Model Uncertainty and Term Premia on Nominal Bonds †

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COMMENTS WELCOME

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‡ Graduate Program 'Finance and Monetary Economics', Goethe University, Mertonstr. 17-21/Uni-Pf 77, D-60054 Frankfurt am Main, Germany, Email: mulrich@wiwi.uni-frankfurt.de. Phone: 00 49 (0) 69 798 23691.
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Abstract

This paper studies how model uncertainty with respect to monetary policy affects the term premium on nominal bond yields. The model uses a multiple-prior continuous-time economy. The ambiguity averse agent faces model uncertainty (Knightian uncertainty) with respect to monetary policy. Our term structure model shows that ambiguity affects the risk factor sensitivity (factor loadings) of the bond yields. Further, we decompose the term premium into its risk contribution and its ambiguity contribution. In the empirical section we take U.S. government bond data and estimate our structural bond model. The resulting model implied term premium on the ten year bond varies between 9.6 and 450.9 basis points. We apply the proposed decomposition for the term premium and find that the expected risk contribution to the term premium fluctuates in a tent shape manner over the business cycle, taking values between 21.2 and 275.7 basis points. The expected ambiguity contribution to the term premium is fluctuating around zero basis points with a sample average of 0.9 basis points. Although its sample average is close to zero it varies between -152 to 403.7 basis points over the business cycle. The time-periods of highest fluctuations in the ambiguity compensation are the periods of the late 1970’s to early 1980’s, and the early 2000’s, both periods of high monetary policy uncertainty. The empirical analysis further suggests that the overall positive slope of the yield curve is due to the pure risk compensation of investors, whereas the strong and volatile fluctuations of the term premium over time are mainly based upon agents’ model uncertainty with respect to monetary policy.

Keywords: Affine term structure model, Term premium, Risk premium, Ambiguity compensation, Model misspecification, Knightian uncertainty

JEL: C13, C32, E43, E44, G12

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1 Introduction

Recently, there has been a growing literature on asset pricing under model uncertainty and under ambiguity aversion. So far, there has not been a model which studies the nominal term structure and term premia implications in a model where agents are ambiguity averse with respect to monetary policy. We close this gap by deriving a term structure model which builds on a simple monetary real business cycle model (monetary RBC).

It is important to distinguish whether bond yields change due to changes in the risk perception or due to changes in the ambiguity perception of investors. On the bond market we observe that bond prices react very sensitively to unpredictable monetary policy actions. Furthermore, we know that bond traders and bond researchers do not know the future distribution of monetary policy actions, which is necessary to specify the future distribution of inflation. To overcome this problem, traders and researchers analyze statements of central banks very precisely in order to gain any clue of how likely different future monetary policy scenarios might be. Traditional structural bond models are not able to distinguish between risk and uncertainty (Knightian uncertainty). It is therefore not possible for existing bond models to explain how bond prices react if traders do not perfectly know the statistical distribution of future monetary policy actions. This is a surprising shortcoming, since it definitely makes a difference to a bond trader whether he invests into a bet where he can specify the future distribution of key price determinants in comparison to bets where he cannot. Since central banks manage the nominal aggregates of the economy, they can decide how predictable they act. Over the past years we have observed that more and more central banks move to a transparent and predictable monetary policy. It is hence also of interest to central bank managers to understand the link between transparency and changes in bond yields.

Our model contributes to the theoretical and empirical asset pricing literature, as well as to the macroeconomic literature. Recent macro-finance research has pointed out that the sensitive behavior of nominal long-term bond yields, with regard to unpredictable monetary policy, are not in accordance with standard general equilibrium models. We present a fairly standard continuous-time general equilibrium model which explains this behavior. We propose a simple monetary real business cycle model, where we introduce the well documented fact that agents do not know the true distribution of future monetary policy actions. Our model allows to decompose the term premium on government bond yields into two parts. One part relates to risk, whereas the other part relates to ambiguity with respect to future monetary policy.

In the empirical section we take U.S. government bond data and estimate our structural bond model. The resulting model implied term premium on the ten year bond varies between 9.6 and 450.9 basis points over the business cycle. We apply the proposed decomposition for the term premium and find that one of the components is a rather slow moving one which fluctuates in a tent shape manner over the business cycle and remains always positive. This component coincides with the expected risk contribution to the term premium. The second component is highly volatile over the business cycle, taking highly positive and highly negative values. This component coincides with the expected ambiguity contribution to the term premium. The expected risk contribution has been varying between 21.2 and 275.7 basis points and seems to be closely related to risk with respect to actual inflation. In sharp contrast to this we find,
that the time-series of the expected ambiguity contribution to the term premium is fluctuating around zero basis points with a sample average of 0.9 basis points. Although its sample average is close to zero it varies between -152 to 403.7 basis points over the business cycle. The time-periods of highest fluctuations in the expected ambiguity contribution are the period of the late 1970’s to early 1980’s, and the early 2000’s. Both periods were characterized by investors who faced high uncertainty about future monetary policy. The highly volatile fluctuations of the expected ambiguity contribution to the term premium seem to be most closely related to long-term expected inflation and to the corresponding model uncertainty with respect to future monetary policy.

The empirical analysis further suggests that the overall positive slope of the yield curve is due to the pure risk compensation of investors, whereas the strong and volatile fluctuations of the term premium over time are mainly based upon agents’ model uncertainty with respect to monetary policy.

Our model draws from several contributions in the robust control, term structure, and macro-finance literature.

In a recent macroeconomic study, Gurkaynak, Sack, and Swanson (2005) show that nominal long-term forward rates respond very sensitively to changes in the underlying economy. The authors demonstrate that the reason for movements in long-term forward rates are unexpected components of monetary policy decisions. Furthermore, Gurkaynak, Sack, and Swanson (2005) point out that these fluctuations are not attributable to time-varying risk premia. It seems instead, that the high sensitivity of nominal forward rates comes from compensation of expected inflation. We add to this field of research, because we propose a general equilibrium model which is able to exactly explain this phenomenon.

Recently, several papers have applied Recursive-Multiple-Prior-Utility (RMPU) and robust control to financial problems. The financial robust control literature has been pioneered by Hansen, Sargent, and Tallarini (1999) and Anderson, Hansen, and Sargent (2003) (AHS) for the discrete and continuous time framework, respectively. Papers like Uppal and Wang (2003), Maenhout (2004), Trojani and Vanini (2004), Hansen, Sargent, Turmhambetova, and Williams (2005), Liu, Pan, and Wang (2005), and others incorporate model uncertainty based upon the work of AHS to study the implications of robust control on asset allocation and asset pricing problems.

In the context of AHS, agents want to protect themselves against model uncertainty. It is assumed that agents do not know for sure the true model. Nevertheless, based on empirical data and experience, agents believe that the true model should be close to a known reference model. Future prospects are evaluated under alternative models. Alternative models are penalized according to their distance from the reference model.

Instead of applying robust control, we follow the ambiguity aversion literature which is based upon Gilboa and Schmeidler (1989), Epstein and Schneider (2003) and Chen and Epstein (2002). Authors like Epstein and Wang (1994), Epstein and Miao (2003), Miao and Wang (2004), Gaagliardini, Paolo, and Trojani (2005), and Shornick (2005) follow this approach. In this type of models agents maximize their utility given a worst-case probability measure. One out of several contributions of their work is that they show that max-min optimization reduces to a
standard maximization problem under an ambiguity adjusted probability measure. In our model, the worst case probability measure is only adjusted with respect to the "true" monetary policy shock, which the central bank fully controls. Uncertainty, which is not controlled by the central bank is not subject to ambiguity adjustments.

Our model is the first model which includes ambiguity averse agents in a monetary general equilibrium model. Further, to the best of our knowledge, the paper is the first which measures the impact that ambiguity has on U.S. government bond yields.

Another literature that we build on studies the interaction of bond yields and macroeconomic variables. Papers like Ang and Piazzesi (2003), Ang, Piazzesi, and Wei (2006), Hördahl, Tristani, and Vestin (2006) and others suggest that macroeconomic variables improve the forecast performance of term structure models. Our simple model adds to this literature because it provides a set up, which endogenously derives the market price of risk and the amount of risk present in bond yields. We solve the model without referring to first- or higher order approximations. Further, we show empirically that ambiguity has a strong impact on the strong term premium fluctuations of long-term nominal bonds.

The structure of our general equilibrium model is most closely related to Bakshi and Chen (1996), Buraschi and Jiltsov (2005), Buraschi and Jiltsov (2006) and Wachter (2006). Bakshi and Chen (1996) study a monetary economy where money is held in equilibrium. They derive closed-form solutions for the nominal term structure. Buraschi and Jiltsov (2005) introduce tax distortions, whereas Wachter (2006) and Buraschi and Jiltsov (2006) introduce habit formation and a correlation between real and nominal shocks in order to study the inflation risk premium. The main differences to our paper are: First, the latter papers study the impact that inflation has on the risk premium (the covariance between the real stochastic discount factor (SDF) and inflation). This is especially important for the term-premium on inflation-indexed bonds. Our paper instead studies how model uncertainty with respect to monetary policy influences the term premium on nominal bonds (covariance between nominal bond yields and the nominal SDF). We do not introduce any frictions like tax distortions, habit formation or correlated real and nominal shocks. Although we do not need this assumption, we work with a money neutral economy, where the Fisher Hypothesis holds. Second, the representative agent in our model faces model uncertainty with respect to future monetary policy. Third, we show that even in this simple and standard framework, ambiguity leads in equilibrium to nominal bond yields and nominal term premia, which react very sensitively to unpredictable monetary policy.

Gaagliardini, Paolo, and Trojani (2005) study a Cox, Ingersoll, and Ross (1985a) (CIR) type model where the dynamics of all state variables are regarded as an approximation to the true data generating process. Our approach is different to theirs. Gaagliardini, Paolo, and Trojani (2005) analyze how interest rate derivatives are influenced by this uncertainty. Our analysis has another focus. We focus explicitly on the question of how nominal bond yields behave in equilibrium if we introduce into a standard monetary RBC model agents who face model uncertainty with respect to future monetary policy. Further, we estimate the bond model based on a panel of U.S. government bond data.

The outline of the paper is as follows: Section 2 sets up the model, characterizes the equi-
librium market prices of risk and the equilibrium nominal term structure. Section 3 determines
the expected excess return (term premium). Section 4 describes the econometric method and
the data. Section 5 summarizes the estimation results. Section 6 concludes.

2 The Model

We consider a continuous-time, \( t \in [0, \infty) \), representative agent recursive multiple-prior economy
with money in the utility function, where the single produced good can either be consumed or
reinvested in a constant-return-to-scale production. Agents in this economy hold money because
of its positive marginal productivity. For notational convenience, the basic model structure of
the paper follows Bakshi and Chen (1996) and Buraschi and Jiltsov (2005).

