense in $\mathbb{R}_{++}$, then, given continuity of $\Psi(.)$, we can choose a value Ψ'' arbitrarily close to Ψ' such that $\Psi'' > \Psi'$, in which case it has also to be that $T(\Psi'') > T(\Psi')$. Given that $T(\Psi) = b_0\Psi$ for all $\Psi \in S^{\alpha,0}_{\Psi_0}$ and $T(\Psi) = b_1\Psi$ for all $\Psi \in S^{\alpha,0}_{\Psi_1}$, where b_0 and b_1 are not necessarily equal, then for any value Ψ'' arbitrarily close to Ψ' such that $\Psi'' > \Psi'$ the condition $T(\Psi'') > T(\Psi') \Leftrightarrow b_1\Psi'' > b_0\Psi'$ is satisfied only if $b_1 \geq b_0$. Otherwise if $b_1 < b_0$ we will have $\frac{b_0}{b_1} > 1$ and therefore it will always be possible to identify a value of Ψ'' sufficiently close to Ψ' such that $\frac{b_0}{b_1} > \frac{\Psi''}{\Psi'} > 1$.

Repeating the same argument for any value $\Psi'' \in S^{\alpha,0}_{\Psi_1}$ arbitrarily close to $\Psi' \in S^{\alpha,0}_{\Psi_0}$ such that $\Psi'' < \Psi'$ we get the restriction $b_1 \leq b_0$. Combining both conditions we get $b_1 = b_0$.

Clearly the same logic can be applied for all $\Psi', \Psi'' \in \mathbb{R}_{--}$. It has to be remarked that when $S^{\alpha,0}_{\Psi_0} \subset \mathbb{R}_+$ and $S^{\alpha,0}_{\Psi_1} \subset \mathbb{R}_-$ then $b_0 > 0$ and $b_1 > 0$ can be different.

To expand the proof in order to hold also for to and $\beta \neq 0$ we consider the translated variable $\tilde{\Psi} : \Psi - \beta$ instead of Ψ. Applying to $\tilde{\Psi}$ the same logic followed in the previous part of the proof we get the characterization of the invariance condition for the general case where $\alpha \neq 0$:

$$T(\Psi) = \begin{cases} b_0\Psi + (1-b_0)\,\beta & \text{for all } \Psi > \beta, \\ b_1\Psi + (1-b_1)\,\beta & \text{for all } \Psi < \beta, \quad \text{where } b_0, b_1 > 0. \\ \beta & \text{if } \ \Psi = \beta \end{cases}$$

We are left with the last case, i.e. $\beta \neq 0$ and $\alpha = 0$. In this case for all $\mathbf{x}, \mathbf{y} \in \Omega$ we have $\beta \ln(n) + \Psi(\Xi^n(\mathbf{x})) = (>)\beta \ln(m) + \Psi(\Xi^m(\mathbf{y})) \Leftrightarrow \beta \ln(n) + T[\Psi(\Xi^n(\mathbf{x}))] = (>)\beta \ln(m) + T[\Psi(\Xi^m(\mathbf{y}))]$. Taking an exponential transformation on both sides of the condition and letting $\tilde{\Psi} = \exp(\Psi)$ we get that for all $\mathbf{x}, \mathbf{y} \in \Omega$:

$$n^\beta \tilde{\Psi}(\Xi^n(\mathbf{x})) = (>)m^\beta \tilde{\Psi}(\Xi^m(\mathbf{y})) \Leftrightarrow n^\beta \exp\{T[\Psi(\Xi^n(\mathbf{x}))]\} = (>)m^\beta \exp\{T[\Psi(\Xi^m(\mathbf{y}))]\}.$$

Since the exponential transformation is strictly increasing the new invariance condition turns out to be equivalent to the original one. Then, letting $\Psi(\Xi^n(\mathbf{x})) = \ln[\tilde{\Psi}(\Xi^m(\mathbf{y}))]$ and denoting $T'(.) = \exp \circ T \circ \ln(.)$ we get that for all $\mathbf{x}, \mathbf{y} \in \Omega$:

$$n^\beta \tilde{\Psi}(\Xi^n(\mathbf{x})) = (>)m^\beta \tilde{\Psi}(\Xi^m(\mathbf{y})) \Leftrightarrow n^\beta T'[\tilde{\Psi}(\Xi^n(\mathbf{x}))] = (>)m^\beta T'[\tilde{\Psi}(\Xi^m(\mathbf{y}))].$$

Following the same logic previously adopted for the $\beta = 0$ and $\alpha = 0$ there we get that when $\beta \neq 0$ then $T'[\tilde{\Psi}(.)] = b \cdot \tilde{\Psi}(.)$ where $b > 0$. Note that here by construction $\tilde{\Psi}(.) > 0$.

Rewriting explicitly $\tilde{\Psi}(.)$ we get $T'[\exp \Psi(.)] = b \cdot [\exp \Psi(.)]$. Rewriting now $T'[.]$ in terms of $T[.]$ we get:

$$\begin{aligned} \exp\{T[\ln(\exp \Psi(.))]\} &= b \cdot [\exp \Psi(.)] \Leftrightarrow \exp\{T[\Psi(.)]\} = b \cdot [\exp \Psi(.)] \\ \text{that is } T[\Psi(.)] &= \ln(b) + \Psi(.). \end{aligned}$$

Letting $\ln(b) = a \in \mathbb{R}$ we get that for all $\Psi(.)$ the admissible transformation is $T[\Psi(.)] = a + \Psi(.)$ for all $a \in \mathbb{R}$. ■

A.1.3 Proof of Proposition 3.2

Applying O-GAfPR, from Proposition 3.1 we obtain that $W(\mathbf{x}) = G \circ (W_{\alpha,\beta}(\mathbf{x}))$ for some increasing transformation $G = F^{-1}$.

