Testing for Instability in Factor Structure of Yield Curves

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This Draft: April 30, 2007
Very Preliminary. DO NOT CIRCULATE.
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ABSTRACT

A widely relied upon but a formally untested consideration is the issue of stability in factors underlying the term structure of interest rates. In testing for stability, practitioners as well as academics have employed ad-hoc techniques such as splitting the sample into a few sub-periods and determining whether the factor loadings have appeared to be similar over all sub-periods. Various authors have found mixed evidence on stability in the factors. In this paper we develop a series of Hypotheses and statistically evaluate the factor structure stability of the US zero coupon yield term structure. We find that the level, slope, and curvature factors were indeed unstable during the sample period considered. Common structural changes affecting all interest rate maturities have fostered instability in the level factor. The slope instability was caused due to structural changes affecting only the short rates; and the curvature instability was caused due to structural changes affecting only the long rates. We find evidence of the presence of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were not affected by any common instabilities.

JEL classification: C12; C13; C14; C51

Keywords: Stability, Factor Structure, Principal Component Analysis, Term Structure of Interest Rates
1 Introduction

The statistical models using factor decomposition techniques such as principal component analysis and factor analysis, where the yield curve dynamics can appropriately be summarized in a few principal factors, have been highly favored in modelling interest rates. The term structure literature using statistical models have all used graphical methods for analysing the stability of the factors. The standard procedure implemented in this regard has been to divide the sample data into sub-periods and to identify the factor loading for the corresponding sub-periods. If the explanatory power of the factor loadings appeared to be similar over all sub-periods, then the factors were said to be stable over time. There have been no other formal tests conducted in this respect except a recent paper by Audrino et al. (2005) that concluded instability in the filtered innovations of the principal factors governing the US Discount bond yields. The instability detected in the paper could not however be interpreted directly as instability in the level, slope, or curvature factors.

In this paper, we formally test for instabilities underlying the level, slope, and curvature factor structure of the US zero coupon yield term structure. The variables underlying the factor structures are estimated via principal component analysis and are effectively the eigenspace variables including the eigenvalues, eigenvectors, and the factor loadings. We develop a series of Hypotheses testing for instabilities in the eigenspace variables of the level, slope, and curvature factors. To anticipate the results, we find that the eigenvalues (volatility) of the level, slope, and curvature factors were unstable over the sample period considered. The level instability was caused due to structural changes common to all interest rate maturities; the slope instability was caused due to structural changes affecting the short rates; and the curvature instability was caused due to structural changes affecting the long rates. We find evidence of the presence
of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were not affected by any common instabilities.

The remainder of this paper is structured as follows. Section 2 provides an account of the instability in yield curves documented in literature. We motivate the formal testing of instability in the yield factor structures by examining the evolution of eigenspace variables (eigenvalues, eigenvectors, and factor loadings) graphically. In Section 3 we present the factor analysis framework for the term structure level, slope, and curvature factors, estimated using the principal component analysis. We provide the asymptotic properties of the estimated eigenspace variables for the three factors, which is applied into developing the stability testing procedure in the subsequent Section. In Section 4 we formulate six Hypotheses for statistically evaluating the stability in the eigenspace variables governing the level, slope, and curvature factors of the yield curves and device the test statistics for evaluating each Hypothesis. Section 5 describes the dataset used and presents the results of the testing procedure developed in Section 4. Section 6 concludes with the summary and findings of the study.

2 Yield Curve Dynamics Instability

Modelling the dynamics of interest rates is vital in trading fixed income securities that are sensitive to movements in interest rates. The concern is to fit the interest rates data within a framework (model) that is able to capture the future evolution of the term structure of interest rates. This is important for valuation of securities such as interest rate derivatives. Also, given that we understand the process that governs the interest movements, we are able to analyse and alter the risk exposures at a given point of time. Bliss and Smith (1997) argue
that model selection and stability of the parameters underlying the process are closely related. The paper illustrates by critically examining the findings of Chan et al. (1992) and show that the unaccounted structural break, caused due to Fed change in the monetary policy, has indeed affected the conclusions drawn.

Following the seminal work of Hamilton (1989) that introduced modelling the short rates using a regime switching process, authors such as Lewis (1991), Evans and Lewis (1995), Garcia and Perron (1996), Gray (1996), and Ang and Bekaert (1998) have studied regime switches in interest rate models. Empirical evidence suggest that not only the short rates but also the whole term structure of interest rates might experience shifts in regimes caused due to business cycle expansions and contractions, changes in monetary policies and regime changes in economic variables such as consumption and inflation. Bansal and Zhou (2002) show that term structure models incorporating regime shifts provide considerable improvements over multifactor CIR and affine models. They develop a model allowing for regime switches in both the state vector and the risk premium and show that the model accommodates for the conditional joint dynamics (the conditional distribution) of short and long yields.

In the case of term structure of interest rates, stability analysis should also been carried out on the factor structure of interest rates. One of the earlier works in factor analysis of term structure of interest rates is the Nelson and Siegel (1987) model. This parsimonious representation is very popular among practitioners for calibrating the yield curve. Since the model is linear in coefficients, they are estimated using ordinary least squares. The coefficients of the yield curves were interpreted to be level, slope, and curvature. Various other authors have found the same statistical interpretation to the coefficients estimated via statistical techniques such as the principal component analysis and factor analysis. Litterman and Scheinkman (1991) show that the three principal factors, explaining around 99 percent of the changes in treasury bond
yields, could be interpreted to be the level (or parallel movement component), slope (or slope oscillation component), and curvature component. The level factor or the parallel movement component alone was the most important factor that accounted for an average of 89 percent of the variations observed in the yield changes data.

Given the widespread use of factor analysis for term structure of interest rates, there arises a need to evaluate the factor structure stability of interest rates. Many authors have assumed that the principal factors driving the evolution of interest rates are stable or robust through time. Some use ad-hoc methods to investigate factor stability. For instance, Bliss (1997) divided the sample period January 1970 – December 1995 into three sub-periods of arbitrary lengths and investigated the change in the factor loadings. Since the factor loadings patterns in the different sub-periods seemed similar in the case of all three factors, the factors were concluded to be stable. However, the factor volatilities were found to fluctuate over the sub-periods considered.

In the forecasting setting using the Nelson-Siegel model, Diebold and Li (2006) found similar results with stable factors and time-varying factor volatilities. Since the parameters were stable over time, the proposed model produced much accurate forecasts at both the short and long horizons than other standard forecasting benchmarks. Since empirical results noted the time-varying nature of volatility associated with the factors, Perignon and Villa (2006) accounted for a time-varying covariance matrix when estimating the factor structure of interest rates. Using the U.S. term structure data between January 1960 and December 1999, Perignon and Villa observed that the factor structure (factor loadings) remained constant across sub-periods considered but the volatility (eigenvalues) of the factors varied through time. Reisman and Zohar (2004) use the yield to maturity data of US discount bonds from 1982. They found that the first two principal components were quite stable; the third component was marginally stable; and the fourth component was unstable. Fabozzi et al. (2005) used the Nelson and
Siegel (1987) model to parameterize twelve monthly yields term structure data from June 7, 1994 to September 5, 2003. They plot the factor loadings from the model, and observed that the level and slope coefficients of the model seemed stable, while the curvature coefficient showed instability. Chantziara and Skiadopoulos (2005) evaluated stability in the principal factors of the term structure of petroleum futures by performing the principal component analysis (PCA) individually on two sub-periods before and after May 1997, the cut-off date being identified as the beginning of the Asian crisis. Since the PCA results for the two sub-periods were not different from the results obtained for the whole sample, the paper concluded stability in the factor structure over the whole sample period.

As it appears empirically, the stability analyses on factors were carried out by graphically plotting the factor loadings and by weighing the similarity in results over time. The standard procedure implemented in this regard was to divide the data into sub-periods and to identify the factor loading for the corresponding periods. If the explanatory power of the factor loadings appeared to be similar over all periods, then the factors were concluded to be stable over time. There were no other formal tests conducted in this regard. The first formal test (to the best of our knowledge) in evaluating stability of factors governing interest rates was introduced in Audrino et al. (2005) that considered a three-factor model with conditional heteroskedastic factors. The paper found contradicting conclusions that the factor loadings of the US discount bond yields were in fact unstable over the period January 1986 to May 1995. The paper used independent filtered innovations in order to find the principal factors for the different sub-periods considered and then using a regression framework on the filtered innovations, tested the Hypothesis that the regression coefficients (factor loadings) in the different sub-periods are indeed equal. Since the authors constructed factors on the filtered innovations, the instability detected could not be interpreted as instability of the level, slope, or curvature factors. Since there exist no formal
instability detecting methods for individual factors, the modest contribution of this paper is to introduce a testing procedure that would enable us to investigate the instability present in the factor structure of level, slope, and curvature.

2.1 A First Examination of Factor Structure Instability

As evident in the term structure literature, the instability risks present in the yield curves can persist also within its factor structures. As a first examination to this argument, we carry out some graphical analyses for the term structure of US zero coupon bond yields between Jan 1999 and May 2006. The specifications of the dataset used is detailed in Section 5.1.

First, we arbitrarily split the seven and half year’s bond yield data into three approximately equal, two and half year subsample periods; Jan ’99 - June ’01, July ’01 - Dec ’03, and Jan ’04 - May ’06 and graphically investigate whether the eigensystem has remained stable over the three subperiods. We perform the principal component analysis (PCA) on the 5 year holding period returns data for the three subsamples, in order to extract the level, slope, and curvature factors that drive the evolution of change in interest rates data. We concur with Litterman and Scheinkman (1991) when we consider the first three principal factors in explaining the evolution of term structure of interest rates. In order to extract the three principal factors using PCA, we perform the following steps:

1. Form the covariance matrix from the change in yields panel data for the three subsample periods considered.

2. Compute the eigenvalues and the corresponding eigenvectors from the covariance matrix for each period using the eigen decomposition. The eigenvectors are the principal components and the eigenvalues present the explanatory power of the corresponding eigenvectors.

Second, we graphically investigate instability along the short end, medium term, and long
end of the yield curve separately over the three subsample periods considered. For this, we draw
the direction of the principal axes (which are the eigenvectors), along with the scatter plot of the
original yield changes data for the three subsample periods. In order to visualize the direction
of the eigenvectors, we have to limit our analysis to the two dimensional plots. We use the three
month and six month rate as a proxy for the short end of the curve; the five year and seven year
rate as a proxy for the medium term of the curve; and the ten and twelve year rate as a proxy
for the long end of the curve.