2.1 Information Structure and Representative Agent’s Perception

Risk is represented by a complete filtered probability space \((\Omega, \mathcal{F}, \mathbf{F}, P)\), which satisfies the usual
conditions (right-continuous, increasing, and augmented). The probability measure \( P \) is not
necessarily the ”true” measure. It is only needed to define the null events of the economy. There
is no disagreement or ambiguity about which events belong to the null set of \( P \). Without loss of
generality and for ease of interpretation, we treat \( P \) as the ”true” benchmark measure. There
is a 4-dimensional standard Brownian motion \( W = (W_t) = \{(W^a_t, W^\omega_t, W^v_t, W^M_t) : t \in [0, T]\} \)
deﬁned on \((\Omega, \mathcal{F}, \mathbf{F}, P)\). The shock \( W^a_t \) drives all real variables, whereas \( W^\omega_t, W^v_t, W^M_t \) enter the
economy through the money supply rule and affect only the nominal variables in the economy. We
model \( W^M_t \) as the pure monetary policy shocks about which the agent is uncertain (Knightian
Uncertainty) about. The ﬁltration \( \mathbf{F} = \{\mathcal{F}_t\} \) is the augmentation under \( P \) of the ﬁltration
generated by \( W \) and represents the information available to the agent at time \( t \). I assume
that the true states of nature are completely characterized by the sample paths of \( W \) on \([0, T]\),
\( \mathcal{F} = \mathcal{F}_T = \sigma(\cup_{0 \leq t \leq T} \mathcal{F}_t) \) and that \( \mathcal{F}_0 \) is trivial, containing only events of probability zero or one.

All stochastic processes in the sequel are progressively measurable with respect to \( \mathbf{F} \) and
all the equalities involving random variables (stochastic processes) are assumed to hold \( P \)-a.s.
\((dt \otimes dP \text{ a.s.})\).

2.2 Utility Function

We follow Buraschi and Jiltsov (2005) and assume a representative agent with time separable
and logarithmic preferences in consumption holdings \( c_t \) and real monetary holdings \( m_t \). Real
monetary holdings \( m_t \) are equal to the ratio of nominal money supply \( M_t \) and the price level
\( p_t \) in the economy. We follow the assumption that real monetary holdings provide a transaction
service. The size of this transaction service is modeled through the parameter \( \gamma \). If \( \gamma = 0 \) money
does not provide a transaction service, whereas \( \gamma = 1 \) means that the agent has to hold one
unit of money for every unit of consumption holdings. For \( \gamma \equiv 0 \) this framework specializes to
the standard RBC model. The representative agent maximizes his RMPU through choosing his optimal allocations as described in Epstein and Schneider (2003) and Chen and Epstein (2002):

\[
V_t = \max_{c_s, m_s} \min_{Q \in \mathcal{P}^\Theta} \mathbb{E}^Q \left[ \int_t^\infty e^{-\rho s} (\log c_s + \gamma \log m_s) \, ds \middle| \mathcal{F}_t \right], \quad \gamma \in [0, 1]; \quad m_s \equiv \frac{M_s}{p_s}
\]

\[
V_t = \max_{c_s, m_s} \mathbb{E}^{Q^\text{min}} \left[ \int_t^\infty e^{-\rho s} (\log c_s + \gamma \log m_s) \, ds \middle| \mathcal{F}_t \right]
\]

where \(\mathcal{P}^\Theta\) is the set of \(P\)-equivalent rectangular Priors that are defined through a set of density generators \(\Theta\). The set \(\Theta\) of correspondences from \(\Omega\) to \(R^4\) fulfills the usual characteristics (\(\Theta\) is uniformly bounded, compact-convex valued, \(B([0, s]) \times \mathcal{F}_s, s \in [0, T]\) measurable, \(P \in \mathcal{P}^\Theta\), which means that if there is no model uncertainty about the money supply rule, the multiple priors model reduces to the standard expected utility framework). Any multiple prior in the economy is defined by

\[
\frac{dQ^\theta}{dP} \equiv z_\theta^T, \quad \theta \in \Theta, \quad z_\theta^T \equiv e^{-\frac{1}{2} \int_0^T ||\theta_s||^2 \, ds + \int_0^T \theta_s \cdot dW_s}, \quad z_0^\theta \equiv 1
\]

and fulfills Novikov’s condition \(E^P \left[ e^{\int_0^T ||\theta_s||^2 \, ds} \right] < \infty\).

It is known from Grisanov’s Theorem, that \(z_\theta^T\) generates a Brownian motion on the probability space \((\Omega, \mathcal{F}_T, Q^\theta)\). The ambiguity aversion of the agent can be interpreted as uncertainty of whether the shock in the economy is driven by \(dW\) or by any other shock satisfying \(dW^\theta_t = dW_t - \theta_t \, dt\).

Most papers which model the behavior of ambiguity averse agents assume a constant density generator \(\theta_s(\omega)\). Chen and Epstein (2002) show formally that this is a special case of the general ambiguity modeling framework. The most general modeling framework would assume that the density generator \(\theta_s(\omega)\) is time and state dependent. It is the goal of the paper to apply such a general density generator in order to study the impact that risk and ambiguity with respect to monetary policy have on nominal bond yields.

### 2.3 Real Side of the Economy

The representative agent owns an all-equity firm and observes unambiguously the technology shock \(a_t\), output growth \(\frac{dy_t}{yt}\), and the capital accumulation process \(K_t\). Unlike Buraschi and Jiltsov (2005), we stick to the neo-classical view that monetary policy shocks have no impact on real dynamics. This assumption means that money is neutral in the short as well as in the long run. In order to keep the context as simple as possible we assume the following dynamics for \(a_t, \frac{dy_t}{yt}\), and \(K_t\) which are similar to Cox, Ingersoll, and Ross (1985a), Cox, Ingersoll, and Ross (1985a).

\(^1\)Compare Chen and Epstein (2002).
(1985b), Longstaff and Schwartz (1992) and Buraschi and Jiltsov (2005):

\[
d a_t = \kappa_a (\theta_a - a_t) \, dt + \sigma_a \sqrt{a_t} dW^a_t \\
\frac{d y_t}{y_t} = (\mu_y + \nu_y a_t) \, dt + \sigma_y \sqrt{a_t} dW^a_t \\
\frac{d K_t}{K_t} = \frac{d y_t}{y_t} - \left( \frac{c_t}{K_t} + \frac{m_t}{K_t} + \delta \right) \, dt
\]  

(2.4)

(2.5)

(2.6)

where \( W^a_t \) is the productivity shock realized in \( t \in [0, T] \), \( c_t, m_t, \delta \) represent real consumption, real monetary holdings and the rate of depreciation in the economy. The coefficients \( \kappa_a, \theta_a, \sigma_a, \sigma_y, \mu_y, \nu_y \) are positive constants. To be consistent with empirical evidence, we model the capital stock process as being non-stationary. This is achieved through \( \mu_y \). The technology shock \( a_t \) is unconditionally stationary.

2.4 Nominal Side of the Economy

Agents believe that the monetary authority follows a Taylor type rule for base money. The degree of output- and inflation targeting are represented by \( q_1 \) and \( q_2 \), respectively. Both variables are determined by the central bank. If \( q_1 \) and \( q_2 \) are different from zero, both the equilibrium capital growth rate \( \frac{d \hat{K}_t}{K_t} \) as well as the equilibrium inflation rate \( \frac{d \hat{\pi}_t}{\hat{\pi}_t} \) influence the drift of the money supply process. We assume that the agent is ambiguity averse with respect to the true monetary policy shocks \( W^\hat{M} \). Since \( W^\hat{M} \) is a Brownian shock, ambiguity with respect to the monetary policy shock introduces uncertainty with respect to the drift of the money supply process. We model the basic structure of the money supply rule under the physical \( P \)-measure (historical measure) as Buraschi and Jiltsov (2005):

\[
\frac{d M_t}{M_t} = \omega_t dt + \frac{d \hat{K}_t}{K_t} - \hat{k} dt + q_1 \left( \frac{d \hat{K}_t}{K_t} - \hat{k} dt ight) + q_2 \left( \frac{d \hat{\pi}_t}{\hat{\pi}_t} - \hat{\pi} dt \right) + \rho_{Mv} \sigma_M \sqrt{v_t} dW^v_t \\
+ \sqrt{1 - \rho_{Mv}^2 \sigma_M \sqrt{v_t} dW^\hat{M}_t}, \quad \rho_{Mv} \in [0, 1],
\]

\[
d \omega_t = \kappa_\omega \left( \theta_\omega - \omega_t \right) \, dt + \sigma_\omega \sqrt{\omega_t} dW^\omega_t, \\
d v_t = \kappa_v \left( \theta_v - v_t \right) \, dt + \sigma_v \sqrt{v_t} dW^v_t
\]

(2.7)

(2.8)

(2.9)

where \( \hat{K}_t \) is the equilibrium capital stock and \( \hat{\pi}_t \) is the equilibrium price level at time \( t \). First, the monetary authority tries to meet its inflation target \( \hat{\pi} \) and real capital growth target \( \hat{k} \). The constant coefficients \( q_1 \) and \( q_2 \), model the importance of the specific target. Second, the monetary authority does only imperfectly control the monetary aggregate. Therefore, the money supply rule is affected by two other stochastic processes, \( \omega_t \) and \( v_t \). The first influences the long-term money growth rate, whereas the latter influences the conditional volatility of the money growth rate.

We assume that our agent has learned everything he could about the central banks’ policy function. The still remaining model uncertainty with respect to the true monetary policy shock is labeled as ambiguity with respect to the money supply rule.
2.5 Parameterizing Ambiguity Adjustment

In the model we assume that the agent faces only ambiguity with respect to the drift of the money supply process. Without loss of generality we assume that $W^M$ is the true monetary policy shock which is fully controlled by the central bank. By assumption this is the only shock that the agent is uncertain about.

Equation (2.6) shows that we assume capital to be not influenced by the monetary shock $W^M$. From an economic point of view this is a nice feature since most economists agree that in the long run money is neutral. The introduction of ambiguity aversion with respect to the money supply rule does not create any effects on the real side of the economy.

The equilibrium real consumption process is not affected by nominal shocks. As a result, real allocations are not influenced by the ambiguity aversion of the agent. On the other hand, expected inflation and the price of nominal bonds are affected by the ambiguity. Whenever agents take expectations with respect to the “true” monetary policy shock $W^M$, the corresponding probability measure $P$ must be transformed to the worst-case measure $Q^{\min}$. Hence, the pricing of nominal bonds and the determination of expected inflation are calculated under the worst-case measure $Q^{\min}$. The determination of the agents’ equilibrium allocation, real risk-free interest rate and real market price of risk, on the other hand, can be calculated under the historical $P$ measure. According to equation (2.3), this means that the density generator $z^{\theta^*}_t$ of the ambiguity-adjusted probability measure $Q^{\min}$ reduces to

$$\frac{dQ^{\min}}{dP} = z^{\theta^*}_t, \quad \theta^* \in \Theta,$$

$$z^{\theta^*}_t = e^{-\frac{1}{2} \int_0^t |(0,0,0,\kappa_s')|^2 ds + \int_0^t \kappa_s' dW^M_s}, \quad z^{\theta^*}_0 \equiv 1,$$

which simplifies to

$$\frac{dQ^{\min}}{dP} |_{\mathcal{F}_t} = z^{\theta^*}_t = e^{-\frac{1}{2} \int_0^t \kappa_s^2 ds + \int_0^t \kappa_s dW^M_s}$$

$$dz^{\theta^*}_t = z^{\theta^*}_t \kappa_t dW^M_t.$$  

(2.10)

Our approach to incorporate ambiguity aversion into an otherwise standard monetary real business cycle model can be seen as a state dependent application of the concept of $\kappa$-Ignorance, as introduced by Chen and Epstein (2002) in subsection 3.3.