Consider the increasing function $B : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} B(t) &:= \begin{cases} \beta + |t|^{|\alpha|} & \text{if } t \geq 0 \\ \beta - |t|^{|\alpha|} & \text{if } t < 0 \end{cases} \quad \text{if } \alpha \neq 0; \\ B(t) &:= |\beta| \cdot \ln(t) \text{ if } \alpha = 0, \beta \neq 0; \; B(t) := t \text{ if } \alpha = \beta = 0. \end{aligned} \tag{36}$$

and the increasing function $\bar{\Upsilon} : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \bar{\Upsilon}(\zeta) &:= \begin{cases} |\Psi(\zeta) - \beta|^{\frac{1}{|\alpha|}} & \text{if } \Psi(\zeta) \geq \beta \\ -|\Psi(\zeta) - \beta|^{\frac{1}{|\alpha|}} & \text{if } \Psi(\zeta) < \beta \end{cases} \quad \text{if } \alpha \neq 0, \\ \bar{\Upsilon}(\zeta) &:= \{\exp[\Psi(\zeta)]\}^{\frac{1}{|\beta|}} \qquad \text{if } \alpha = 0, \beta \neq 0, \\ \bar{\Upsilon}(\zeta) &:= \Psi(\zeta) \qquad \text{if } \alpha = \beta = 0. \end{aligned} \tag{37}$$

Then letting $H = F^{-1} \circ B$ we get that $W(\mathbf{x}) = F^{-1} \circ (W_{\alpha,\beta}(\mathbf{x})) = H \circ [n^k \cdot \bar{\Upsilon}(\Xi^n(\mathbf{x}))]$ where $k \in \{-1, 0, 1\}$. Thus there exist a one to one relationship between k and $\bar{\Upsilon}(.)$ on one hand and $\Psi(.)$ and the values of α, β on the other hand. In particular $k = 0$ if $\alpha = \beta = 0$; $k = -1$ if either $\alpha < 0$ or $\alpha = 0$ and $\beta < 0$ while $k = 1$ in all other cases. ■

A.1.4 Proof of Corollary 3.2

From Proposition 3.1 we obtain that $W(\mathbf{x}) = H(W_{\alpha,\beta}(\mathbf{x}))$ for some increasing transformation H. Applying N we get $W(\zeta) = H(\Psi(\zeta)) = \zeta$ for all $\zeta \in \mathbb{R}$ thus $H(t) = \Psi^{-1}(t)$ for all $t \in \mathbb{R}$, where the function Ψ is continuous and increasing.

Substituting we obtain $W(\mathbf{x}) = \Psi^{-1}(W_{\alpha,\beta}(\mathbf{x}))$, then letting

$$\begin{aligned} \Upsilon(\zeta) &:= \begin{cases} |\Psi(\zeta) - \beta|^{\frac{1}{\alpha}} & \text{if } \Psi(\zeta) \geq \beta \\ -|\Psi(\zeta) - \beta|^{\frac{1}{\alpha}} & \text{if } \Psi(\zeta) < \beta \end{cases} \quad \text{if } \alpha \neq 0 \text{ and} \\ \Upsilon(\zeta) &:= [\exp(\Psi(\zeta))]^{\frac{1}{\beta}} \qquad \text{if } \alpha = 0, \beta \neq 0 \end{aligned} \tag{38}$$

we identify the continuous and monotonic function $\Upsilon : \mathbb{R} \to \mathbb{R}$ such that $\Upsilon^{-1}(n \cdot \Upsilon(\Xi^n(\mathbf{x}))) = \Psi^{-1}(W_{\alpha,\beta}(\mathbf{x}))$ for any Ψ. Note that there exists a one to one relation between functions Ψ and Υ and that the latter is not necessarily increasing, indeed it is decreasing if $\alpha < 0$. Moreover, note that by construction, given that $\Psi(.)$ is continuous and (strictly) increasing then $\Psi(\mathbb{R})$ is open, and (i) if $\alpha = 0$ and $\beta \neq 0$ or $\alpha \neq 0$ and $\beta \in \Psi(\mathbb{R})$ then $W_{\alpha,\beta}(\Omega) = \mathbb{R}$ otherwise (ii) if $\alpha \neq 0$ and $\inf \Psi(\mathbb{R}) \geq \beta$ then $W_{\alpha,\beta}(\Omega) = (\beta, +\infty)$ while (iii) if $\alpha \neq 0$ and $\sup \Psi(\mathbb{R}) \leq \beta$ then $W_{\alpha,\beta}(\Omega) = (-\infty, \beta)$. If $\alpha = 0$ and $\beta = 0$ instead we have $W_{\alpha,\beta}(\Omega) = \Psi(\mathbb{R})$.

Recalling that $W(\mathbf{x}) = \Psi^{-1}(W_{\alpha,\beta}(\mathbf{x}))$ it follows that $\Psi(\mathbb{R}) = \mathbb{R}$ [case (i)]; or $\Psi(\mathbb{R}) = (\beta, +\infty)$ [case (ii)], or $\Psi(\mathbb{R}) = (-\infty, \beta)$ [case (iii)]. In terms of the function $\Upsilon(.)$ in (38) the range restrictions lead to: $\Upsilon(\mathbb{R}) = \mathbb{R}$ [case (i)]; $\Upsilon(\mathbb{R}) = \mathbb{R}_{++}$ [case (ii)], $\Upsilon(\mathbb{R}) = \mathbb{R}_{--}$ [case (iii)].

The only case left is $\alpha = 0, \beta = 0$ leading to $W(\mathbf{x}) = \Xi^n(\mathbf{x})$. ■