Third, in order to examine the evolution of the entire eigenspace, we conduct recursive PCA
by expanding the estimation window at every run by including one new observation and then
record the evolution of the eigenvalues, eigenvectors, and factor loadings. We undertake two
recursive schemes, namely Forward Recursive Scheme (FRS) and Backward Recursive Scheme
(BRS). The two schemes allow us to evaluate stability in an informal way. The FRS allows us to
visually gauge the impact of adding one extra observation at each recursion and the BRS allows
us to visually gauge the impact of removing one observation at each recursion. The instability
can be seen as the abrupt increase in variability at a point in time in the case of the FRS and a
reduction in variability at a point in time in the case of the BRS. This FRS and BRS patterns
can also be used to check if there are more than one changes affecting the variability in
the recursion.

[Insert Figure 1 here]

The Figures 1 to 7 present the results towards the preliminary study of the issue of instability.
The Figure 1 plots the three principal components determined over the three subsamples. We
observe that, in all the three subsample periods, considering the first three principal components
would be sufficient in explaining the dynamics of the term structure. Though the three factors
vary in detail, the term structure responsiveness to these factors has remained stable over time. This stability result concurs with that recorded by Bliss (1997), Perignon and Villa (2006), and others. However, the bar charts show that the level risks, captured by the first principal component, was the highest in the third subsample period; the slope risks, explained by the second principal component, was the lowest in the third subsample period, and the curvature risks, explained by the third principal component, was the highest in the second subsample period. This means that the shocks to the term structure varied during the subperiods considered, though we are unable to (at the point) make any statistical inference of instability.

Figures 2, 3, and 4 plot the short run, medium term, and long run principal axes (also called directional vectors) for the three subsample period considered. The two directional vectors are orthogonal to each other by construction. The plot shows how well the principal axes explains the variability in yields. The Table below the plot records the eigenvalues (volatility), eigenvectors, and the percentage of variances explained by the two principal components. For the case of short rates, if we compare the direction of the principal axes across the three subsample period, we find that the first principal axis differ across the three subsamples and by the orthogonality condition, so does the second principal axis. Further, we observe that the sample data for the short rates are dispersed distinctly across the three subsample periods. This means there exist different volatility patterns in the three subperiods and supports the argument allowing for distinct time-varying covariance matrices. Therefore considering a constant covariance matrix decomposition of principal components may induce instability in the components. For the case of medium term and long term rates governing the yield curves (3, and 4 respectively), we find that the two eigenvalues have similar directional vectors for the three subsample periods, with
around 99% explanatory power of the variances.

[Insert Figures 5, 6, and 7 here]

Further consider the recursive plots of the eigenvalues, eigenvectors, and factor loadings (Figures 5, 6, and 7). The plots obtained from the recursion clearly show endurance of instability in the eigensystem. In the case of eigenvalues governing the factors (Figure 5), we can clearly see that the dynamics have not remained the same over time even though the percentage variation explained by the eigenvalues have remained the same. The eigenvalues for the level and curvature factors seem to have one prominent change but the eigenvalues governing the slope seem to have more than one abrupt changes. Looking at the recursion patterns for eigenvectors (Figure 6), the level and curvature eigenvectors show two prominent patterns and the slope eigenvector shows three prominent patterns suggesting possible structural changes in the eigenvectors. In the case of factor loadings (Figure 7), the FRS suggest one possible pattern change in the case of level, and two pattern changes in the case of slope and curvature. However, if we also consider the BRS, we can see there exist one possible intermittent blip in all the three level, slope, and curvature factor loadings. The observations of pattern changes surely corroborate the time-varying nature of the eigensystem, which may have caused possible structural breaks in the series. In order to formally conclude instability in the data, we would further require a formal test that would evaluate the significance of the observed blips in data.

Summarizing the preliminary results, we have used various adhoc graphical techniques in order to infer about the term structure stability. By evaluating the three arbitrarily split subsamples, we find that the shocks contributing to the level, slope, and curvature instability risks have varied during the three subsamples. Also, we find that the directional axes of the short end interest rates have varied over time. The forward and backward recursive plots give us an idea
about the evolution of the eigenspace variables and we find indications of instability in them.

In Section 4 we develop a formal statistical test for stability and empirically evaluate a series of hypotheses in order to infer about the instability risks associated with the level, slope, and curvature factors.

3 Framework and Estimation of the Eigenspace

In this Section, we detail the estimation framework and the inferential theory developed for the eigenspace variables (eigenvalues, eigenvectors, and factor loadings) that are estimated via the principal component analysis. The limiting distributions of the eigenspace variables developed in this Section will help us construct the asymptotic test statistics for evaluating the issue of eigensystem instability.

Estimation of panel factor models have been originally developed in order to capture the main sources of variations and covariations among the $N$ independent random variables in a panel framework. These methods were extended by Geweke (1977) and Brillinger (1964) into dynamic factor models and dynamic principal component analysis respectively, that were able to predict the covariation in economic variables by few underlying latent factors. Although the two methods differed for panels with small cross-sectional dimensions ($N$), they gave similar inferences as $N$ increased and was large. Chamberlain and Rothschild (1983) then distinguished the dynamic models into exact and approximate dynamic factor models. In the case of exact dynamic factor models, the idiosyncratic terms are assumed to be mutually uncorrelated whereas the approximate factor models relaxes this restriction and allows for limited correlation among the idiosyncratic terms. Applications in finance particularly favor the approximate factor models
where the idiosyncratic terms are weekly correlated and where large number of cross-sectional
units can be competently summarized by a few common statistical factors.

In this paper, we use the classical principal component analysis framework, which incor-
porates the approximate structure in the cross-sectional correlation among units. The factor
structure is considered static, as we do not assume any dynamic evolution for the factors. Con-
sider the stationary representation for term structure of interest rates with cross-sectional \(N\) and
time-series \(T\) dimension and with \(r\) factors:

\[
Y_t = \gamma' F_t + \varepsilon_t \quad t = 1, 2, \ldots, T
\]  

Let \(Y_t = (Y_{1t}, \ldots, Y_{Nt})'\) be the term structure panel with \(Y_t\) being an \(N \times 1\) vector of cross-
sectional observations from the panel data structure at time period \(t\), \(\gamma\) is an \(r \times N\) matrix of
the factor loadings, \(F_t\) is the \(r \times 1\) vector of common factors for all cross-sectional units at time
period \(t\), and \(\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'\) is the \(N \times 1\) vector of idiosyncratic disturbances. In the term
structure literature, the number of common factors that are sufficient in order to explain the
dynamics of interest rates are commonly established to be equal to three. Therefore we consider
the case of \(r = 3\). The factor loadings matrix loads the factors on to the variables, explaining the
correlation between the factors and the variables. The factor loadings \((\gamma)\) can be computed as
the unit length eigenvectors matrix multiplied by its singular value, which is the square-root of
eigenvalues. Thus \(\gamma\) characterizes the unit length eigenvectors in its true size and encompasses
in them the information of direction as well as magnitude.

The loadings underlying the factor structure of \(Y\), by definition is a function of eigenvalues
and eigenvectors. In order to estimate the loadings, we use the principal component analysis
(PCA) that undertakes the eigen decomposition of the covariance matrix \(\Sigma\) of \(Y\). When \(\Sigma\) is
unknown, we estimate the sample variance covariance matrix whose elements at position $i, j$ is given as
\[
[\Sigma]_{i,j} = \frac{1}{T-1} \sum_{t=1}^{T} (y_{it} - \mu_i) (y_{jt} - \mu_j) \quad i, j = 1, ..., N
\] (2)
where $(y_{i1}, ..., y_{iT})$ for $i = 1, ..., N$ are each independent and identically distributed. The PCA framework is summarized in the Appendix A. In Appendix B, we report the limiting distribution of the eigenvalues and eigenvectors estimated from a covariance matrix, which is Wishart distributed. The results are combined in the following theorem in order to get the limiting distributions of the factor loadings for the case of large time dimension interest rate panels.

**Theorem.** (Limiting distribution of factor loadings) Consider $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ and $\beta_1, \beta_2, ..., \beta_N$ as the first $N$ ordered eigenvalues and their corresponding eigenvectors of $\Sigma$ respectively. Define $\beta_i \lambda_i^{1/2} = \gamma_i$ as the $i^{th}$ factor loading vector where $\gamma_i = (\gamma_{i1}, \gamma_{i2}, ..., \gamma_{iN})'$ and $\gamma = (\gamma_1, ..., \gamma_N)$.

From the theorem above, since $\hat{\lambda}_i - \lambda_i$ is independent of $\hat{\beta}_i - \beta_i$ we can show that
\[
(\hat{\gamma} - \gamma) = O_p(T^{-1/2})
\] (3)
\[
\sqrt{T} (\hat{\gamma} - \gamma) \xrightarrow{d} N(0, \Psi)
\] (4)
where $\Psi = \sum_{i=1}^{N} \sum_{j=1}^{N} (U_{ij} \otimes \Psi_{ij})$ where $U_{ij}$ is an $N \times N$ matrix that has 1 in the $ij^{th}$ position and 0’s elsewhere. The asymptotic covariance matrix
\[
\Psi_{ij} = \begin{cases} 
\lambda_i \Theta_{ij} + \frac{1}{2} \lambda_i \beta_j \beta_i' & \text{for } i = j \\
(\lambda_i \lambda_j)^{1/2} \Theta_{ij} & \text{for } i \neq j
\end{cases}
\]

**Proof:** see Appendix C

The above theorem gives us the rate of convergence and the limiting distribution of the factor loadings matrix, $\hat{\gamma}$. Let $\hat{\Psi}$ be the estimated covariance matrix of the factor loadings.
According to the continuous mapping theorem, as $T \to \infty$, $\hat{\Psi} \to^p \Psi$. $\hat{\Psi}$ is consistent since it is a continuous function of the estimated eigenvalues and eigenvectors that are consistent.

Since we estimate the term structure model using PCA, we find consistent estimates for the factor loadings. This is because the factors loadings are defined in terms of the eigenvalues and eigenvectors obtained via eigen decomposition that are consistent for panels with large number of observations.