Given the CIR dynamics of the underlying economy, we assume an extended affine representation for $\kappa_t$, which measures the distance between the historical measure $P$ and the ambiguity adjusted measure $Q^{\min}$. Such an extended affine representation is often used in empirical

\[ {\text{Cheridito}, \text{Filipovic, and Kimmel (2007) introduce the class of extended affine term structure models. They show that an extended affine specification of the distance between two probability measures, always nests the essentially affine and completely affine specifications.}} \]

\[ {\text{Most affine term structure papers use an A}_0(3) \text{ specification, where all three state variables are driven by Ornstein-Uhlenbeck processes. For these A}_0(3) \text{ processes, essentially affine and extended affine specifications of the distance between two probability measures coincide.}} \]
finance research for the parameterization of the distance between the risk-neutral and the physical probability measure. This parameterization leads to a tractable exponentially affine bond pricing model.

\[
\kappa_t \equiv q_{a_1} \sqrt{v_t} + \frac{q_{a_2}}{\sqrt{v_t}} + \frac{q_{a_3} a_t}{\sqrt{v_t}}, \quad q_{a_1}, q_{a_2}, q_{a_3} \in \mathbb{R}. \tag{2.13}
\]

In the empirical section, we let the U.S. bond data speak and decide whether or not this parameterization, for the distance between the benchmark probability measure \( P \) and the ambiguity-adjusted probability measure \( Q_{\text{min}} \), is supported.

The ambiguity adjustment in (2.13) consists of three parts. The first two parts constitute an essentially affine ambiguity adjustment, whereas the third component makes the ambiguity adjustment to have an extended affine structure. The third part ensures that ambiguity is connected to the state of the real economy. A negative \( q_{a_3} \) ensures that in times of large productivity increases, the agent reduces his ambiguity aversion with respect to the future money supply rule. As long as the economy is in a boom, the agent cares less that he faces model uncertainty with respect to the money supply rule. In the empirical section it turns out that \( q_{a_3} \) is highly negative, which seems to support this parameterization.

### 2.6 Ambiguous Money Supply Rule

Given the assumption about the parametrization of the ambiguity adjustment, the resulting ambiguity adjusted money supply process, as perceived by the agent, follows by straightforward measure transformation\(^5\), i.e.

\[
dM_t = M_t \left( \omega_t dt + q_1 \left( \frac{d\hat{K}_t}{K_t} - \hat{k} dt \right) + q_2 \left( \frac{d\hat{p}_t^{Q_{\text{min}}}}{\hat{p_t}^{Q_{\text{min}}}} - \hat{\pi} dt \right) \right) + M_t \left( \kappa_M(t) dt + \rho_{MV} \sigma_M \sqrt{v_t} dW_t^v + \sqrt{1 - \rho_{MV}^2 \sigma_M^2} \sqrt{v_t} dW_t^{M,Q_{\text{min}}} \right), \tag{2.14}
\]

\[
\kappa_M(t) = \sqrt{1 - \rho_{MV}^2 \sigma_M^2} \cdot \sqrt{v_t} \cdot \kappa_t \tag{2.15}
\]

As discussed earlier, the agent requires only an ambiguity adjustment with respect to the "true" monetary policy shock \( W_t^M \). This shock enters also into the equilibrium inflation rate \( \frac{d\hat{p}_t}{\hat{p_t}} \). Hence, only these variables are perceived under the worst-case measure \( Q_{\text{min}} \). For the other variables, i.e. real variables, the historical measure \( P \) and the ambiguity adjusted measure \( Q_{\text{min}} \) coincide.

### 2.7 Optimization under Model Uncertainty

In our model, ambiguity with respect to the money supply rule has no impact on the budget constraint and utility function of the agent. As a result, we apply a standard Longstaff and Ang and Piazzesi (2003), Hörðahl, Tristani, and Vestin (2006), and others.

\(^5\)A brief sketch on how to do this measure transformation is given in appendix A.
Schwartz (1992) optimization under the physical measure $P$. The optimization with respect to the real variables characterizes the optimal consumption and real monetary holdings as well as the real market price of risk and the real interest rate. Given the optimal real money demand and the money supply equation, we determine the equilibrium ambiguity adjusted price process (price process under $Q^{min}$) as the price dynamic which clears the money market. Having determined the equilibrium values for consumption and inflation, we are able to calculate the nominal short rate, nominal market prices of risk and the term structure of nominal bond yields. Nominal bond yields are affected by ambiguity, because nominal bond yields are affected by expected inflation and expected inflation is affected by ambiguity.

**Definition 1 (Equilibrium)** An equilibrium is defined as a price system $[\hat{p}_t]$ and consumption processes $[\hat{c}_t, \hat{m}_t]$ and a value function $J(K_t, a_t)$ such that the HJB programming problem is solved, subject to the intertemporal budget constraint (2.6), the ambiguity adjusted dynamics of monetary policy (2.14) and the money market clearing condition, $\hat{p}_t = \frac{M_s}{m_t}$.

\[
\rho J(K_t, a_t) = \max_{c_t, m_t} (U(x_t) + AJ(K_t, a_t)) \tag{2.17}
\]

s.t.

\[
U(x_t) = \log(c_t) + \gamma \log(m_t) \tag{2.18}
\]

\[
\lim_{T \to \infty} E^P [e^{-\rho T} | J(K_T, a_T) |] = 0, \tag{2.19}
\]

where $A$ denotes the differential operator applied to function $J(K_t, a_t)$.

**Proposition 1 (Characterization of Equilibrium)** The representative agent equilibrium is given by:

- The value function of the agent

\[
J(K_t, a_t) = Q \log(K_t) + g(a_t) \tag{2.20}
\]

- The optimal allocation of the agent

\[
\hat{c}_t = \frac{\hat{K}_t}{Q} \tag{2.21}
\]

\[
\hat{m}_t = \gamma \hat{c}_t \tag{2.22}
\]

\[
Q = \frac{1 + \gamma}{\rho} \tag{2.23}
\]

- Equilibrium dynamics of capital

\[
\frac{d \hat{K}_t}{\hat{K}_t} = (\mu_y + \nu_y a_t - \rho - \delta) dt + \sigma_y \sqrt{a_t} dW^a_t \tag{2.24}
\]
• Equilibrium dynamics of the price level (CPI)

\[
\frac{d\hat{p}_t}{\hat{p}_t} = \mu_p(t) dt + \sigma_{p,a}(t) dW_t^a + \rho_{Mv}\sigma_{p,M}(t) dW_t^v + \sqrt{1 - \rho_{Mv}^2}\sigma_{p,M} dW_t^{M,Q_{\min}}
\]

(2.25)

\[
\mu_p(t) = \left[\omega_t + (q_1 - 1) (\mu_y + \nu_y a_t - \rho - \delta) - q_1 \hat{k} - q_2 \hat{\pi} + \sigma_y^2 a_t\right] + \kappa_M(t)
\]

(2.26)

\[
\sigma_{p,a}(t) = \frac{(q_1 - 1) \sigma_y \sqrt{a_t}}{1 - q_2}
\]

(2.27)

\[
\sigma_{p,M}(t) = \frac{\sigma_M \sqrt{v_t}}{1 - q_2}.
\]

(2.28)

The proof is contained in appendix B.

The results of the proposition show that the real allocations of resources are the same as in a standard monetary RBC model. However, the perceived equilibrium price dynamic is directly affected by ambiguity. Depending on the current state of the economy, the agent expects either higher or lower inflation compared to a standard RBC model.

Looking at equation (2.25) shows that the central bank has three possibilities to reduce expected inflation in our model. First, if agents believe that the central bank is a strong inflation targeter, \(q_2\) becomes more negative, the ambiguity adjustment \(\kappa_M(t)\) decreases and the denominator in (2.26) increases which leads to decreasing expected inflation. Second, pursuing a gradual policy (which means reducing the volatility of the monetary instrument \(\sigma_M\)) reduces the ambiguity adjustment \(\kappa_M(t)\) and therefore reduces expected inflation. Third, a transparable and predictable monetary policy might reduce agents specific ambiguity coefficients \(q_{a1}, q_{a2},\) and \(q_{a3}\) to zero.\(^6\)

Proposition 2 (Real Interest Rate and Real Market Price of Risk) The real interest rate \(r_t\) and the market price of output risk \(\lambda_{r,y}(t)\) are given by:

\[
r_t = \mu_y + \nu_y a_t - \delta - \sigma_y^2 a_t
\]

(2.29)

\[
\lambda_{r,y}(t) = \sigma_y \sqrt{a_t}.
\]

(2.30)

The details of the derivation are given in appendix C.

The dynamics of the real interest rate and the real market of risk are like in a standard monetary RBC model where money neutrality holds. Equation (2.29) shows that the real interest rate equals the real growth rate of the economy net depreciation and precautionary savings. Equation (2.30) displays that the only real risk premium in the economy is paid for output risk \(W^a\). This means that assets like inflation-indexed bonds contain only a risk premium for output risk.

\(^6\)There is a huge literature on the topic why it might be optimal for central banks to pursue a gradual policy. Although our model does not tackle this question, it provides an intuition that the volatility of the central bank instrument amplifies the effect that ambiguity with respect to this monetary instrument has on expected inflation and on the nominal short rate. For a nice overview on why it might be optimal for central banks to smooth its policy instrument, compare Sack (2000) and Sack and Wieland (2000).
Proposition 3 (Nominal Interest Rate and Nominal Market Price of Risk) The nominal interest rate $R_t$ and the nominal market price for output risk $\lambda_{R,y}(t)$ as well as the nominal market price of monetary risk $\lambda_{R,M}(t)$ and $\lambda_{R,v}(t)$ are given by:

$$R(t) = \mu_y + \nu_y a(t) - \delta - \sigma_y^2 a(t) + \frac{[\omega(t) + (q_1 - 1)(\mu_y + \nu_y a(t)) - (q_1 - 1)\delta - \rho (q_1 - 1) - q_1 \hat{k} - q_2 \hat{\pi} + \sigma_y^2 a(t)]}{1 - q_2}$$

$$+ \kappa_M(t) \left( \frac{(q_1 - 1)^2 \sigma_y^2 a(t)}{(1 - q_2)^2} - \frac{\sigma_M^2 v(t)}{(1 - q_2)^2} - \frac{(q_1 - 1) \sigma_y^2 a(t)}{1 - q_2} \right)$$

$$\lambda_{R,y} = \frac{\sigma_y \sqrt{a(t)} (q_1 - q_2)}{1 - q_2}$$

$$\lambda_{R,v}(t) = \frac{\rho_{Mv} \sigma_M \sqrt{v(t)}}{1 - q_2}$$

$$\lambda_{R,M}(t) = \frac{\sqrt{1 - \rho_{Mv}^2 \sigma_M \sqrt{v(t)}}}{1 - q_2}$$

The details of the derivation are given in appendix D.

The money supply ambiguity adjustment $\kappa_M(t)$ affects the otherwise standard equilibrium nominal interest rate in our economy. The nominal market prices of risk are not affected by agent’s ambiguity. The endogenously determined market prices of risk, represent the equilibrium compensation per unit of risk, that an agent in our economy requires. Ambiguity with respect to the monetary shocks does not influence this compensation. In the next section we will see, that ambiguity with respect to the money supply rule affects the equilibrium amount of priced risk that a nominal bond contains. Since, the term premium coincides with the product of market price per unit of risk times the amount of risk, we will see that ambiguity affects the term premium through its impact on the amount of priced risk.