In the case of estimating factor loadings in stationary approximate factor models using regression methods, it is well known that since the factors ($F_t$) are unknown and usually estimated via principal component analysis, the factor loadings ($\gamma$) estimated in regression are in fact inconsistent for the case of fixed number of cross-sectional units ($N$). The inconsistency arises due to the fact that there is limited information available to estimate the factors when the number of cross-sectional units are fixed and not large. A recent paper Heaton and Solo (2006) evaluated the importance of using large $N$ dimensional panels when using PCA to estimate approximate factor models. The paper showed that the consistency and rate of convergence derived for this case depend on the rate at which the cross-sectional correlation of the model disturbances grow as $N$ grows to infinity (that is, the rate at which the maximum eigenvalues of covariance matrix of the error terms grow with $N$). This means that for panels where there is an increase in the cross-correlation of the disturbances with growing $N$, we would need $N$ really large. The implication of this finding is that the issue of inconsistency in finite dimensional panels is dependent of the panel data under consideration and have to be explored case by case. This conclusion pertains in the case of term structure of interest rates where the cross-sectional interest rate maturities are finite. Since the inconsistency of the estimated factor loadings in a regression framework is induced by inconsistency in estimation of the unobserved factors in the case of
fixed $N$ dimensional panels, we do not estimate the factor loadings in a regression framework but simply as an eigen decomposition problem. Thus the factor loadings, which are defined in terms of the eigenvalues and eigenvectors, are consistent for panels with fixed $N$ and for large number of observations ($T$).

4 Testing for Instability in the Eigensystem

In this Section, we formulate a series of Hypotheses that will enable us to evaluate stability among the eigenspace variables of the yield curves. Since we are primarily concerned with the level, slope, and curvature factors governing the yield curves, we investigate stability in the eigenspace variables of the first three principle factors.

We examine instability by testing the null Hypothesis of no change point against the alternative of at least one change point happening at the unknown time, $\tau$. We define $\tau$ as a fraction of the sample space $T$ such that $\tau = [T\epsilon]$ where $\epsilon = [0, 1]$. We define the eigenvalues, eigenvectors, and the factor loadings for the sample split around the unknown time point $\tau$ as

$$
\Lambda = \begin{cases}
\Lambda^a & \text{for } t = 1, \ldots, \tau \\
\Lambda^b & \text{for } t = \tau + 1, \ldots, T
\end{cases}
$$

$$
\beta_i = \begin{cases}
\beta_i^a & \text{for } t = 1, \ldots, \tau \\
\beta_i^b & \text{for } t = \tau + 1, \ldots, T
\end{cases}
$$

$$
\gamma_i = \begin{cases}
\gamma_i^a & \text{for } t = 1, \ldots, \tau \\
\gamma_i^b & \text{for } t = \tau + 1, \ldots, T
\end{cases}
$$

We test the following Hypotheses in order to gather inference on instability in the underlying eigensystem of the yield curves:
I. $H_0 : \Lambda^a = \Lambda^b$

$H_1 : \Lambda^a \neq \Lambda^b$

II. $H_0 : \lambda_i^a = \lambda_i^b$

$H_1 : \lambda_i^a \neq \lambda_i^b$ for $i = 1, 2, 3$

III. $H_0 : \beta_i^a = \beta_i^b$

$H_1 : \beta_i^a \neq \beta_i^b$ for $i = 1, 2, 3$

IV. $H_0 : \beta_{ip}^a = \beta_{ip}^b$

$H_1 : \beta_{ip}^a \neq \beta_{ip}^b$ for $i = 1, 2, 3$, and $p = 1, 2, ..., N$

V. $H_0 : \gamma_i^a = \gamma_i^b$

$H_1 : \gamma_i^a \neq \gamma_i^b$ for $i = 1, 2, 3$

VI. $H_0 : \gamma_i^a = \gamma_i^b$ and $\gamma_j^a = \gamma_j^b$ for $i, j = 1, 2, 3$ and $i \neq j$

$H_1 : \gamma_i^a \neq \gamma_i^b$ or $\gamma_j^a \neq \gamma_j^b$

**Remarks:**

- In testing the series of Hypotheses formulated above, we aim to study the economic shocks causing structural changes and their impact on the eigensystem of the yield curves. Since the risks associated with the yield curves are summarized within its eigensystem, we are able to envisage which economic shocks have caused what kind of risks.

- In Hypothesis I we test for the stability of the overall eigensystem by testing the restriction on $\Lambda = (\lambda_1, \lambda_2, \lambda_3)'$. The result from this test would help us conclude whether there persist structural changes in the eigensystem of the yield curves.
• A natural extension to this would be asking the question "What kind of structural changes have occurred?". The Hypotheses II, III, and V test for instabilities in the individual level, slope, and curvature factor magnitude, direction, and loadings respectively. The result from these tests would help us conclude whether the instability has been induced by level breaks, slope breaks, or rather curvature breaks. The corollary to this test would be understanding the risks associated with level, slope, and curvature shocks.

• The Hypothesis IV relates to testing for instability in each factor, and understanding which interest rates have experienced structural changes, causing the instability.

• The Hypothesis VI, unlike the previous ones, tests for common structural changes in factors. Since the level, slope, and curvature factors are correlated, the test tries to capture change points in one factor that might ripple into the other factors causing common change points in all factors.

Below we develop the stability test statistics for evaluating the six Hypotheses formulated above. We define \( \Pi = (\Lambda, \beta, \gamma) \) as the parameter space. Let \( \hat{\Pi}^a \) and \( \hat{\Pi}^b \) be the consistent estimators of \( \Pi^a \) and \( \Pi^b \). The limiting distribution of \( \Pi \) for the restricted sample space before the break and after the break, given the change point \( \tau \) is

\[
\sqrt{T} \left( \hat{\Pi}^a - \Pi \right) = \sqrt{T} \begin{pmatrix} \hat{\Lambda}^a - \Lambda \\ \hat{\beta}^a - \beta \\ \hat{\gamma}^a - \gamma \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N(0, \bar{\Lambda}^a) \\ N(0, \bar{\beta}^a) \\ N(0, \bar{\gamma}^a) \end{pmatrix},
\]

and

\[
\sqrt{T} \left( \hat{\Pi}^b - \Pi \right) = \sqrt{T} \begin{pmatrix} \hat{\Lambda}^b - \Lambda \\ \hat{\beta}^b - \beta \\ \hat{\gamma}^b - \gamma \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N(0, \bar{\Lambda}^b) \\ N(0, \bar{\beta}^b) \\ N(0, \bar{\gamma}^b) \end{pmatrix}.
\]
where the superscript \( a \) and \( b \) denote estimation from restricted sample before and after the break respectively. The associated covariance weighting structure \( \tilde{\Theta}^a = \frac{T^a}{c} \), \( \tilde{\Theta}^b = \frac{T^b}{1-c} \), \( \tilde{\Psi}^a = \frac{\Psi^a}{c} \) and \( \tilde{\Psi}^b = \frac{\Psi^b}{(1-c)} \).

In testing the Hypothesis I, we construct the Wald test statistic under the null Hypothesis of no structural change in \( \Lambda \) against the alternative of atleast one structural change in \( \Lambda \) can be constructed as under:

\[
W_I(\tau) = \left( (\hat{\Lambda}^a - \Lambda) - (\hat{\Lambda}^b - \Lambda) \right)' \left[ (\tilde{\Theta}^a + (\tilde{\Theta}^b)]^{-1} \left( (\hat{\Lambda}^a - \Lambda) - (\hat{\Lambda}^b - \Lambda) \right) \right. \\
\left. \quad \overset{d}{\rightarrow} Z' \left[ (\tilde{\Theta}^a + (\tilde{\Theta}^b)]^{-1} Z \right.
\]

where \( Z \sim N(0, \tilde{\Psi}^a + \tilde{\Psi}^b) \).

Define \( \tilde{\Psi} = \tilde{\Theta}^a + \tilde{\Theta}^b \) where \( \tilde{\Psi} \) is positive definite. Using Cholesky decomposition, we have \( \tilde{\Psi} = LL' \) and \( \tilde{\Psi}^{-1} = L^{-1}L^{-1}' \) where \( L \) is a lower triangular matrix with strictly positive diagonal entries. Premultiplying \( Z \) by the inverse of \( L \),

\[
L^{-1}Z \sim N(0, L^{-1}\tilde{\Psi}L^{-1}') \\
\quad = N(0, L^{-1}LL'LL^{-1}) \\
\quad = N(0, I_r)
\]

Therefore using this result, we can show asymptotically

\[
W_I(\tau) \overset{d}{\rightarrow} Z' \tilde{\Psi}^{-1}Z = Z'L^{-1}L^{-1}Z = Q(\tau)
\]

(5)

where for a given \( \tau = [Te] \), \( Q(\tau) \sim \chi^2(q) \) with the degrees of freedom \( q \) corresponding to the number of restrictions being tested. Thus the distribution of our test statistic under the null is asymptotically pivotal.
Since the eigenvectors and the factor loadings are also asymptotically normal (see Appendix B), we can test all the other five Hypotheses using the statistic $W(\tau)$, which when normalized with their respective asymptotic variances, can again be shown to converge to a chi-squared as above. The form of the Wald statistics corresponding to the five Hypotheses are given below.

$$W_{II}(i, \tau) = \frac{(\hat{\lambda}_i^a - \hat{\lambda}_i^b)^2}{\left[2\left(\hat{\lambda}_i^a\right)^2 / \epsilon \right] + \left[2\left(\hat{\lambda}_i^b\right)^2 / (1 - \epsilon)\right]}$$

$$W_{III}(i, \tau) = \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right)' \left[\hat{\Theta}_ii + \hat{\Theta}_ii\right]^{-1} \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right)$$

$$W_{IV}(i, p, \tau) = \frac{(\hat{\beta}_{ip} - \hat{\beta}_{ip})^2}{\left[\hat{\Theta}_{ii,pp}^a + \hat{\Theta}_{ii,pp}^b\right]}$$

$$W_V(i, \tau) = \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right)' \left[\hat{\Psi}_{ii}^a + \hat{\Psi}_{ii}^b\right]^{-1} \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right)$$

$$W_{VI}(i, j, \tau) = \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right)' \left[\begin{array}{cc}
(\hat{\Psi}_{ii}^a + \hat{\Psi}_{ii}^b) & (\hat{\Psi}_{ij}^a + \hat{\Psi}_{ij}^b) \\
(\hat{\Psi}_{ij}^a + \hat{\Psi}_{ij}^b) & (\hat{\Psi}_{jj}^a + \hat{\Psi}_{jj}^b)
\end{array}\right]^{-1} \left(\hat{\gamma}_j^a - \hat{\gamma}_j^b\right)$$