### 2.8 Term Structure of Nominal Interest Rates

Given the equilibrium price dynamic and the equilibrium allocations under the ambiguity adjusted $Q^{\min}$ measure, we solve for the term structure of nominal bonds. The nominal bond price satisfies the following standard Euler equation:

$$N_t(\tau) = e^{-\rho \tau} E_t^Q \left[ \frac{u_c(\hat{c}_{t+\tau}, \bar{m}_{t+\tau})}{u_c(\hat{c}_t, \bar{m}_t)} \frac{\hat{p}_t}{\hat{p}_{t+\tau}} \right], \forall \tau \in \mathbb{R}^+.$$  

(2.35)

Proposition 4 (The Nominal Term Structure) The closed-form solution for the equilibrium price of a nominal zero coupon bond $N(t, \tau)$ with time to maturity $\tau$ is given by a log-linear function of the productivity $a_t$ and the nominal state variables $v_t$, $\omega_t$:

$$N(t, \tau) = Z(\tau) e^{-b_a(\tau) a_t - b_v(\tau) \omega_t - b_{\omega_t}(\tau) v_t}, \forall \tau \in \mathbb{R}^+;$$

(2.36)

\footnote{For more details consult Bakshi and Chen (1996).}
where \( Z(\tau), b_a(\tau), b_\omega(\tau), b_v(\tau) \) are deterministic functions of the structural parameters of the economy.

The characterization of the deterministic functions and the proof of the proposition can be found in appendix E.

The nominal yield curve, \(-\frac{1}{\tau} \ln(N_t(\tau))\) is linear in the three state variables. The RMPU framework in our monetary RBC model preserves the simple affine bond pricing structure, which is known from models like Cox, Ingersoll, and Ross (1985b), Vasěřek (1977), and others. As in Buraschi and Jiltsov (2005), the inflation target and output target of the central bank affects the intercept of the yield curve but not the slope with respect to the state variables. Said differently, the Taylor-type money rule in our model does not give the central bank an impact on the slope or curvature of the yield curve. On the other hand, as a consequence of our assumption in equation (2.13), ambiguity aversion influences not only the intercept but also the slope and curvature of the yield curve with regard to the state variables \( v_t \) and \( a_t \), i.e. through the factor sensitivity functions \( b_v \) and \( b_a \). If the ambiguity adjustment was constant, ambiguity would only affect the intercept of the yield curve. This shows that due to the assumed ambiguity adjustment in (2.13), the central bank influences the slope and curvature of the yield curve. As in other multifactor term structure models, such as Longstaff and Schwartz (1992), Constantinides (1992), Buraschi and Jiltsov (2005), our model is able to capture different shapes of the yield curve.

2.9 Completely versus Essentially Affine Model Structure.

Backus, Telmer, and Wu (1999), Dai and Singleton (2000), Duarte (2000), Duffee (2002) show that if the market price of risk is a constant multiple of the local volatility, the resulting affine models ("completely affine") fail to match conditional second moments of bond yields. Due to this linear connection between risk and risk premium, completely affine models fail to generate a high variance of excess returns with a small unconditional mean. Further, this class of models is not able to reproduce the size of the failure of the expectations hypothesis. Forecast errors are not only large but also negatively correlated with the slope of the term structure. Duffee (2002) proposes the application of "essentially affine" term structure models, which parameterize the market price of risk in a way that it is not simply a constant multiple of the priced factor volatility. Among other things this allows the price of risk to become negative, which results in better empirical performance.

To be in accordance with empirical evidence, the model must be able to produce a ratio of expected excess bond returns and interest rate volatility,

\[
E_t^{Q_{\text{min}}} \left[ \frac{\text{d}N_t(\tau)}{N_t(\tau)} - R_t \text{d}t \right] / \sqrt{\text{Var}(R_t)},
\]

(2.37)

close to zero. In our model, the nominal state variable \( \omega_t \), affects the conditional expected value of the money supply process and of the inflation process. Further, it affects the level and volatility of bond yields. But at the same time, \( \omega_t \) does not affect bonds term premium. Our economy supports a nominal market price of risk which is not a constant multiple of the conditional
interest rate volatility. It represents therefore, an equilibrium framework which supports an essentially affine nominal market price of risk.

3 Term Premia on Nominal Bonds

As in Piazzesi and Schneider (2006) and Veronesi and Yared (2000) it is important to distinguish between the pricing of nominal bonds and the pricing of inflation-indexed bonds. Inflation-indexed bonds are priced with the real stochastic discount factor, whereas nominal assets, like nominal bonds, are priced with the nominal stochastic discount factor. This is especially important if one wants to determine the expected excess return (term premium) of nominal assets. In this section we are going to determine the term premium on nominal bonds. It does not come as a surprise that the term premium on a \( \tau \)-period maturity bond is given by the negative covariance of the corresponding nominal bond yield with the growth rate of the nominal stochastic discount factor.\(^8\) Nevertheless, we provide a supplementary argument in order to determine the term premium which also underpins very nicely that model uncertainty with respect to monetary policy does not influence the time-varying market price of risk, but instead influences the price sensitivity of nominal bonds with regard to macroeconomic news.

Intuitively, ambiguity with respect to monetary policy enters into the factor loadings \( b_a(\tau) \) and \( b_v(\tau) \) of the nominal term structure. This makes the yield curve react more sensitive to the risk factors \( a_t \) and \( v_t \) of the economy in comparison to an economy without model uncertainty. This behavior of the model is supported by empirical evidence. A recent study by Gurkaynak, Sack, and Swanson (2005) shows that nominal long-term forward rates react very sensitive to changes in the economy. More remarkably, the authors demonstrate that the reason for movements in long-term forward rates are unexpected components of monetary policy decisions. Further, Gurkaynak, Sack, and Swanson (2005) point out that these volatile movements are not attributable to time-varying risk premia. It seems instead, that the strong sensitivity of nominal forward rates comes from compensation of expected inflation. We show in the next section that our model predicts exactly this empirical finding. Further, we show in the empirical exercise that our model implied risk compensation will be rather slow moving over the business cycle, whereas the ambiguity compensation for unpredictable monetary policy is on average zero in the sample, but strongly time-varying over the business cycle, varying between -152 basis points and 403.7 basis points.

3.1 Model Implied Term Premia on Nominal Bonds

We denote \( e_a^N(t, \tau) \) to be the equilibrium expected excess return (term premium) at time t of a \( \tau \)-maturity nominal bond. The superscript "a" denotes that the expected excess return contains risk and ambiguity compensation. There are at least two ways of determining the term premium. In financial economics it is more common to find the term premium by the negative covariance

\(^8\)A good overview of real and nominal bond pricing is given in Piazzesi and Schneider (2006) and Veronesi and Yared (2000).
of the corresponding nominal bond yield with the growth rate of the nominal stochastic discount factor, i.e.

\[ e^a_N(t, \tau) = -\text{cov}^Q_{\min} \left( \frac{dN(t, \tau)}{N(t, \tau)} \frac{d\xi^a(t)}{\xi^a(t)} \right). \] (3.1)

We present another way, which shows more naturally that ambiguity affects the bond price sensitivity to changes in economic fundamentals. This approach relies on changing probability measures as in standard option pricing theory or as in general equilibrium analysis under heterogeneous agents.

In order to determine the term premium on nominal bond yields, we apply the closed-form solution for the price of the nominal bond from equation (2.36) with the nominal market prices of risk from equation (2.32) and (2.33) and the nominal short rate \( R_t \) from equation (2.31). The risk-neutral dynamic of the bond price equals

\[ \frac{dN(t, \tau)}{N(t, \tau)} = R_t dt - b_v(\tau)\sigma_v\sqrt{v_t}d\hat{W}_t^v - b_\omega(\tau)\sigma_\omega\sqrt{\omega_t}d\hat{W}_t^\omega - b_a(\tau)\sigma_a\sqrt{a_t}d\hat{W}_t^a, \] (3.2)

where \( \hat{W} \) denotes a Brownian motion under the risk-neutral pricing measure.

According to proposition 3, only \( dW^a \) and \( dW^v \) risk are priced. The resulting dynamic of the bond price under the historical measure \( P \) is given by

\[ \frac{dN(t, \tau)}{N(t, \tau)} = (R_t - b_v(\tau)\sigma_v\sqrt{v_t} \cdot \lambda_{R,v}(t) - b_a(\tau)\sigma_a\sqrt{a_t} \cdot \lambda_{R,y}(t)) dt - b_v(\tau)\sigma_v\sqrt{v_t}dW_t^v - b_\omega(\tau)\sigma_\omega\sqrt{\omega_t}dW_t^\omega - b_a(\tau)\sigma_a\sqrt{a_t}dW_t^a, \] (3.3)

where \( \lambda_{R,y} \) denotes the nominal market price of output risk and \( \lambda_{R,v} \) denotes the nominal market price of monetary risk.

The terms \(-b_a(\tau)\sigma_a\sqrt{a_t}\) and \(-b_v(\tau)\sigma_v\sqrt{v_t}\) represent the amount of priced output risk and priced monetary risk of the nominal bond.

The bond price in (3.3) evolves under the historical measure \( P \). While the risk-neutral dynamics \( \hat{W} \) are used for pricing assets, the historical dynamics \( W \) are used for the econometric analysis of the bond model. In the beginning we assumed that there is no ambiguity with respect to \( W^a, W^v \) and \( W^\omega \) shocks. If the bond price evolution in (3.3) is transformed to the ambiguity adjusted \( Q_{\min} \) measure we see that the structure of the stochastic differential equation is equal to the structure of the stochastic differential equation under the historical measure \( P \).

To explain this from another angle, note that equation (3.3) can be written as

\[ dN(t, \tau) = \left[ R_tN_t(\tau) + \frac{\partial N_t(\tau)}{\partial v_t}\sigma_v\sqrt{v_t} \cdot \lambda_{R,v}(t) + \frac{\partial N_t(\tau)}{\partial a_t}\sigma_a\sqrt{a_t} \cdot \lambda_{R,y}(t) \right] dt + \sum_{i_t \in S_t} \frac{\partial N_t(\tau)}{\partial i_t} \sigma(i_t) dW_t^i \] (3.4)

where \( S_t \) is the state vector containing the state variables \((a_t, v_t, \omega_t)\) and \( \sigma(i_t) \) is the volatility of the corresponding state variable.
Intuitively, equation (3.4) denotes that the price change of a nominal zero-coupon bond with payoff $1 at time $t + \tau$ accounts for the nominal risk-free interest rate $R_t$, the risk premium $\lambda_{R,y}$ and $\lambda_{R,v}$ multiplied by the corresponding amount of priced risk, i.e. $\frac{\partial N_t(\tau)}{\partial a_t} \sigma_{a} \sqrt{v_t}$ and $\frac{\partial N_t(\tau)}{\partial v_t} \sigma_{v} \sqrt{v_t}$, plus random fluctuations due to the stochastic state variables $\sum_{i \in S_t} \frac{\partial N_t(\tau)}{\partial s_{it}} \sigma(i_t) dW_i$.

We have already seen that ambiguity is not a priced risk factor. Instead, ambiguity enters the yield curve factor loadings $b_a(\tau)$ and $b_v(\tau)$. Since $\frac{\partial N_t(\tau)}{\partial a_t}$ equals $-b_a(\tau) N_t(\tau)$ and $\frac{\partial N_t(\tau)}{\partial v_t}$ equals $-b_v(\tau) N_t(\tau)$, it becomes clear how ambiguity enters the term premium of the nominal $\tau$-period bond. Ambiguity influences the nominal bond sensitivity with respect to the priced sources of risk $a_t$ and $v_t$, i.e. $\frac{\partial N_t(\tau)}{\partial a_t}$, $\frac{\partial N_t(\tau)}{\partial v_t}$.

To summarize, once we assume an ambiguity averse agent with ambiguity adjustment as specified in (2.13), nominal bond yields and expected excess returns (term premia) are affected by ambiguity. Ambiguity affects nominal bond yields and expected excess returns through expected inflation only. Since the price of inflation-indexed bonds is not affected by expected inflation, ambiguity does not enter into the yield curve factor $b_a(\tau)$ and $b_v(\tau)$ of inflation-indexed bonds. All of these characteristics are supported by the study of Gurkaynak, Sack, and Swanson (2005).

The expected excess return $e^a_N(t, \tau)$ that an ambiguity averse agent requires for holding the nominal $\tau$-period bond over an infinitesimal time interval is equal to

$$e^a_N(t, \tau) \equiv E_t \left[ \frac{dN_t(\tau)}{N_t(\tau)} - R_t \right] = \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial a_t} \sigma_{a} \sqrt{v_t} \cdot \lambda_{R,y} + \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial v_t} \sigma_{v} \sqrt{v_t} \cdot \lambda_{R,v} \right]$$

$$= -b_a(\tau) \sigma_{a} \rho_{Ma} q_1 - q_2 \rho_{Ma} \sigma_{v} v_t \left( \frac{1}{1 - q_2} \right) - b_v(\tau) \rho_{Mv} \sigma_{v} v_t \left( \frac{1}{1 - q_2} \right)$$

(3.5)

where ambiguity enters into the expected excess return through the factor loadings $b_a(\tau)$ and $b_v(\tau)$. If the agent is not ambiguity averse, $q_{a1}$, $q_{a2}$ and $q_{a3}$ in (2.13) are zero. Hence, the expected excess return of a nominal $\tau$-period bond, $e^a_N(t, \tau)$, of an agent who is not ambiguity averse equals

$$e^a_N(t, \tau) \equiv E_t \left[ \frac{dN_t(\tau)}{N_t(\tau)} - R(t) dt \right] = \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial a_t} \sigma_{a} \sqrt{v_t} \cdot \lambda_{R,y} + \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial v_t} \sigma_{v} \sqrt{v_t} \cdot \lambda_{R,v}$$

$$= -b_a(\tau) \sigma_{a} q_1 - q_2 \rho_{Mv} \sigma_{v} v_t \left( \frac{1}{1 - q_2} \right) - b_v(\tau) \rho_{Mv} \sigma_{v} v_t \left( \frac{1}{1 - q_2} \right)$$

(3.6)

$$= \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial a_t} \sigma_{a} \sqrt{v_t} \cdot \lambda_{R,y} + \frac{1}{N_t(\tau)} \frac{\partial N_t(\tau)}{\partial v_t} \sigma_{v} \sqrt{v_t} \cdot \lambda_{R,v}$$

(3.7)

3.2 Ambiguity Contribution to the Term Premium

In the empirical section, we estimate the bond model (2.36) with a panel of U.S. bond data. As an approximation of the contribution that ambiguity has on the term premium we determine the
difference between the term premium which contains risk and ambiguity compensation, $e^a_N(t, \tau)$, and the term premium which contains only risk compensation, $e^N(t, \tau)$. This difference leaves us with the pure ambiguity contribution to the term premium. The ambiguity contribution to the term premium on the nominal $N(t)$ bond is hence given by

$$e^a_N(t, \tau) - e^N_N(t, \tau) = -(b_a(\tau) - \hat{b}_a(\tau))\sigma_a\sigma_y \frac{q_1 - q_2}{1 - q_2} a_t - (b_v(\tau) - \hat{b}_v(\tau))\sigma_v\rho_{Mv}\sigma_M \frac{1}{1 - q_2} v_t. \quad (3.9)$$

4 Estimation

4.1 Methodology

We estimate the model by Quasi Maximum Likelihood. The log-likelihood function $L_T(\Omega_P)$ of the bond model is fully characterized in the appendix. During the estimation we match six maturities (1, and 6 month, 1, 2, 5, and 7 years). We numerically maximize the log-likelihood function through repeatedly evaluating this function for different parameter values of $\Omega_P$. For each parameter combination $\omega_p \in \Omega_P$, we unfold the unobservable state variables $[a_t, v_t, \omega_t]$ by inverting the affine yield relationship for the three-month, 3-year, and ten-year zero-coupon bonds (Chen and Scott (1993), Fisher and Gilles (1996), Duffee (2002)). Moreover, we assume that the measurement error shocks are conditionally joint normal distributed and orthogonal to the shocks to the unobservable states.

We estimate the asymptotic estimates of the variance-covariance matrix of the parameters through the outer product of the score. The score of the log-likelihood for observation $t$ is obtained through applying finite differences to the gradient $\frac{\partial L_T(\omega_p)}{\partial \omega_p}$.

4.2 Data

The sample consists of smoothed continuously compounded Fama-Bliss yields for the period January 1970 to December 2003. The interest rate data comprises 9 different maturities (1, 3 and 6 month as well as 1, 2, 3, 5, 7 and 10 years).\textsuperscript{9}

5 Empirical Analysis

We estimate our structural bond model with a cross-section of nine continuously compounded Fama-Bliss yields. The parameter estimates are presented in Table 2. Some statistics for the cross-sectional model fit are given in Table 1.

Understanding the determinants for movements in long-term nominal bond yields is especially important for central bankers as well as for bond traders. In a recent study Gurkaynak, Sack, and Swanson (2005) show that nominal long-term forward rates react excessively sensitive

\textsuperscript{9}We thank Rob Bliss for sharing his data and programs with us.
to changes in the economy. The authors demonstrate that the reason for movements in long-term forward rates are unexpected components of monetary policy decisions. Further, Gurkaynak, Sack, and Swanson (2005) point out that these excessive movements are not attributable to time-varying risk premia. It seems instead, that the excessive sensitivity of nominal forward rates comes from compensation of expected inflation. We show in the next section that our model predicts exactly this behavior.

5.1 Ten Year Term Premium

Figure 1 compares the data implied excess return for the ten year bond over the three month bond with the corresponding model implied expected excess return (term premium). The term premium in the data is highly volatile, fluctuating between -426 and +527.6 basis points. The model implied term premium does a good job in tracking its empirical counterpart. The model implied expected excess return varies between 9.6 and 450.9 basis points. The overall qualitative shape of the model implied term premium is close to the excess return as observed in the data. The quantitative size of the model implied expected excess return comes especially close to its empirical counterpart in times where the yield curve is upward sloping.

5.2 Ten Year Term Premium Decomposition

We have shown in equation (3.9) that our model is able to decompose the model implied term premium into its expected risk and its expected ambiguity contribution. We have performed this decomposition with our data set. The result is presented in Figure 2. Figure 2 contains in principle the same information as the previous Figure 1. The only difference is that we do not show the model implied term premium, but instead its two components. One of the components is rather slow moving over the business-cycle and remains always positive. This component coincides with the expected risk contribution to the term premium. The second component is highly volatile over the business cycle, taking highly positive and highly negative values. This component coincides with the expected ambiguity contribution to the term premium. Both components add up to the model implied term premium.

Looking first at the risk contribution to the term premium (red line) we see that it is time-varying over the business cycle, taking values in the range of 21.2 to 275.7 basis points. Its time-series of the expected risk contribution is tent-shaped. Between the early and late 1970’s its fluctuating narrowly around 140 basis points. During the late 1970’s and early 1980’s it lifts up to around 275.7 basis points, which marks the top of the tent. After the early 1980’s, until the late 1980’s it decreases again to approximately 140 basis points. From the late 1980’s to the late 1990’s it remains fluctuating in a tight range between 130 and 150 basis points. Beginning in the late 1990’s until December 2003 it decreased steadily to approximately 40 basis points. This shows that the pure risk contribution to the term premium has always stayed positive, giving the yield curve a positive slope. The pure risk compensation was highest in the high inflation

\[^{10}\text{The three-dimensional CIR structure of the economy makes it impossible to generate negative expected excess returns. Duffee (2002) explains the reasons. Nevertheless, we will later show that we can even explain the negative term premiums, which are times where the yield curve is inverted.}\]
period of the early 1980’s, during the Fed experiment, and lowest in the early 2000’s, where U.S. inflation was lowest.

In sharp contrast to the time-series of the pure risk contribution is the time-series of the expected ambiguity contribution. The ambiguity contribution to the term premium is fluctuating around zero basis points with a sample average of 0.9 basis points. Although its sample average is close to zero it varies between -152 to 403.7 basis points. The time-periods of highest fluctuations are the period of the late 1970’s to early 1980’s, and the early 2000’s. Both periods are characterized by high investor uncertainty with regard to monetary policy. The period of the early 1980’s is characterized by high monetary uncertainty. The U.S. Fed experimented with different monetary regimes and investors were uncertain about the future paths of monetary policy. The blue line in Figure 1 shows that the ambiguity contribution to the term premium has in this period been fluctuating between -152 to +146.5 basis points. During the mid and late 1990’s, when the U.S. economy was very robust, the ambiguity contribution to the term premium decreased steadily from zero to -50 basis points, leading to a flattening of the yield curve. In the late 1990’s however, the expected ambiguity contribution increased again and sky-rocketed after the technology bubble crash in 2001, peaking at 403.7 basis points in late 2003. This period was also characterized by investors nervously monitoring the behavior of the U.S. Fed. At that time investors were very concerned about how the Fed was going to achieve a soft-landing of the U.S. economy. Our model captures this uncertainty via the ambiguity contribution to the term premium.

5.3 Risk Contribution vs. Ambiguity Contribution to the Term Premium

In the following paragraphs we present some figures which again present the structurally different behavior on the expected ambiguity contribution and the expected risk contribution of the ten year nominal term premium.

Figures 3, 4, 5 and 6 plot the expected risk and the expected ambiguity contribution on the ten year nominal term premium. These figures show nicely how the expected risk contribution moves slowly over the business cycle, staying always positive, whereas the ambiguity contribution to the term premium fluctuates wildly around zero and peaking in periods which are characterized by high monetary policy uncertainty.

6 Conclusion

This paper presents a dynamic equilibrium model of the nominal term structure where agents are ambiguity averse with respect to monetary policy. The theoretical part of the paper presents five results.

First, we characterize the equilibrium in a Recursive-Multiple-Prior economy, where the representative agent faces model uncertainty with respect to monetary policy.

Second, the paper shows that in equilibrium, expected inflation is directly affected by ambiguity. The state variables of the economy determine whether expected inflation is higher or
lower compared to a standard expected utility framework.

Third, we provide closed-form solutions for the equilibrium values of the real and nominal interest rate, real and nominal market prices of risk as well as nominal bond prices.

Fourth, the paper shows that if agents are ambiguity averse with respect to the unknown future distribution of money supply, it is the money supply volatility which amplifies the effects of ambiguity. Hence, even a central bank which only cares for output and inflation stabilization might want to pursue a smooth policy in order to reduce the volatility of its monetary instrument.