Note that the Wald test in the above framework are equivalent to the F test or Chow type tests. One could also use the langrange multiplier (LM) or the likelihood ratio (LR) tests in order to test the linear restrictions. It is possible to show that Wald, LM and LR type tests have the same asymptotic distributions. When the date of the structural change is unknown but known to fall within a finite range, Andrews (1993) and Andrews and Ploberger (1994) introduced the $Sup$, $Exp$, and $Avg$ tests for the Wald, LM and LR test statistics and derived its asymptotic distributions. If we define the Wald test statistic as $W$ for a break occurring at
time \( \tau \), then

\[
Sup W = \max_{t_1 < \tau < t_2} W 
\]

(11)

\[
Avg W = \frac{1}{t_2 - t_1 + 1} \sum_{\tau = t_1}^{t_2} W 
\]

(12)

\[
Exp W = \ln \left[ \frac{1}{t_2 - t_1 + 1} \sum_{\tau = t_1}^{t_2} \exp \left( \frac{1}{2} W \right) \right] 
\]

(13)

where the breakpoint \( \tau \) lies between \( t_1 \) and \( t_2 \) such that \( t_1 = [T \epsilon_1], t_2 = [T \epsilon_2], t_1 \neq t_2, \epsilon_2 = 1 - \epsilon_1 \), and \( t_1 \) is bounded away from zero and \( t_2 \) is bounded away from \( T \). This condition is required since the proposed test statistic is unbounded in limit at the boundaries. Andrews (1993) suggested the restricted interval \( t_1 = 0.15T \) and \( t_2 = 0.85T \) such that \( \epsilon_1 \) and \( \epsilon_2 \) lies in the interval \([0.15, 0.85]\) and \( T \) denotes the number of observations in the sample. Therefore the test statistics does not capture the breakpoints occuring at the end of sample.

Under the null of no structural change, according to the continuous mapping theorem the asymptotic distributions of the test statistics converge as follows:

\[
Sup W \overset{d}{\to} \max_{\epsilon_1 < \tau < \epsilon_2} Q(\epsilon) 
\]

(14)

\[
Avg W \overset{d}{\to} \int_{\epsilon_1}^{\epsilon_2} Q(\epsilon) d\epsilon 
\]

(15)

\[
Exp W \overset{d}{\to} \ln \left[ \int_{\epsilon_1}^{\epsilon_2} \exp \left( \frac{1}{2} Q(\epsilon) \right) d\epsilon \right] 
\]

(16)

where if we know the break point fraction \( \epsilon, Q(\epsilon) \) will be \( \chi^2(q) \) with the degrees of freedom \( q \) corresponding to the number of restrictions being tested. In the GMM estimation framework, Andrews (1993) and Andrews and Ploberger (1994) provide the critical values for \( Sup, Avg \), and \( Exp \) of the Wald test statistic using Monte Carlo simulation. The Monte Carlo critical values provided by Andrews and Ploberger break down under various circumstances as documented by Diebold and Chen (1996), Hansen (2000), and O’Reilly and Whelan (2005). O’Reilly and
Whelan (2005) proposed a wild bootstrap approach for generating critical values for the $Sup$ statistic in dynamic time series models.

In providing inference on the eigensystem stability, we rely upon the test statistic distribution obtained using the bootstrap methodology. In this, we bootstrap the space vector of $N$ maturities by resampling across time. It is well established that the bootstrap procedures provide much accurate and reliable inferences than asymptotics based inferences. Andrews (1993) and Andrews and Ploberger (1994) provide asymptotic critical values for the $Sup, Avg,$ and $Exp$ of optimal tests based on a regression type framework. We know that the principal component analysis provides a different solution than the least squares solution, in which the least squares problem minimizes the vertical distance between the points and the principal component analysis problem minimizes the orthogonal distances between the points. Since the estimation framework developed in this paper is based on orthogonal rotation of axes and different to the regression type framework in Andrews’ papers, we use the bootstrapped critical values rather than asymptotic critical values provided by Andrews.

Appendix D provide the bootstrapped critical values of the test statistics developed in this Section. Since the Wald test statistic is asymptotically pivotal, the asymptotic distribution of the test statistics does not depend on a particular data generating process under the null. Therefore bootstrap distribution can consistently estimate the asymptotic distribution of the test statistics and provide more reliable inference than asymptotically based inferences by removing the finite sample biases. Davidson and MacKinnon (1999) find that for asymptotically pivotal test statistics using critical values from the bootstrap will produce smaller size distortions (reduced by an order of $T^{-1/2}$) than when using the critical values obtained from the first order asymptotics. Using the bootstrapped critical values, one may be able to mimic the skewness and kurtosis of
the empirical distribution that is not captured by the first order limiting distribution.

In order to construct the bootstrap distribution of the test statistics, we undertake the following steps:

1. For a given value of \( \epsilon \), the break fraction; randomly draw the vector of maturities from the \( T \times N \) term structure data in order to construct the \( T \times N \) bootstrapped data.

2. Then construct the covariance matrix for the bootstrapped data and conduct the principal component analysis in order to estimate the eigenspace variables \( \hat{\Lambda}, \hat{\beta}, \hat{\gamma} \).

3. Compute the Wald statistics \( W_k(.) \) for \( k = 1, 2, ..., 6 \) and calculate the weighted measures \( \Sup, \Avg,\) and \( \Exp \) of the Wald statistics.

4. Repeat steps 1 through 3 for \( BR \) number of bootstrap replications.

The procedure generates \( BR \) number of bootstrap statistics of \( \Sup, \Avg,\) and \( \Exp \) of \( W_k(.) \).

For \( \epsilon = 0.15 \) and for significance level \( \alpha = 0.05 \) we conduct 1000 iterations (\( BR = 1000 \)) and in each iteration we resample the term structure panel, which is of 1923 by 21 dimension. The Tables 1 and 2 provide the bootstrap critical values for \( \Sup, \Avg,\) and \( \Exp \) of the Wald statistics in equations 5 - 10.
5 Empirical Results

5.1 Data

We use the term structure of US zero coupon bond yields obtained from Datastream. The term structure of zeros are extremely useful in fixed income applications such as pricing bonds, swaps, and other fixed income derivatives; financial engineering the interest rates exposures; obtaining the forward rate curves, par yield curves; and so on. In the Table 3, we summarize the datasets used in the previous studies that directly or in passing evaluates term structure stability.

[Insert Figure 8 here]

Our dataset consist of yields of the following maturities: 3, 4, ..., 11 months, 1, ..., 12 years. The matrix plot in Figure 8 shows the strong and cross-sectionally varying correlation structures among the different interest rate maturities. The sample period extends from 11 Jan 1999 to 31 May 2006, with daily frequency (1927 observations). The data period covers both the period of downturn (during the technology stock boom in 2001) and upswings where the risk aversion of the investors are high causing gains in the bond markets. The bond yields data for maturities less than 3 months were filtered out in order to reduce the market microstructure effects and avoid liquidity issues. On the same note, we use the 5 day change in yields (the 5 day holding period returns) in order to perform the eigen decomposition on its covariance structure as recommended by Lardic et al. (2003) and as commonly used in factor analysis literature of term structure of interest rates. Working with changes rather than levels allow us to reduce the high autocorrelation and work in a stationary framework.
5.2 Stability Testing Results

Tables 4 and 5 record the results from implementing the $Sup$, $Avg$, and $Exp$ test statistics for the six Hypotheses formulated above. We test the linear restrictions of equality in eigenspace variables for a given change point occurring at time $\tau$, using the Wald test. In practice since we do not know this change point $\tau$, we calculate the weighted statistics $Sup$, $Avg$, and $Exp$ for all possible change points within the restricted sample period. The tests are evaluated for significant structural changes within the restricted sample period $[0.15T, 0.85T]$. We avoid the boundaries since the test statistics produce unstable results at the boundaries as documented by Andrews (1993). The conclusions are drawn based on results from all the three weighted measures $Sup$, $Avg$, and $Exp$ that concur.

*Investigating stability in the overall eigensystem*

Evaluating the weighted test statistics for $W_I(\tau)$, we reject the null in favor of the alternative that $\Lambda^a \neq \Lambda^b$. Thus, $Sup$, $Avg$, and $Exp$ test statistics of $W_I(\tau)$ infer that significant changes persist in the eigensystem of the yield curves. Instability in the vector of eigenvalues would mean structural instability in the variance process governing the factors. The instability detected have also been concluded in literature by Bliss (1997), Audrino et al. (2005) among others.

It is worth mentioning that the conclusions on instability of the factors governing the volatility is indeed different to the conclusions drawn in this paper where we evaluated the volatility governing the factors. The distinction lies within the fact that the information extracted (using eigen decomposition) from the covariance matrix of the yields are different than the information summarized in the covariance matrix of unobserved volatility. In regard to the latter, Perignon and Villa (2006) document the time-varying nature of the volatility governing the factors and
Bliss (1997) reported instability present in the factor volatility structures using graphical methods.

*Investigating stability in eigensystem of the level factor*

Evaluating the weighted test statistics for $W_{II}(1, \tau), W_{III}(1, \tau), \text{ and } W_{V}(1, \tau)$ we reject the null in favor of the alternative that $\lambda_1^a \neq \lambda_1^b, \beta_1^a \neq \beta_1^b,$ and $\gamma_1^a \neq \gamma_1^b$ respectively. Thus according to all the three weighted measures ($\text{Sup}, \text{Avg}, \text{ and } \text{Exp}$) for the various Hypotheses, we can conclude that all the three eigenspace variables (eigenvalues, eigenvectors, and factor loadings) governing the level factor have had statistically significant structural changes inducing instability. The result differs to the graphical inferences gathered by several authors such as Reisman and Zohar (2004) who have drawn stability conclusions for the level factor of discount bond yields.

In order to gauge which interest rate maturities have contributed to structural instability in the level factor, we evaluated the weighted test statistics for $W_{IV}(1, \tau)$. According to all the three weighted measures $\text{Sup}, \text{Avg}, \text{ and } \text{Exp}$ we can conclude that the structural instability was common and evident in all the 21 interest rate maturities governing the level factor. This means that the structural change in the level factor has been caused by economic shocks that eminently influenced the whole yield curve (short end as well as the long end maturities).