Fifth, we derive a tractable exponentially affine bond pricing model which allows to decompose the term premium on bonds into their expected risk and their expected ambiguity contribution.

In the empirical section, we estimate the model with a panel of U.S. Treasury bond data, ranging from 1970.1 to 2003.12, with maturities from one month to ten years. Our model implied ten year term premium does a good job in tracking its empirical counterpart. The model implied expected excess return varies between 9.6 and 450.9 basis points over the business cycle. We apply the proposed decomposition for the term premium and find that one of the components is a rather slow moving one which fluctuates in a tent shape manner over the business cycle and remains always positive. This component coincides with the the expected risk contribution to the term premium. The second component is highly volatile over the business cycle, taking highly positive and highly negative values. This component coincides with the expected ambiguity contribution to the term premium. Both components of course, add up to the model implied term premium. The expected risk contribution has been varying between 21.2 and 275.7 basis points and seems to be closely related to actual inflation. In sharp contrast to this we find, that the time-series of the expected ambiguity contribution to the term premium is fluctuating around zero basis points with a sample average of 0.9 basis points. Although its sample average is close to zero it varies between -152 to 403.7 basis points. The time-periods of highest fluctuations in the expected ambiguity contribution are the period of the late 1970’s to early 1980’s, and the early 2000’s. Both periods were characterized by investors who faced high uncertainty about future monetary policy. The highly volatile fluctuations of the expected ambiguity contribution to the term premium seem to be most closely related to long-term expected inflation and to the corresponding model uncertainty with respect to future monetary policy.

The empirical analysis further suggests that the overall positive slope of the yield curve is due to the pure risk compensation of investors, whereas the strong and volatile fluctuations of the term premium over time are mainly based upon agents’ model uncertainty with respect to monetary policy.

The theoretical and empirical results show that relaxing the assumption that agents know for sure the distribution of future monetary policy is important for understanding the volatile long-term nominal term premium and the high sensitivity of long-term yields with regard to unexpected macroeconomic news (especially unexpected monetary news). Traditional theoretical and empirical term structure models do not take ambiguity into account. Our empirical analysis shows that taking ambiguity with respect to monetary policy into account can explain why nominal bond yields in comparison to inflation-indexed bonds, react so sensitive to unexpected monetary news.
For future research, it is beneficial to further analyze the link between model uncertainty with respect to monetary policy and bond yield behavior. It is especially interesting to do similar empirical studies with U.K. data and to take explicitly inflation-indexed bonds into account. Further it seems very promising to do the same analysis with a higher data frequency, for example daily frequency.
A Determine Ambiguous Money Supply Rule

In order to determine the ambiguity-adjusted money supply rule we need two ingredients. First, our economy is money-neutral. From that, we immediately know, that the equilibrium real money demand as well as the equilibrium capital stock is not affected by the monetary policy shock $W^\hat{M}$. Second, in equilibrium, the money market is cleared, which means that nominal money supply equals nominal money demand, or equivalently

$$\hat{p}_t = \frac{M^s_t}{m_t}. \quad (A.1)$$

The model assumes that agents are only ambiguity averse with respect to $W^\hat{M}$. Therefore, need to find in the money supply equation all terms that are affected by $dW^\hat{M}$. The money supply equation 2.7 together with our two ingredients from above shows that only $q_2 \frac{dp_t}{p_t}$ and $\sqrt{1 - \rho^2_M \sigma_M \sqrt{v_t}}$ are related to $dW^\hat{M}$.

We are going to determine how $q_2 \frac{dp_t}{p_t}$ is affected by $dW^\hat{M}$, by applying Ito’s lemma to (A.1).

$$d\hat{p}_t = \frac{1}{m_t} dM^s_t - \frac{M^s_t}{m_t^2} dm_t + \frac{M^s_t}{m_t^3} dm_t dm_t - \frac{1}{m_t^2} dM_t dm_t. \quad (A.2)$$

Due to money neutrality, we already see that only the first component $\frac{1}{m_t} dM^s_t$ can contain the monetary policy shock $dW^\hat{M}$. We therefore, write the last equation as

$$d\hat{p}_t = \frac{1}{m_t} dM^s_t + A \quad (A.3)$$

$$A \equiv -\frac{M^s_t}{m_t^2} dm_t + \frac{M^s_t}{m_t^3} dm_t dm_t - \frac{1}{m_t^2} dM_t dm_t. \quad (A.4)$$

Plugging in the dynamic for money supply, according to 2.7, and using the money market clearing condition, gives

$$d\hat{p}_t = \hat{p}_t \left( \omega_t dt + q_1 \left( \frac{d\hat{K}_t}{K_t} - \hat{k} dt \right) + q_2 \left( \frac{dp_t}{p_t} - \hat{\pi} dt + \rho_M \sigma_M \sqrt{v_t} dW^\nu_t \right) \right)$$

$$+ \hat{p}_t \left( \sqrt{1 - \rho^2_M \sigma_M \sqrt{v_t}} dW^\hat{M}_t \right) + A \quad (A.5)$$

The term $d\hat{p}_t$ appears on the left and on the right hand sight. We collect these terms and arrive at

$$d\hat{p}_t = \frac{\hat{p}_t}{1 - q_2} \sqrt{1 - \rho^2_M \sigma_M \sqrt{v_t}} dW^\hat{M}_t + \frac{A}{1 - q_2}$$

$$+ \frac{\hat{p}_t}{1 - q_2} \left( \omega_t dt + q_1 \left( \frac{d\hat{K}_t}{K_t} - \hat{k} dt \right) - q_2 \hat{\pi} dt + \rho_M \sigma_M \sqrt{v_t} dW^\nu_t \right) \quad (A.6)$$
We now have determined the term within $d\hat{p}_t$, which is affected by the monetary shock $dW^M$. Now, we go back to the money supply equation (2.7),

$$
\frac{dM_t}{M_t} = \omega_t dt + q_1 \left( \frac{dK_t}{K_t} - \hat{\kappa} dt \right) + q_2 \left( \frac{d\hat{p}_t}{\hat{p}_t} - \hat{\pi} dt \right) + \rho_{Mv} \sigma_M \sqrt{\hat{\nu}_t} dW^v_t \\
+ \sqrt{1 - \rho_{Mv}^2 \sigma_M \sqrt{\hat{\nu}_t}} dW^M_t
$$

and plug (A.6) into it. After doing this, we see that only one term remains within the money supply equation, which is affected by the monetary shock $dW^M$, namely

$$
1 - q_2 \sigma_M \sqrt{1 - \rho_{Mv}^2 \sqrt{\hat{\nu}_t}} dW^M_t
$$

By applying Girsanov theorem and (2.11), i.e. $dW_t^{\hat{M},Q_{\min}} = dW_t^M - \kappa_t$ to the term $1 - q_2 \sigma_M \sqrt{1 - \rho_{Mv}^2 \sqrt{\hat{\nu}_t}} dW^M_t$ we get

$$
1 - q_2 \sigma_M \sqrt{1 - \rho_{Mv}^2 \sqrt{\hat{\nu}_t}} dW^M_t = \kappa_t \cdot 1 - q_2 \sigma_M \sqrt{1 - \rho_{Mv}^2 \sqrt{\hat{\nu}_t}} dW^M_t
$$

The last summand coincides with the ambiguity adjustment in the money supply equation. We define it to be $\kappa_M(t)$ and arrive therefore at equation (2.14).

**B Equilibrium Allocations, Price and Capital Dynamics**

The representative agent owns an all-equity company and decides how much to consume and to invest. Holding cash does not imply any cash outflows as it is the case with borrowed capital. The only cost associated with the production technology are depreciation costs. The agent solves the optimal consumption-investment plan:

$$
\max_{c_t, m_t} E_{Q_{\min}} \left[ \int_0^{\infty} e^{-\rho s} \left( \log(c_s) + \gamma \log(m_s) \right) ds | \mathcal{F}_0 \right]
$$

s.t. the budget constraint (2.6).

Ambiguity aversion with respect to the money supply rule does not influence the evolution of capital. In this case the $Q_{\min}$ measure coincides with the historical measure $P$. The consumption-investment problem simplifies to a standard one. For this framework it is known that the value function $J(\cdot)$ takes the form

$$
J(K_t, a_t) = Q \log(K_t) + g(a_t).
$$

In equilibrium, there exists a value function $J(K_t, a_t)$ and control variables $c_t, m_t$ that solve the following Benveniste-Scheinkman condition

$$
\rho J(K_t, a_t) = \max_{c_t, m_t} \left( \log(c_t) + \gamma \log(m_t) + AJ(K_t, a_t) \right),
$$

where $A$ denotes the differential operator applied to $J(K_t, a_t)$. Function $J(\cdot)$ is linear in its state variables. Their quadratic variation terms vanish. Hence,

$$
AJ(\cdot) = \frac{\partial J(\cdot)}{\partial a} \mu_a(t) + \frac{\partial J(\cdot)}{\partial K} \mu_K(t) + \frac{1}{2} \frac{\partial^2 J(\cdot)}{\partial K^2} < dK dK >_t,
$$

22
where $J(\cdot) \equiv J(K_t, a_t)$ and $<dK dK>_t$ denotes the quadratic variation of $dK_t$.

First-order conditions yield the standard result

$$
\hat{c}_t = \frac{K_t}{Q}, \quad (B.5)
$$

$$
\hat{m}_t = \frac{\gamma K_t}{Q}, \quad (B.6)
$$

To verify that the guess is correct and to determine the constant $Q$, plug the optimal policy functions in the Benveniste-Scheinkman conditions (B.3). Matching coefficients on the constant terms and $[\log(K_t), a_t]$ of the Benveniste-Scheinkman condition gives the missing parameter values for $Q$ and $g(a_t)$. Doing this gives $Q = (1 + \gamma)/\rho$. Hence, the optimal policy functions are given by

$$
\hat{c}_t = \frac{\rho K_t}{1 + \gamma}, \quad (B.7)
$$

$$
\hat{m}_t = \frac{\gamma \rho K_t}{1 + \gamma}. \quad (B.8)
$$

Plugging the optimal policy functions into the budget constraint (2.6) gives the equilibrium dynamic for the accumulated capital process (2.24). Applying Ito’s Lemma to the market clearing condition for monetary holdings gives us the equilibrium price dynamic (2.25). The guess of $J(\cdot)$ can be verified by substituting the equilibrium values into the Hamilton-Jacobi-Bellman equation. Coefficient matching shows that $(Q, g(\cdot))$ are uniquely determined and independent of $K_t$ and $a_t$.