*Investigating stability in eigensystem of the slope factor*

In the case of the slope factor, we find that the eigenvalues or volatility governing the factor have incurred structural changes. Using all the three weighted statistics for $W_{II}(2, \tau)$, we reject the null in favor of the alternative of $\lambda_2^a \neq \lambda_2^b$. However, by evaluating $W_{III}(2, \tau)$ and $W_{V}(2, \tau)$ we find that the eigenvectors and the factor loadings governing the slope factor have remained
stable over time. By evaluating the weighted test statistic of $W_{IV}(2, \tau)$ for the slope factor, we can find that the short term interest rates (3 months - 10 months) governing the factor were unstable whereas the medium-long term interest rates (2 years - 12 years) governing the factor were tested to be stable over time. The test results for the slope factor do not concur with Reisman and Zohar (2004) that document stability of the slope factor.

In light of the tests, we can infer that, for the sample period considered, the instability of the slope factor lie in the volatility governing the factor structure; and since the eigenvalues and eigenvectors have a one to one correspondence with each other, we can infer that the instability in the volatility of the slope factor is caused due to economic shocks causing structural changes within the short term interest rates.

Investigating stability in eigensystem of the curvature factor

In the case of testing for instability in the eigenspace variables of the curvature factor, we find similar results to that of the slope factor. Using the $Sup, Avg,$ and $Exp$ for $W_{II}(3, \tau), W_{III}(3, \tau),$ and $W_{V}(3, \tau)$ we find that the curvature eigenvalue (volatility) has been subject to statistically significant structural changes but the corresponding eigenvectors and factor loadings have remained stable through time. By evaluating stability in the interest rates governing the curvature factor (using $W_{IV}(3, \tau))$, we find that the medium and long term rates (2 years - 12 years) have contributed to the structural change in the volatility of the curvature factor. Unlike the slope factor, we find that the short and the medium term (3 months - 1 year) interest rates were stable through time. In literature, Reisman and Zohar (2004) document marginal stability of the curvature factors using graphical tools.

Thus we can conclude that, as in the case of the slope factor structure, the volatility governing the curvature factor has incurred statistically significant structural changes. Evaluating the
interest rates governing the eigenvectors, we find instability present in the medium and long end of the yield curves, unlike the case of the slope factor. Since the instability in the slope and curvature factors have been caused due to economic shocks influencing different ends of the yield curve, we can say that the slope factors are sensitive to movements and shocks affecting in the short rates and the curvature factors are sensitive to movements and shocks affecting in the long rates.

*Investigating common instability in factor loadings*

Since we have found that the eigenspace variables for the level, slope, and curvature factors have incurred instability and since the three factors are correlated with each other, the economic shocks affecting one factor could also have affected the other. Therefore we investigate the presence of common structural changes due to common shocks in factors. By evaluating the weighted test statistics of \( W_{VI} (1, 2, \tau) \) we do not reject the null of presence of common structural changes in level and slope factor loadings. Thus we can conclude that there exist statistically significant change points common to the level and slope factors. Combining this result with the instability conclusions found for the level and slope eigenvectors, we can identify the common sources of instability within the level and slope factors as the economic shocks that have caused structural changes in the short term interest rates (3 months - 10 months). Since the \( Sup, Avg, \) and \( Exp \) for \( W_{VI} (1, 3, \tau) \) provide variant conclusions from testing common instabilities in level and curvature factor loadings, we cannot infer any presence of common structural changes. In the case of evaluating common instabilities present in the slope and curvature factor loadings, we reject the weighted test statistics of \( W_{VI} (2, 3, \tau) \) in favor of the alternative that no common structural changes exist between the slope and curvature factor loadings. Thus we can conclude that the slope and curvature factors behave dissimilarly to economic shocks that may have
caused structural instabilities in them separately.

6 Conclusion

This paper explores the important question of whether the yield factor structures are stable through time. Several authors have either assumed stability or relied upon graphical tools to make inferences. We have formulated six Hypotheses for statistically evaluating the stability in the eigenspace variables (eigenvalues, eigenvectors, and factor loadings) governing the level, slope, and curvature factors of the yield curves. Using statistical tests for identifying the structural change points there may exist in the eigenspace variables, we formally make stability inferences on the US zero coupon bond yield factor structures between January 1999 and May 2006. We find that the overall factor structure of the yield curves have been unstable over time. For the level factor structure, we find instability in the volatility of the factor caused due to instability affecting all the interest rate maturities governing the factor; in the case of the slope factor structure, the instability in the volatility of the factor is caused due to instability affecting only the short term and medium term maturities (3 months - 1 year); and in the case of the curvature factor structure, the instability in the volatility of the factor is caused due to instability affecting only the medium and long term rates (2 years - 12 years). In investigating the presence of common structural changes in factors, we find statistically significant breaks common to level and slope factors and no statistically significant common breaks in the slope and curvature factors.
References


Heaton, C. and V. Solo (2006). Estimation of approximate factor models: Is it important to have a large number of variables? Technical report.


Appendices

A Principal Component Analysis Framework

In Principal Component Analysis we estimate the eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ of the matrix $\Sigma$ satisfying the equality

$$|\Sigma - \Lambda I| = 0$$

(17)

where $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ and their corresponding vectors $\beta_1, \beta_2, ..., \beta_N$ satisfying the two conditions

$$\Sigma \beta_i = \lambda_i \beta_i$$

(18)

$$\beta'_i \beta_i = 1$$

(19)

The conditions ensure that the characteristic vectors $\beta_i$ for $i = 1, 2, ..., N$ are orthogonal to each other and are of unit length.

The estimated vectors $\beta_1, \beta_2, .., \beta_i, ..., \beta_N$ are such that the vector $\beta'_i Y$ is the directional vector that captures the maximum variability in $Y$. Therefore the estimation of $\beta_i$ can be seen as solution to the optimization problem

$$\max E(\beta'_i Y' \beta_i)$$

$$= \max \beta'_i \Sigma \beta_i$$

subject to the conditions $\beta'_i \beta_i = 1$ and $\beta_i Y' \perp \beta_j Y'$ for $i < j$. The orthogonality condition between the characteristic vectors means that

$$0 = E \left[ \left( \beta'_j Y' \right) \left( \beta'_i Y' \right) \right] = E (\beta'_j Y' Y \beta_i) = \beta'_j \Sigma \beta_i$$
The lagrangian equation to be maximized is therefore

\[ L_j = \beta_j^\top \Sigma \beta_j - \xi (\beta_j^\top \beta_j - 1) - 2 \sum_{i=1}^{j-1} \phi_i \beta_i^\top \Sigma \beta_i \]

where \( \xi \) and \( \phi = (\phi_1, ..., \phi_{j-1}) \) are the lagrange multipliers and \( j = 1, 2, ..., N \). The solution to this optimization problem satisfies the equation (18) and (17) and therefore the eigenvalues \( \lambda_i \) summarize the amount of variability captured by the corresponding eigenvector \( \beta_i \).

### B Asymptotic Properties of the Eigenspace Variables

We provide the inferential theory for the eigenvalues, and eigenvectors that are estimated using the classical principal component analysis. Let \( Z = (z_1', ..., z_T' ) \) be \( N \times T \) matrix such that

\[ ZZ' = (T - 1) \hat{\Sigma} \]

in equation (2). Therefore

\[ \hat{\Sigma} = \frac{1}{T-1} \sum_{t=1}^{T} z_t z_t' \]

where \( z_t = (y_t - \bar{y}) \) is the demeaned vector and \( z_t \sim N_N(0, \Sigma) \).

**Definition.** \((N - variate wishart distribution)\) Let \( x_1, ..., x_k \) be \( k \)-independent \( N \)-vectors. Suppose each \( x_i \sim N_N(0, \Sigma) \). Let \( U = x_1 x_1' + x_2 x_2' + ... + x_k x_k' \). Then \( U \) is said to have a \( N - variate \) Wishart Distribution with \( k \) degrees of freedom and covariance matrix \( \Sigma \). That is,

\[ U \sim W_N(\Sigma, k) \]

According to the above definition, \( \hat{\Sigma} (T - 1) = \sum_{t=1}^{T} z_t z_t' = \sum_{t=1}^{T} y_{it} \cdot y_{jt} \sim W_N(\Sigma, T - 1) \).

Therefore

\[ \hat{\Sigma} \sim W_N((T - 1)^{-1} \Sigma, T - 1) \]  

(20)
The density function of matrix $\Sigma$ is
\[
f(\Sigma) = \left(\frac{1}{2}\right)^{N(T-2)} \frac{(T-N-2)!}{\pi^\frac{1}{2}(T-1)^{2(N-1)} |\Sigma|^\frac{1}{2}} \prod_{i=1}^N \Gamma \left(\frac{1}{2}(T-i)\right)
\]
where $\Gamma(.)$ is the gamma function.

The following proposition provides the rate of convergence and the limiting distribution of the eigenvalues and eigenvectors decomposed from a covariance matrix $\Sigma$.

**Proposition.** (Limiting distribution of eigenvalues and eigenvectors) Let $y_1, ..., y_T$ be independently distributed, each being an $N \times 1$ vector of $N \times N (0, \Sigma)$. Define $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ a $N \times 1$ vector of independent eigenvalues and $\beta = (\beta_1, \beta_2, ..., \beta_N)$ a $N \times N$ matrix of orthogonal eigenvectors. The sample covariance matrix $\hat{\Sigma}$ is such that $\hat{\Sigma} \sim W_N ((T-1)^{-1}\Sigma, T-1)$. Then as $T \to \infty$,

\[
\begin{align*}
(\hat{\Lambda} - \Lambda) &= O_p(T^{-1/2}) \quad (21) \\
(\hat{\beta} - \beta) &= O_p(T^{-1/2}) \quad (22)
\end{align*}
\]

where the sequence $(\hat{\Lambda} - \Lambda)$ and $(\hat{\beta} - \beta)$ are independent to each other. The limiting distribution is given by

\[
\sqrt{T} (\hat{\Lambda} - \Lambda) \overset{d}{\to} N(0, \Upsilon) \quad (23)
\]

where $\Upsilon = \text{diag} (2\lambda_1^2, 2\lambda_2^2, ..., 2\lambda_N^2)$ and

\[
\sqrt{T} (\hat{\beta} - \beta) \overset{d}{\to} N(0, \Theta) \quad (24)
\]

where $\Theta = \sum_{i=1}^N \sum_{j=1}^N (U_{ij} \otimes \Theta_{ij})$ with

\[
\Theta_{ij} = \begin{cases} \\
\frac{\lambda_i}{\lambda_i - \lambda_j} \frac{1}{(\lambda_i - \lambda_j)} \beta_k \beta_k' & \text{for } i = j \\
-\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_j \beta_j' & \text{for } i \neq j
\end{cases}
\]
and $U_{ij}$ is an $N \times N$ matrix that has 1 in the $ij^{th}$ position and 0’s elsewhere.

The results mentioned in this proposition have been proved almost simultaneously by Girshick (1939), Hsu (1939), Fisher (1939), Roy (1939), Mood (1951), Anderson (1963) and widely known in multivariate statistics literature. For the proof, we refer the reader to any of the above papers or book by Anderson 2003 pp.546.

C Proof to Theorem

We know from the proposition that as $T \to \infty$,

$$\sqrt{T} \left( \hat{\lambda}_i - \lambda_i \right) \xrightarrow{d} N(0, 2\lambda_i^2)$$

and

$$\sqrt{T} \left( \hat{\beta}_i - \beta_i \right) \xrightarrow{d} N(0, \Theta_{ii})$$

$$\sqrt{T} \left( \left( \hat{\beta}_i - \beta_i \right) \left( \hat{\beta}_j - \beta_j \right) \right) \xrightarrow{d} N(0, \Theta_{ij})$$

where

$$\Theta_{ij} = \begin{cases} 
\lambda_i \sum_{\substack{k=1 \\
k \neq i}}^{N} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} \beta_k \beta_k' & \text{for } i = j \\
-\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_j \beta_j' & \text{for } i \neq j
\end{cases}$$

We define the error in estimation of the eigenvalues $\left( \hat{\lambda}_i - \lambda_i \right)$ as $\varepsilon_{\lambda_i}$ and the error in estimation of the eigenvectors $\left( \hat{\beta}_i - \beta_i \right)$ as $\varepsilon_{\beta_i}$. Note that $E \left( \varepsilon_{\lambda_i} \varepsilon_{\beta_i} \right) = 0$. 
\[ \hat{\lambda}_{i}^{1/2} = (\lambda_i + \varepsilon_{\lambda_i})^{1/2} = \lambda_i^{1/2} \left(1 + \frac{\varepsilon_{\lambda_i}}{\lambda_i}\right)^{1/2}. \]

Using Taylor expansion up to the first order, \[ \hat{\lambda}_{i}^{1/2} = \lambda_i^{1/2} \left(1 + \frac{\varepsilon_{\lambda_i}}{2\lambda_i}\right) + o_p(1). \]

Therefore we can write \[ \hat{\lambda}_{i}^{1/2} - \lambda_{i}^{1/2} = \frac{\varepsilon_{\lambda_i}}{2\lambda_i}. \]
Since we know the limiting distribution of the \( \varepsilon_{\lambda_i} \), we have

\[ \sqrt{T} \left( \hat{\lambda}_{i}^{1/2} - \lambda_{i}^{1/2} \right) \overset{d}{\rightarrow} N \left(0, \frac{1}{2} \lambda_i\right) \quad (25) \]

We define \( \hat{\lambda}_{i}^{1/2} - \lambda_{i}^{1/2} \equiv \tilde{\varepsilon}_{\lambda_i} \). Therefore we can write

\[ \hat{\lambda}_{i}^{1/2} \tilde{\beta}_i = \left(\lambda_i^{1/2} + \tilde{\varepsilon}_{\lambda_i}\right) \left(\beta_i + \varepsilon_{\beta_i}\right) \]
\[ = \lambda_i^{1/2} \beta_i + \lambda_i^{1/2} \varepsilon_{\beta_i} + \beta_i \tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i} \varepsilon_{\beta_i} \quad (26) \]

Therefore

\[ \hat{\lambda}_{i}^{1/2} \tilde{\beta}_i - \lambda_{i}^{1/2} \beta_i = \lambda_i^{1/2} \varepsilon_{\beta_i} + \beta_i \tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i} \varepsilon_{\beta_i} \]

We know, \( \varepsilon_{\beta_i} = O_p(T^{-1/2}), \tilde{\varepsilon}_{\lambda_i} = O_p(T^{-1/2}) \) and \( \tilde{\varepsilon}_{\lambda_i} \varepsilon_{\beta_i} = O_p(T^{-1}). \) Therefore \( \hat{\lambda}_{i}^{1/2} \tilde{\beta}_i - \lambda_{i}^{1/2} \beta_i = O_p(T^{-1/2}) \). This proves equation (3).

From the above, \( \sqrt{T} \left( \lambda_{i}^{1/2} \varepsilon_{\beta_i} \right) \overset{d}{\rightarrow} N(0, \lambda_i \Theta_{ii}) \) and \( \sqrt{T} \left( \beta_i \tilde{\varepsilon}_{\lambda_i} \right) \overset{d}{\rightarrow} N \left(0, \frac{1}{2} \lambda_i \beta_i \beta_i'\right) \)

Define \( X = \lambda_i^{1/2} \varepsilon_{\beta_i} \) and \( Y = \beta_i \tilde{\varepsilon}_{\lambda_i} \). From the above, we know that \( X + Y \) has a limiting distribution that is Normal. The first moment of the distribution is \( E(X + Y) = 0 \). The second moment (variance) can be calculated as follows:

\[ E \left((X + Y) (X + Y)'ight) = E \left[X'X\right] + E \left[Y'Y\right] + E \left[X'Y\right] + E \left[Y'X\right] \]

We know \( E \left[X'X\right] = \lambda_i \Theta_{ii}, E \left[Y'Y\right] = \frac{1}{2} \lambda_i \beta_i \beta_i', E \left[X'Y\right] = E \left[Y'X\right] = E \left[\lambda_i^{1/2} \varepsilon_{\beta_i} \tilde{\varepsilon}_{\lambda_i} \beta_i'\right] = \lambda_i^{1/2} E \left[\varepsilon_{\beta_i} \tilde{\varepsilon}_{\lambda_i}\right] \beta_i' = 0. \)
Therefore,

\[ E\left[ \left( \left( \lambda_i^{1/2} \varepsilon_{\beta_i} \right) + (\beta_i \tilde{\varepsilon}_{\lambda_i}) \right) \left( \left( \lambda_i^{1/2} \varepsilon_{\beta_i} \right) + (\beta_i \tilde{\varepsilon}_{\lambda_i}) \right) \right] = \lambda_i \Theta_i + \frac{1}{2} \lambda_i \beta_i \beta_i' \]

From the above,

\[ \sqrt{T} \left( \lambda_i^{1/2} \tilde{\beta}_i - \lambda_i^{1/2} \beta_i \right) \xrightarrow{d} N \left( 0, \lambda_i \Theta_i + \frac{1}{2} \lambda_i \beta_i \beta_i' \right) + Q + o_p(1) \]

where \( \frac{1}{\sqrt{T}} (\hat{\varepsilon}_{\lambda_i} \varepsilon_{\beta_i}) \xrightarrow{d} Q \) where \( Q \) is a distribution of the product of two mean zero independent normal variates. As \( T \to \infty \), the effect of \( \frac{1}{\sqrt{T}} (\hat{\varepsilon}_{\lambda_i} \varepsilon_{\beta_i}) = O_p(T^{-1/2}) \) is negligible and therefore

\[ \sqrt{T} \left( \lambda_i^{1/2} \tilde{\beta}_i - \lambda_i^{1/2} \beta_i \right) \xrightarrow{d} N \left( 0, \lambda_i \Theta_i + \frac{1}{2} \lambda_i \beta_i \beta_i' \right) \]

This proves equation (4).

The asymptotic covariance matrix for \((\tilde{\gamma}_i - \gamma_i) (\tilde{\gamma}_j - \gamma_j), i \neq j:\)

\[
\text{Cov} \left( (\tilde{\gamma}_i - \gamma_i) , (\tilde{\gamma}_j - \gamma_j) \right) \\
= E \left[ (\tilde{\gamma}_i - \gamma_i) (\tilde{\gamma}_j - \gamma_j) \right] \\
= E \left[ \lambda_i^{1/2} \tilde{\beta}_i \left( \lambda_j^{1/2} \tilde{\beta}_j - \lambda_j^{1/2} \beta_j \right) \right] - E \left[ \lambda_i^{1/2} \beta_i \left( \lambda_j^{1/2} \tilde{\beta}_j - \lambda_j^{1/2} \beta_j \right) \right] \\
= I - II
\]

Substituting for the estimators of \( \lambda_l^{1/2} \tilde{\beta}_l \) for \( l = i, j, \) we solve the two parts below:

**I** :

\[
E \left[ \lambda_i^{1/2} \tilde{\beta}_i \left( \lambda_j^{1/2} \tilde{\beta}_j - \lambda_j^{1/2} \beta_j \right) \right] \\
= E \left[ \lambda_i^{1/2} \beta_i + \lambda_i^{1/2} \varepsilon_{\beta_i} + \beta_i \tilde{\varepsilon}_{\lambda_i} + \varepsilon_{\beta_i} \tilde{\varepsilon}_{\lambda_i} \left( \lambda_j^{1/2} \beta_j + \lambda_j^{1/2} \varepsilon_{\beta_j} + \beta_j \tilde{\varepsilon}_{\lambda_j} + \varepsilon_{\beta_j} \tilde{\varepsilon}_{\lambda_j} - \lambda_j^{1/2} \beta_j \right) \right] \\
= E \left[ \lambda_i^{1/2} \lambda_j^{1/2} \varepsilon_{\beta_i} \varepsilon_{\beta_j} \right]
\]
\[II:\]

\[E \left[ \lambda_i^{1/2} \beta_i \left( \hat{\lambda}_j^{1/2} \hat{\beta}_j - \lambda_j^{1/2} \beta_j \right) \right] = E \left[ \lambda_i^{1/2} \beta_i \left( \lambda_j^{1/2} \beta_j + \lambda_j^{1/2} \varepsilon \beta_j + \beta_j \lambda_j + \bar{\varepsilon} \lambda_j \varepsilon \beta_j - \lambda_j^{1/2} \beta_j \right) \right] = 0\]

Therefore

\[\text{Cov} \left( (\hat{\gamma}_i - \gamma_i), (\hat{\gamma}_j - \gamma_j) \right) = E \left[ \lambda_i^{1/2} \lambda_j^{1/2} \varepsilon \beta_i \varepsilon \beta_j \right] = \lambda_i^{1/2} \lambda_j^{1/2} \Theta_{ij}\]
D Bootstrapped Critical Values

Tables 1 and 2 gather the critical values of the bootstrapped distributions of $Sup$, $Avg$, and $Exp$ of the test statistics $W(\tau) = (W_I(\tau), W_{II}(i, \tau), W_{III}(i, \tau), W_{IV}(i, \tau), W_{V}(i, \tau), W_{VI}(i, \tau))$ associated with the six Hypotheses formulated in equations 5 - 10. The critical values correspond to testing the null Hypotheses of stability in the eigenspace variables against the alternative of the presence of atleast one point of instability in the eigenspace variables for parameters $\epsilon = 0.15$ and significance level, $\alpha = 0.05$