### C Interest Rate and Market Price of Risk

The interest rate is given by the negative drift of the real stochastic discount factor. The volatility of the real stochastic discount factor coincides with the market price of risk. Due to money neutrality in our economy, there are no effects of model uncertainty with respect to the money supply rule on the real interest rate or the real market price of risk. The real stochastic discount factor $\xi^r_t$ is given by

$$
\xi^r_t = e^{-\rho t} \frac{\partial u(c_t, m_t)}{\partial c_t}.
$$

(C.1)

Applying Ito’s Lemma to the optimal consumption rule (2.21), gives the equilibrium dynamic of $c_t$. Given this dynamic, it is straightforward to apply Ito’s Lemma to (C.1) and to obtain

$$
-\frac{d\xi^r_t}{\xi^r_t} = (\mu_y + \nu_y a_t - \delta - \sigma_y^2 a_t) dt + \sigma_y \sqrt{a_t} dW^a_t.
$$

(C.2)

The drift of the last equation coincides with the real spot interest rate. The volatility of the last expression coincides with the real market price of output risk.
D Nominal Interest Rate and Nominal Market Price of Risk

As in Piazzesi and Schneider (2006) and Veronesi and Yared (2000) we define the nominal stochastic discount factor, which prices all nominal assets in the economy, as

$$\xi^n(t) \equiv \frac{\xi^r(t)}{p_t} \quad (D.1)$$

The agent in our economy is ambiguity averse with respect to future monetary policy. He therefore regards the equilibrium price process $\hat{p}(t)$ under his ambiguity adjusted probability measure $Q_{\text{min}}$. This is different to Piazzesi and Schneider (2006) and Veronesi and Yared (2000) who work under the assumption that the agent knows perfectly the future distribution of inflation.

The dynamics of the real stochastic discount factor $\xi^r_t$ are derived in appendix C. The equilibrium dynamic of the ambiguity adjusted price level is summarized in equation (2.25), in proposition (1). An application of Ito’s lemma reveals the dynamic of the nominal stochastic discount factor, i.e.

$$-d\xi^n(t) = \left(\mu_y + \nu_y a(t) - \delta - \sigma_y^2 a(t)\right) dt$$

$$+ \left(\omega(t) + (q_1 - 1)(\mu_y + \nu_y a(t)) - (q_1 - 1)\delta - \rho (q_1 - 1) - q_1 \hat{k} - q_2 \hat{\pi} + \sigma_y^2 a(t)\right) dt$$

$$+ \frac{\kappa_M(t) - (q_1 - 1)^2 \sigma_y^2 a(t)}{(1 - q_2)^2} - \frac{\sigma_M^2 \nu(t)}{(1 - q_2)^2} - \frac{(q_1 - 1) \sigma_y^2 a(t)}{1 - q_2} dt$$

$$+ \frac{\sigma_y \sqrt{a(t)} (q_1 - q_2)}{1 - q_2} dW_a(t) + \rho_{Mv} \frac{\sigma_M \sqrt{v(t)}}{1 - q_2} dw^v + \sqrt{1 - \rho_{Mv}^2} \frac{\sigma_M \sqrt{v(t)}}{1 - q_2} dW_t^{M,Q_{\text{min}}}. \quad (D.2)$$

The drift of the latter equation coincides with the nominal short-term interest rate, whereas the volatility terms coincide with the nominal market prices of risk.

E Nominal Term Structure

The equilibrium price of a nominal zero-coupon bond $N_t(\tau)$ with time to maturity $\tau$ equals the ambiguity adjusted conditional expected value of the intertemporal marginal rate of consumption substitution times the real payoff at maturity:

$$N_t(\tau) = e^{-\rho \tau} E_t^{Q_{\text{min}}} \left[ \frac{u_c(\hat{c}_{t+\tau}, \hat{m}_{t+\tau})}{\hat{p}_t} \frac{\hat{p}_t}{\hat{p}_{t+\tau}} \right]. \quad (E.1)$$

Plugging in the log utility function together with the optimal consumption policy (2.21) and defining $\hat{\kappa}_t \equiv \rho t + \ln(\hat{K}_t)$ yields

$$N_t(\tau) = \frac{1}{\exp(-\hat{\kappa}_t)} E_t^{Q_{\text{min}}} \left[ \frac{\exp(-\hat{\kappa}_{t+\tau})}{\hat{p}_{t+\tau}} \right]. \quad (E.2)$$
The no-arbitrage price at time t of a zero-coupon bond maturing in \( t + \tau \) solves the stochastic problem in (E.2). To get a closed-form solution we apply Feynman-Kac's Theorem and solve the dual parabolic PDE:\(^\text{11}\)

\[
\frac{\partial N(\cdot, \tau)}{\partial \tau} = AN(\cdot, \tau) \tag{E.3}
\]

s.t. \( N(\cdot, 0) = 1, \) \( \tag{E.4} \)

where \( N(\cdot, \tau) \equiv N(\hat{k}_t, \hat{p}_t, v_t, \omega_t, a_t; \tau) \) and \( A \) represents the second-order differential operator applied to function \( N(\cdot, \tau) \). Define \( \phi(\hat{k}_t, \hat{p}_t, v_t, \omega_t, a_t; \tau) \) to be the solution of the stochastic problem:

\[
\phi(\hat{k}_t, \hat{p}_t, v_t, \omega_t, a_t; \tau) = E_t^{Q_{\min}} \left[ \frac{\exp(-\hat{k}_t - \tau)}{\hat{p}_t} \right] . \tag{E.5}
\]

Since our economy has logarithmic preferences with constant return to scale production, we guess that the solution has the form:

\[
\phi(\hat{k}_t, \hat{p}_t, v_t, \omega_t, a_t; \tau) = e^{-\hat{k}_t} Z(\tau) \frac{e^{-b_v(\tau)v_t - b_\omega(\tau)\omega_t - b_a(\tau)a_t}}{\hat{p}_t} . \tag{E.6}
\]

If (E.6) solves the stochastic problem than it also solves the PDE

\[
\frac{\partial \phi(\cdot, \tau)}{\partial \tau} = A\phi(\cdot, \tau) \tag{E.7}
\]

s.t. \( \lim_{\tau \to 0} \phi(\cdot, \tau) = \frac{\exp(-\hat{k}_t)}{\hat{p}_t} , \) \( \tag{E.8} \)

where \( \phi(\cdot, \tau) \equiv \phi(\hat{k}_t, \hat{p}_t, v_t, \omega_t, a_t; \tau) \).

Differentiating (E.6) with respect to \( \tau \) yields

\[
\frac{\partial \phi(\cdot, \tau)}{\partial \tau} = \left[ \frac{d}{d\tau} Z(\tau) \right] - \left( \frac{d}{d\tau} b_v(\tau) \right) v_t - \left( \frac{d}{d\tau} b_\omega(\tau) \right) \omega_t - \left( \frac{d}{d\tau} b_a(\tau) \right) a_t \right] \phi(\cdot, \tau) . \tag{E.9}
\]

If we apply the differential operator to (E.6) we end up with

\[
A\phi(\cdot, \tau) = \sum_{i \in L} \frac{\partial \phi(\cdot, \tau)}{\partial i} \mu_i(t) + \frac{1}{2} \sum_{i \in L} \frac{\partial^2 \phi(\cdot, \tau)}{\partial i^2} < di di >_t + \sum_{i \in L \setminus \{v_t, a_t\}} \frac{\partial^2 \phi(\cdot, \tau)}{\partial \hat{p}_t \partial i} < d\hat{p} di >_t + \frac{\partial^2 \phi(\cdot, \tau)}{\partial a_t \partial k_t} < d\hat{a} dk >_t , \tag{E.10}
\]

where \( L \) is the set consisting of \( \{a_t, v_t, \omega_t, \hat{p}_t, \hat{k}_t\} \) and \( < di dj > \) stands for the instantaneous covariation of process \( i \) with process \( j \). Collecting terms shows that the latter formula is an affine function of the underlying state variables \( \{v_t, \omega_t, a_t\} \)

\[
A\phi(\cdot, \tau) = \left[ \partial_{\hat{p}_t} + \partial_{\omega_t} v_t + \partial_{\omega_t} \omega_t + \partial_{a_t} a_t \right] \phi(\cdot, \tau) , \tag{E.11}
\]

\(^{11}\)A similar solution strategy is performed in Buraschi and Jiltsov (2005).
where the slope coefficients and $\theta_0$ are deterministic functions, depending only on the structural parameters of the model (summarized as $\Xi$) and on the bond factors $b_v(\tau)$, $b_\omega(\tau)$ and $b_a(\tau)$

$$\theta_0 = \iota_0(\Xi) + \iota_a(\Xi)b_a(\tau) + \iota_\omega(\Xi)b_\omega(\tau) + \iota_v(\Xi)b_v(\tau) \quad \text{(E.12)}$$

$$\theta_i = \psi^0_i(\Xi) + \psi^1_i(\Xi)b_i(\tau) + \psi^2_i(\Xi)b^2_i(\tau); \quad i = \{a, v, \omega\}. \quad \text{(E.13)}$$

All $\iota$ and $\psi$ functions are nonlinear in the structural parameters $\Xi$ of the economy. In order to save space we do not present the functional forms. Matching the coefficients of the state variables in (E.9) and (E.11) gives a system of ODEs:

$$\frac{d}{d\tau} Z(\tau) = \theta_0 \quad \text{(E.14)}$$

$$-\frac{d}{d\tau} b_i(\tau) = \theta_i; \quad i = v, \omega, a. \quad \text{(E.15)}$$

This system can be solved by standard methods. Lemma 1 contains the general solution.

**Lemma 1 (Solution of the System of ODEs)** The solution to the ODE in (E.15) can be verified by direct substitution:

$$b_i(\tau) = \frac{1}{2} \left( -\psi^i_1 - \tan \left( \frac{1}{2} \tau \sqrt{D^i} - \arctan \left( \frac{\psi^i_1}{\sqrt{D^i}} \right) \right) \right) \psi^i_2^{-1} \quad \text{(E.16)}$$

$$D^i \equiv 4\psi^i_0 \psi^i_2 - \psi^i_1^2. \quad \text{(E.17)}$$

In order to solve ODE (E.14) let us guess the following solution:

$$Z(\tau) = e^{A_0 \tau} a_v(\tau) a_\omega(\tau) a_a(\tau). \quad \text{(E.18)}$$

Since we need an expression for $\frac{d}{d\tau} Z(\tau)/Z(\tau)$ we make the following calculations

$$\frac{d}{d\tau} Z(\tau)/Z(\tau) = \frac{d(\ln(Z(\tau))}{d\tau} \quad \text{(E.19)}$$

Plugging this into (E.14) and equating the individual restrictions gives

$$A_0 = \iota_0(\Xi) \quad \text{(E.20)}$$

$$\frac{d}{d\tau} a_i(\tau)/a_i(\tau) = \iota_i(\Xi)b_i(\tau); \quad i \in a, v, \omega. \quad \text{(E.21)}$$

Direct substitution verifies the following result

$$a_i(\tau) = e^{-1/2 \frac{\iota_i(\Xi)}{\psi^i_2}} \left( 1 + \left( \tan \left( \frac{1}{2} \tau \sqrt{D^i} - \arctan \left( \frac{\psi^i_1}{\sqrt{D^i}} \right) \right) \right)^2 \right)^{-1/2 \frac{\iota_i(\Xi)}{\psi^i_2}} \times \left( \frac{D^i + \psi^i_1^2}{D^i} \right)^{-1} \quad \text{;} \quad i \in a, v, \omega \quad \text{(E.22)}$$
\[
D^i \equiv 4 \psi^i_0 \psi^i_2 - \psi^i_1^2. \tag{E.23}
\]

To ensure that bond prices are arbitrage free we restrict \(D^i\) to be negative for all \(i \in \{a, v, \omega\}\). If \(D^i \geq 0\), the Riccati equation (E.15) is periodic and hence arbitrage opportunities arise.