<table>
<thead>
<tr>
<th></th>
<th>$Sup$</th>
<th>$Avg$</th>
<th>$Exp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_I(\tau)$</td>
<td>0.10647</td>
<td>0.016155</td>
<td>-0.6376</td>
</tr>
<tr>
<td>$W_{II}(1, \tau)$</td>
<td>0.027513</td>
<td>0.0048876</td>
<td>-0.6485</td>
</tr>
<tr>
<td>$W_{II}(2, \tau)$</td>
<td>0.044315</td>
<td>0.0060717</td>
<td>-0.64735</td>
</tr>
<tr>
<td>$W_{II}(3, \tau)$</td>
<td>0.067158</td>
<td>0.0080533</td>
<td>-0.64545</td>
</tr>
<tr>
<td>$W_{III}(1, \tau)$</td>
<td>0.17855</td>
<td>0.041762</td>
<td>-0.61261</td>
</tr>
<tr>
<td>$W_{III}(2, \tau)$</td>
<td>69573</td>
<td>1494.9</td>
<td>628.23</td>
</tr>
<tr>
<td>$W_{III}(3, \tau)$</td>
<td>38165</td>
<td>1341.9</td>
<td>660.17</td>
</tr>
<tr>
<td>$W_{V}(1, \tau)$</td>
<td>0.16094</td>
<td>0.039824</td>
<td>-0.61471</td>
</tr>
<tr>
<td>$W_{V}(2, \tau)$</td>
<td>1.999</td>
<td>0.85389</td>
<td>0.1815</td>
</tr>
<tr>
<td>$W_{V}(3, \tau)$</td>
<td>1.9993</td>
<td>0.79494</td>
<td>0.1429</td>
</tr>
<tr>
<td>$W_{VI}(1, 2, \tau)$</td>
<td>2.2677</td>
<td>0.92023</td>
<td>0.24683</td>
</tr>
<tr>
<td>$W_{VI}(1, 3, \tau)$</td>
<td>2.0918</td>
<td>0.81725</td>
<td>0.16398</td>
</tr>
<tr>
<td>$W_{VI}(2, 3, \tau)$</td>
<td>5.5412</td>
<td>1.6858</td>
<td>1.2748</td>
</tr>
</tbody>
</table>

Table 1: Bootstrap Critical Values (for $\alpha = 0.05$)
<table>
<thead>
<tr>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{IV}(1, 1, \tau)$</td>
<td>0.036288</td>
<td>0.0063495</td>
<td>-0.64709</td>
<td>$W_{IV}(2, 1, \tau)$</td>
<td>0.051577</td>
<td>0.0065321</td>
<td>-0.64708</td>
<td>$W_{IV}(3, 1, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 2, \tau)$</td>
<td>0.029168</td>
<td>0.0053343</td>
<td>-0.64805</td>
<td>$W_{IV}(2, 2, \tau)$</td>
<td>0.12512</td>
<td>0.02597</td>
<td>-0.62804</td>
<td>$W_{IV}(3, 2, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 3, \tau)$</td>
<td>0.027792</td>
<td>0.0050012</td>
<td>-0.64838</td>
<td>$W_{IV}(2, 3, \tau)$</td>
<td>0.077094</td>
<td>0.013121</td>
<td>-0.64057</td>
<td>$W_{IV}(3, 3, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 4, \tau)$</td>
<td>0.04486</td>
<td>0.0084218</td>
<td>-0.64509</td>
<td>$W_{IV}(2, 4, \tau)$</td>
<td>0.32226</td>
<td>0.083682</td>
<td>-0.57185</td>
<td>$W_{IV}(3, 4, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 5, \tau)$</td>
<td>0.057391</td>
<td>0.0104422</td>
<td>-0.64315</td>
<td>$W_{IV}(2, 5, \tau)$</td>
<td>0.61571</td>
<td>0.17536</td>
<td>-0.48234</td>
<td>$W_{IV}(3, 5, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 6, \tau)$</td>
<td>0.065338</td>
<td>0.011304</td>
<td>-0.64231</td>
<td>$W_{IV}(2, 6, \tau)$</td>
<td>0.85115</td>
<td>0.25188</td>
<td>-0.40749</td>
<td>$W_{IV}(3, 6, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 7, \tau)$</td>
<td>0.063509</td>
<td>0.010929</td>
<td>-0.64268</td>
<td>$W_{IV}(2, 7, \tau)$</td>
<td>0.87334</td>
<td>0.27685</td>
<td>-0.38139</td>
<td>$W_{IV}(3, 7, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 8, \tau)$</td>
<td>0.056745</td>
<td>0.0095981</td>
<td>-0.64395</td>
<td>$W_{IV}(2, 8, \tau)$</td>
<td>0.82397</td>
<td>0.26617</td>
<td>-0.3939</td>
<td>$W_{IV}(3, 8, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 9, \tau)$</td>
<td>0.05125</td>
<td>0.0084614</td>
<td>-0.64505</td>
<td>$W_{IV}(2, 9, \tau)$</td>
<td>0.72726</td>
<td>0.2428</td>
<td>-0.41755</td>
<td>$W_{IV}(3, 9, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 10, \tau)$</td>
<td>0.047699</td>
<td>0.0075604</td>
<td>-0.64592</td>
<td>$W_{IV}(2, 10, \tau)$</td>
<td>0.63881</td>
<td>0.21369</td>
<td>-0.44538</td>
<td>$W_{IV}(3, 10, \tau)$</td>
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<tr>
<td>$W_{IV}(1, 11, \tau)$</td>
<td>0.07329</td>
<td>0.011633</td>
<td>-0.642</td>
<td>$W_{IV}(2, 11, \tau)$</td>
<td>1.9012</td>
<td>0.66653</td>
<td>0.008678</td>
<td>$W_{IV}(3, 11, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 12, \tau)$</td>
<td>0.062959</td>
<td>0.010263</td>
<td>-0.64332</td>
<td>$W_{IV}(2, 12, \tau)$</td>
<td>3.6191</td>
<td>1.2345</td>
<td>0.64373</td>
<td>$W_{IV}(3, 12, \tau)$</td>
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<tr>
<td>$W_{IV}(1, 13, \tau)$</td>
<td>0.0588</td>
<td>0.0096068</td>
<td>-0.64396</td>
<td>$W_{IV}(2, 13, \tau)$</td>
<td>6.5092</td>
<td>2.0605</td>
<td>1.6257</td>
<td>$W_{IV}(3, 13, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 14, \tau)$</td>
<td>0.054342</td>
<td>0.0091606</td>
<td>-0.64437</td>
<td>$W_{IV}(2, 14, \tau)$</td>
<td>7.6617</td>
<td>2.3896</td>
<td>2.0536</td>
<td>$W_{IV}(3, 14, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 15, \tau)$</td>
<td>0.052668</td>
<td>0.0089185</td>
<td>-0.64462</td>
<td>$W_{IV}(2, 15, \tau)$</td>
<td>8.122</td>
<td>2.5765</td>
<td>2.2694</td>
<td>$W_{IV}(3, 15, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 16, \tau)$</td>
<td>0.049518</td>
<td>0.0083841</td>
<td>-0.64513</td>
<td>$W_{IV}(2, 16, \tau)$</td>
<td>7.2262</td>
<td>2.4504</td>
<td>2.0487</td>
<td>$W_{IV}(3, 16, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 17, \tau)$</td>
<td>0.046074</td>
<td>0.0079147</td>
<td>-0.64558</td>
<td>$W_{IV}(2, 17, \tau)$</td>
<td>5.9639</td>
<td>2.0917</td>
<td>1.6156</td>
<td>$W_{IV}(3, 17, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 18, \tau)$</td>
<td>0.044357</td>
<td>0.0075259</td>
<td>-0.64596</td>
<td>$W_{IV}(2, 18, \tau)$</td>
<td>4.9673</td>
<td>1.7805</td>
<td>1.2349</td>
<td>$W_{IV}(3, 18, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 19, \tau)$</td>
<td>0.040922</td>
<td>0.0073153</td>
<td>-0.64615</td>
<td>$W_{IV}(2, 19, \tau)$</td>
<td>4.0719</td>
<td>1.4843</td>
<td>0.88861</td>
<td>$W_{IV}(3, 19, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 20, \tau)$</td>
<td>0.040231</td>
<td>0.0072198</td>
<td>-0.64625</td>
<td>$W_{IV}(2, 20, \tau)$</td>
<td>3.7643</td>
<td>1.3731</td>
<td>0.77263</td>
<td>$W_{IV}(3, 20, \tau)$</td>
</tr>
<tr>
<td>$W_{IV}(1, 21, \tau)$</td>
<td>0.039394</td>
<td>0.0070053</td>
<td>-0.64646</td>
<td>$W_{IV}(2, 21, \tau)$</td>
<td>3.427</td>
<td>1.249</td>
<td>0.63609</td>
<td>$W_{IV}(3, 21, \tau)$</td>
</tr>
</tbody>
</table>

Table 2: Bootstrap Critical Values (for $\alpha = 0.05$)
<table>
<thead>
<tr>
<th>Authors</th>
<th>Term Structure (rates considered)</th>
<th>Source</th>
<th>Frequency</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0.25, 0.5, 1, 2, 3, 5, 7, 10, 15, and 20 yrs)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diebold and Li (2003)</td>
<td>Govt bonds (3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, and 120 months)</td>
<td>CRSP</td>
<td>Monthly</td>
<td>Jan 1985 - Dec 2000</td>
</tr>
<tr>
<td>Perignon and Villa (2005)</td>
<td>T-bill (1,...,6m) and Fama-Bliss Discount bond yields</td>
<td>CRSP</td>
<td>Monthly</td>
<td>Jan 1960 - Dec 1999</td>
</tr>
<tr>
<td></td>
<td>bond yields (1,...,5 yrs)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.25, 0.5, 1, 2, 3, 5, 7, and 10 yrs)</td>
<td></td>
<td>&amp; weekly</td>
<td></td>
</tr>
<tr>
<td>Fabozzi et al (2005)</td>
<td>Swap rates (0.25, 0.5, 1-5, 7, 10, 15, 20, 30 yrs)</td>
<td>-</td>
<td>Monthly</td>
<td>June 1994 - Sept 2003</td>
</tr>
</tbody>
</table>