## F Quasi Maximum Likelihood

We assume that the 3 month, 3 year, and 10 year bond yields are perfectly observed. All the other maturities (1, and 6 month, 1, 2, 5, 7 years) (we call them \(\tau_m\)), are observed with measurement errors. This widely used methodology goes back to Chen and Scott (1993) and Fisher and Gilles (1996).

### F.1 Likelihood Function of the State Variables

Let \(S_t \in \mathbb{R}^3\) be the vector of state variables \([v_t, \omega_t, a_t]'\). Each component of the state vector \(S\) (we call it \(X\)) follows a CIR dynamic, i.e.

\[
dX_t = \kappa(\theta - X_t)dt + \sigma \sqrt{X_t}dW_t. \tag{F.1}
\]

The first two (unconditional) moments of the CIR sde are given by

\[
E[X_t] = \theta + e^{-\kappa t}(E(X_0) - \theta) \quad \lim_{t \to \infty} E[X_t] = \theta \tag{F.2}
\]

\[
E[X_t^2] = \theta^2 + \frac{\sigma^2 \theta}{2\kappa} + (2\theta + \frac{\sigma^2}{\kappa})(E(X_0) - \theta)e^{-\kappa t} + (E(X_0^2) + \theta + \frac{\sigma^2}{\kappa})(\theta - 2E(X_0))e^{-2\kappa t} \tag{F.3}
\]

The first moment is obtained by taking expectations in (F.1) and solving the resulting linear ode. The second moment is obtained by applying Ito formula to \(X_t^2\), taking expectation, using the result for \(E[X_t]\) and solving the resulting linear ode.

The transition density of CIR variables is non-central chi-square. However, if two consecutive time points \(t\) and \(t+1\) are close to each other we can approximate its transition density by a Gaussian distribution with mean \(m(X_t|X_{t-1})\) and variance \(v(X_t|X_{t-1})\), i.e.\(^{12}\)

\[
E(X_t|X_{t-1}) = X_{t-1}e^{-\kappa \Delta} + \theta(1 - e^{-\kappa \Delta}) =: m(X_t|X_{t-1}); \tag{F.5}
\]

\[
V(X_t|X_{t-1}) = X_{t-1}\frac{\sigma^2}{\kappa}(e^{-\kappa \Delta} - e^{-2\kappa \Delta}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa \Delta})^2 =: v(X_t|X_{t-1}) \tag{F.6}
\]

\(^{12}\)More advanced numerical approximations, which are especially useful if two consecutive time points are far away from each other and if it is difficult to sample directly from the true transition density, can be found in Kloeden and Platen (2001) and Kloeden, Platen, and Schurz (1997).
where $\Delta$ denotes the distance between two consecutive realizations of the state variable.

The Quasi Maximum Log-Likelihood function (for the whole sample) is given by

$$l_X(\theta) \equiv \frac{1}{T} \sum_{t=2}^T \log(f(X_t|X_{t-1}; \theta)) \quad (F.7)$$

$$\equiv \frac{1}{T} \sum_{t=2}^T \left( -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(v(X_t|X_{t-1})) - \frac{1}{2} \frac{(X_t - m(X_t|X_{t-1}))^2}{v(X_t|X_{t-1})} \right). \quad (F.8)$$

Our three state variables are orthogonal to each other. Therefore, their joint (conditional) likelihood $f(S_t|S_{t-1}; \Omega_P)$ is just the product of their (conditional) marginals $f(X_t|X_{t-1}; \Omega_P)$.

### F.2 Likelihood Function of the Model

The yield at time $t$ of a zero coupon bond maturing in $t + \tau$ is given by $y_t(\tau) = -\frac{1}{\tau} \ln(N_t(\tau)) = -\frac{1}{\tau} \ln(Z(\tau)) + \frac{1}{\tau} (b_v(\tau)v_t + b_\omega(\tau)\omega_t + b_a(\tau)a_t)$, where $Z(\tau)$ is given in (2.36). We collect the state vectors $[v_t, \omega_t, a_t]'$ in the vector $S_t$ and define matrix $B(\tau)$ as

$$B(\tau) = \begin{bmatrix} b_v(\tau_1) & b_\omega(\tau_1) & b_a(\tau_1) \\ b_v(\tau_2) & b_\omega(\tau_2) & b_a(\tau_2) \\ b_v(\tau_3) & b_\omega(\tau_3) & b_a(\tau_3) \end{bmatrix}$$

, where $\tau_1$, $\tau_2$, and $\tau_3$ are the three yield maturities that are perfectly observed (3 month, 3 year, and 10 year). Inverting the affine yield function backs out the time-series of the implied state vectors $v_t$, $\omega_t$, and $a_t$. Given these states, we assume that the measurement errors of the yields (6 month, 1, 2, 5, 7 year) (model implied yields minus empirically observed yields) are orthogonal to the innovations in the implied state variables.

The likelihood function is the product of the conditional transition probabilities from time $t - 1$ to $t$. Hence, the log-likelihood can be written as

$$L_T \equiv \frac{1}{T} \sum_{t=2}^T l_t(\Omega_P). \quad (F.9)$$

The orthogonality assumption allows to decompose the joint conditional density (F.9) into two parts,

$$l_t(\Omega_P) = \log(f(Y_t|Y_{t-1}; \Omega_P)) + \log(f(\epsilon_t|\epsilon_{t-1}; \Omega_P)). \quad (F.10)$$

The fist sum characterizes the conditional distribution of observing the yield vector

$$Y_t = \begin{pmatrix} y(t, \tau_1) \\ y(t, \tau_2) \\ y(t, \tau_3) \end{pmatrix}$$
conditional on the last observation \( Y_{t-1} \). The second sum denotes the conditional distribution of the measurement errors. Both conditional distributions depend on the state variables \( S_{t-1} \) which we assume to have joint (conditional) distribution \( f(S_t|S_{t-1}; \Omega_P) \). The conditional density \( f(Y_t|Y_{t-1}; \Omega_P) \) is obtained via a change of variable

\[
f(Y_t|Y_{t-1}; \Omega_P) \equiv f(S_t|S_{t-1}; \Omega_P) \frac{1}{|\det J(\tau)|},
\]

where \( J(\tau) \) is the Jacobian. In the affine yield model the Jacobian is given by \( J(\tau) \equiv B(\tau) \).

The conditional distribution of measurement errors (F.10) is assumed to be multivariate normal with an orthogonal covariance matrix of dimension \( 6 \times 6 \).
Table 1: Goodness of Fit

This table presents an in-sample goodness-of-fit summary for the estimated structural model (QML). The fitting errors are defined as model-generated bond yields minus observed bond yields for the period January 1970 to December 2003.

Average cross-sectional fitting errors

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean Absolute Error</th>
<th>Median Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>56.96 bp</td>
<td>54.88 bp</td>
</tr>
<tr>
<td>6 months</td>
<td>24.00 bp</td>
<td>22.03 bp</td>
</tr>
<tr>
<td>1 year</td>
<td>25.23 bp</td>
<td>23.16 bp</td>
</tr>
<tr>
<td>2 years</td>
<td>17.42 bp</td>
<td>17.26 bp</td>
</tr>
<tr>
<td>5 years</td>
<td>3.09 bp</td>
<td>2.99 bp</td>
</tr>
<tr>
<td>7 years</td>
<td>16.89 bp</td>
<td>7.82 bp</td>
</tr>
</tbody>
</table>

In-sample forecasting performance

<table>
<thead>
<tr>
<th>Bond maturity</th>
<th>Forecast Horizon (months)</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 months</td>
<td>3</td>
<td>0.9296</td>
</tr>
<tr>
<td>2 years</td>
<td>3</td>
<td>0.7808</td>
</tr>
<tr>
<td>10 years</td>
<td>3</td>
<td>0.6488</td>
</tr>
<tr>
<td>6 months</td>
<td>6</td>
<td>1.3811</td>
</tr>
<tr>
<td>2 years</td>
<td>6</td>
<td>1.1355</td>
</tr>
<tr>
<td>10 years</td>
<td>6</td>
<td>1.0187</td>
</tr>
<tr>
<td>6 months</td>
<td>12</td>
<td>1.8970</td>
</tr>
<tr>
<td>2 years</td>
<td>12</td>
<td>1.5817</td>
</tr>
<tr>
<td>10 years</td>
<td>12</td>
<td>1.4578</td>
</tr>
</tbody>
</table>
Table 2: Parameter Estimates of Bond Model based on Quasi Maximum Likelihood Estimation

This table presents the Quasi Maximum Likelihood parameter estimates of the structural model and their standard errors. The estimation is done with smoothed Fama-Bliss yields (1, 3, 6 month and 1, 2, 3, 5, 7, 10 years). The sample period is January 1970 to December 2003.

<table>
<thead>
<tr>
<th>QML Estimates and (Standard Errors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ</td>
</tr>
<tr>
<td>θ_a</td>
</tr>
<tr>
<td>γ</td>
</tr>
<tr>
<td>q_{a1}</td>
</tr>
<tr>
<td>k</td>
</tr>
<tr>
<td>κω</td>
</tr>
<tr>
<td>κ_v</td>
</tr>
<tr>
<td>κ_a</td>
</tr>
<tr>
<td>μ_y</td>
</tr>
<tr>
<td>q_1</td>
</tr>
<tr>
<td>q_2</td>
</tr>
<tr>
<td>\hat{π}</td>
</tr>
<tr>
<td>ρ</td>
</tr>
<tr>
<td>ρ_Mv</td>
</tr>
<tr>
<td>σ_M</td>
</tr>
<tr>
<td>σ_{1,ω}</td>
</tr>
<tr>
<td>σ_v</td>
</tr>
<tr>
<td>σ_a</td>
</tr>
<tr>
<td>σ_y</td>
</tr>
<tr>
<td>θ_ω</td>
</tr>
<tr>
<td>ν_y</td>
</tr>
<tr>
<td>\theta_v</td>
</tr>
<tr>
<td>q_{a2}</td>
</tr>
<tr>
<td>q_{a3}</td>
</tr>
</tbody>
</table>
This figure plots the excess return (term premium) for the ten year bond over the three month bond versus the model implied expected excess return (term premium). The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
This figure presents the excess return (term premium from data) for the ten year bond over the three month bond with its model implied counterpart. The model implied counterpart contains the expected ambiguity- (blue line) and expected risk contribution (red line). The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
Figure 3: Expected Risk Contribution and Expected Ambiguity Contribution, 1970.1 - 2003.12

This figure decomposes the model implied term premium into its components. The blue line presents the expected risk contribution to the term premium whereas the green line presents the expected ambiguity contribution to the term premium. The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
Figure 4: Excess Return and Expected Risk Contribution, 1970.1 - 2003.12

This figure plots the excess return (term premium from data) for the ten year bond over the three month bond versus the model implied expected risk contribution to the term premium. The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
Figure 5: Excess Return and Expected Ambiguity Contribution, 1970.1 - 2003.12

This figure plots the excess return (term premium from data) for the ten year bond over the three month bond versus the model implied expected ambiguity contribution to the term premium. The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
Figure 6: Excess Return and Its Decomposition 1970.1 - 2003.12

This figure presents the excess return (term premium from data) for the ten year bond over the three month bond with its model implied counterparts. The model implied counterparts contains the expected excess return (blue line) and its expected ambiguity- (red line) and risk contribution (green line). The model is estimated via QML and uses continuously compounded smoothed Fama-Bliss yields from 1970.1 until 2003.12.
References