Table 3: List of datasets used in studies that evaluated the issue of PCA factor’s stability
Table 4: \( Sup, \ Avg, \) and \( Exp \) values for the various test statistics

<table>
<thead>
<tr>
<th>Testing overall system</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_I(\tau) )</td>
<td>0.96701</td>
<td>0.26218</td>
<td>-0.38973</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing the level factor</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{II}(1, \tau) )</td>
<td>0.24475</td>
<td>0.062922</td>
<td>-0.59174</td>
</tr>
<tr>
<td>( W_{III}(1, \tau) )</td>
<td>0.87365</td>
<td>0.22575</td>
<td>-0.43265</td>
</tr>
<tr>
<td>( W_V(1, \tau) )</td>
<td>0.70327</td>
<td>0.24609</td>
<td>-0.41435</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing the slope factor</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{II}(2, \tau) )</td>
<td>0.34214</td>
<td>0.090797</td>
<td>-0.5643</td>
</tr>
<tr>
<td>( W_{III}(2, \tau) )</td>
<td>249.64</td>
<td>10.664</td>
<td>118.11</td>
</tr>
<tr>
<td>( W_V(2, \tau) )</td>
<td>1.9815</td>
<td>0.77043</td>
<td>0.11505</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing the curvature factor</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{II}(3, \tau) )</td>
<td>0.38012</td>
<td>0.10846</td>
<td>-0.54758</td>
</tr>
<tr>
<td>( W_{III}(3, \tau) )</td>
<td>335.49</td>
<td>5.6174</td>
<td>160.9</td>
</tr>
<tr>
<td>( W_V(3, \tau) )</td>
<td>1.9899</td>
<td>0.44182</td>
<td>-0.20286</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing the common factors</th>
<th>Sup</th>
<th>Avg</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{V_I}(1, 2, \tau) )</td>
<td>3.4721</td>
<td>1.2964</td>
<td>0.62638</td>
</tr>
<tr>
<td>( W_{V_I}(1, 3, \tau) )</td>
<td>2.393</td>
<td>0.70712</td>
<td>0.063935</td>
</tr>
<tr>
<td>( W_{V_I}(2, 3, \tau) )</td>
<td>5.4103</td>
<td>1.4711</td>
<td>0.83286</td>
</tr>
<tr>
<td>Sup</td>
<td>Avg</td>
<td>Exp</td>
<td>Sup</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>$W_1 V(1, 1, \tau)$</td>
<td>0.084479</td>
<td>0.008324</td>
<td>-0.64515</td>
</tr>
<tr>
<td>$W_1 V(1, 2, \tau)$</td>
<td>0.13845</td>
<td>0.019008</td>
<td>-0.63475</td>
</tr>
<tr>
<td>$W_1 V(1, 3, \tau)$</td>
<td>0.087745</td>
<td>0.011164</td>
<td>-0.64241</td>
</tr>
<tr>
<td>$W_1 V(1, 4, \tau)$</td>
<td>0.1796</td>
<td>0.031018</td>
<td>-0.62314</td>
</tr>
<tr>
<td>$W_1 V(1, 5, \tau)$</td>
<td>0.23842</td>
<td>0.058936</td>
<td>-0.59595</td>
</tr>
<tr>
<td>$W_1 V(1, 6, \tau)$</td>
<td>0.39888</td>
<td>0.087758</td>
<td>-0.56738</td>
</tr>
<tr>
<td>$W_1 V(1, 7, \tau)$</td>
<td>0.52171</td>
<td>0.105</td>
<td>-0.54994</td>
</tr>
<tr>
<td>$W_1 V(1, 8, \tau)$</td>
<td>0.53798</td>
<td>0.11548</td>
<td>-0.53989</td>
</tr>
<tr>
<td>$W_1 V(1, 9, \tau)$</td>
<td>0.46991</td>
<td>0.1161</td>
<td>-0.53983</td>
</tr>
<tr>
<td>$W_1 V(1, 10, \tau)$</td>
<td>0.42221</td>
<td>0.11239</td>
<td>-0.54391</td>
</tr>
<tr>
<td>$W_1 V(1, 11, \tau)$</td>
<td>0.5422</td>
<td>0.078327</td>
<td>-0.57431</td>
</tr>
<tr>
<td>$W_1 V(1, 12, \tau)$</td>
<td>0.5252</td>
<td>0.090457</td>
<td>-0.56329</td>
</tr>
<tr>
<td>$W_1 V(1, 13, \tau)$</td>
<td>0.49403</td>
<td>0.09326</td>
<td>-0.56105</td>
</tr>
<tr>
<td>$W_1 V(1, 14, \tau)$</td>
<td>0.41073</td>
<td>0.08511</td>
<td>-0.56987</td>
</tr>
<tr>
<td>$W_1 V(1, 15, \tau)$</td>
<td>0.3606</td>
<td>0.08916</td>
<td>-0.5662</td>
</tr>
<tr>
<td>$W_1 V(1, 16, \tau)$</td>
<td>0.3292</td>
<td>0.086432</td>
<td>-0.56912</td>
</tr>
<tr>
<td>$W_1 V(1, 17, \tau)$</td>
<td>0.30412</td>
<td>0.080299</td>
<td>-0.57535</td>
</tr>
<tr>
<td>$W_1 V(1, 18, \tau)$</td>
<td>0.27069</td>
<td>0.074814</td>
<td>-0.5807</td>
</tr>
<tr>
<td>$W_1 V(1, 19, \tau)$</td>
<td>0.24245</td>
<td>0.06894</td>
<td>-0.58645</td>
</tr>
<tr>
<td>$W_1 V(1, 20, \tau)$</td>
<td>0.2327</td>
<td>0.064932</td>
<td>-0.59037</td>
</tr>
<tr>
<td>$W_1 V(1, 21, \tau)$</td>
<td>0.22566</td>
<td>0.060909</td>
<td>-0.5943</td>
</tr>
</tbody>
</table>

Table 5: $Sup$, $Avg$, and $Exp$ values for the various test statistics
Figure 1: The three principal components for the three subsample periods

The First Principal Component
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
3m 5m 7m 9m 11m 2yr 4yr 6yr 8yr 10yr 12yr
Percentage variation explained by first factor

Jan ’99 - Jun ’01  Jul ’01 - Dec ’03  Jan ’04 - May ’06
72.65% 71.71% 83.69%

The Second Principal Component
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
3m 5m 7m 9m 11m 2yr 4yr 6yr 8yr 10yr 12yr
Percentage variation explained by second factor

Jan ’99 - Jun ’01  Jul ’01 - Dec ’03  Jan ’04 - May ’06
20.82% 21.59% 12.68%

The Third Principal Component
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
3m 5m 7m 9m 11m 2yr 4yr 6yr 8yr 10yr 12yr
Percentage variation explained by third factor

Jan ’99 - Jun ’01  Jul ’01 - Dec ’03  Jan ’04 - May ’06
3.48% 4.33% 1.81%

- blue subperiod: Jan ’99 - Jun ’01
- red subperiod Jul ’01 - Dec ’03
- red subperiod: Jan ’01 - May ’06
Figure 2: Short Term Rates (Yield Changes) and Principal Axes for the Whole Sample Period and the Three Subsample Periods

<table>
<thead>
<tr>
<th>Full Sample Period</th>
<th>First Subperiod: Jan 99 - Jun 01</th>
<th>Second Subperiod: Jul 01 - Dec 03</th>
<th>Third Subperiod: Jan 04 - May 06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Eigenvectors</td>
<td>% Explained</td>
<td>Eigenvalues</td>
</tr>
<tr>
<td></td>
<td>Vector 1</td>
<td>Vector 2</td>
<td>% Explained</td>
</tr>
<tr>
<td>0.0015</td>
<td>-0.5366</td>
<td>-0.8439</td>
<td>88.5676</td>
</tr>
<tr>
<td>0.0002</td>
<td>-0.8439</td>
<td>0.5365</td>
<td>11.4324</td>
</tr>
<tr>
<td>0.003</td>
<td>-0.4008</td>
<td>-0.6769</td>
<td>87.3595</td>
</tr>
<tr>
<td>0.0004</td>
<td>-0.8768</td>
<td>0.4808</td>
<td>12.6405</td>
</tr>
</tbody>
</table>
Figure 3: Medium Term Rates (Yield Changes) and Principal Axes for the Whole Sample Period and the Three Subsample Periods

<table>
<thead>
<tr>
<th>Full Sample Period</th>
<th>First Subperiod: Jan 99 - Jun 01</th>
<th>Second Subperiod: Jul 01 - Dec 03</th>
<th>Third Subperiod: Jan 04 - May 06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td>Eigenvectors</td>
<td>% Explained</td>
<td>Eigenvalues</td>
</tr>
<tr>
<td>0.0018</td>
<td>-0.7523 -0.6589</td>
<td>99.3849</td>
<td>0.0007643</td>
</tr>
<tr>
<td>0</td>
<td>0.7523 0.6151</td>
<td>99.3849</td>
<td>0.0000047</td>
</tr>
<tr>
<td>0</td>
<td>0.6499 0.7609</td>
<td>99.3849</td>
<td>0.4299</td>
</tr>
</tbody>
</table>
Figure 4: Long Term Rates (Yield Changes) and Principal Axes for the Whole Sample Period and the Three Subsample Periods

<table>
<thead>
<tr>
<th>Full Sample Period</th>
<th>First Subperiod: Jan 99 - Jun 01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvectors</td>
<td>Eigenvectors</td>
</tr>
<tr>
<td>Vector 1</td>
<td>Vector 2</td>
</tr>
<tr>
<td>0.0012</td>
<td>-0.7265</td>
</tr>
<tr>
<td>0</td>
<td>-0.6671</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Second Subperiod: Jul 01 - Dec 03</th>
<th>Third Subperiod: Jan 04 - May 06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvectors</td>
<td>Eigenvectors</td>
</tr>
<tr>
<td>Vector 1</td>
<td>Vector 2</td>
</tr>
<tr>
<td>0.0019</td>
<td>-0.7315</td>
</tr>
<tr>
<td>0</td>
<td>-0.6618</td>
</tr>
</tbody>
</table>
Figure 5: Forward Recursive Scheme and Backward Recursive Scheme for the Eigenvalues
Figure 6: Forward Recursive Scheme and Backward Recursive Scheme for the Eigenvectors
Figure 7: Forward Recursive Scheme and Backward Recursive Scheme for the Factor Loadings
Figure 8: Matrix Plot of Term Structure of Interest Rates