Vasicek’s Model of Distribution of Losses in a Large, Homogeneous Portfolio

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Vasicek’s Model

• Important method for calculating distribution of loan losses:
  ✓ widely used in banking
  ✓ used in Basel II regulations to set bank capital requirements

• Motivation linked to distance-to-default analysis

• But, model of dependence is Gaussian Copula again

• Key assumptions (apart from Gaussian dependence)
  ✓ homogeneous portfolio (equal investment in each credit)
  ✓ very large number of credits
Motivation: Merton’s Model

- In Merton model value of risky debt depends on firm value and default risk is correlated because firm values are correlated (e.g., via common dependence on market factor).
- Value of firm $i$ at time $T$:

$$V_{T,i} = V_i \exp \left( (\mu_i - (1/2)\sigma_{V,i}^2)T + \frac{\sigma_{V,i}}{\sqrt{T}} \hat{\varepsilon}_i \right)$$

where $\hat{\varepsilon}_i \sim N(0,1)$

- We will assume that correlation between firm values arises because of correlation between surprise in individual firm value ($\varepsilon_i$) and market factor ($m$)

Correlation structure: Gaussian Copula

- Suppose correlation between each firm’s value and the market factor is the same and equal to $\sqrt{\rho}$.
- This means that we may model correlation between the $\varepsilon$’s as

$$\varepsilon_i = \sqrt{\rho} m + \sqrt{1-\rho} v_i, \quad i = 1,\ldots,N$$

and

$$\text{corr}(\varepsilon_i, \varepsilon_j) = \rho$$

- Where $m$ and $v_i$ are independent $N(0,1)$ random variables and $\rho$ is common to all firms
- Notice that if $v_i \sim N(0,1)$ and $m \sim N(0,1)$ then $\varepsilon_i \sim N(0,1)$
Structural Approach, contd.

- From our analysis of *distance-to-default*, we know that under the Merton Model a firm defaults when:

  \[ \varepsilon_i \leq \left( R_{D,i} - \left( \mu_i - \frac{1}{2} \sigma_{V,i}^2 \right) T \right) / \sigma_{V,i} \sqrt{T} \text{ where } R_{D,i} = \ln(B_i / V_i) \]

- The **unconditional** (natural) probability of default, \( p \), is therefore:

  \[ p \equiv \text{Prob} \left( \varepsilon_i < \frac{R_{D,i} - \left( \mu_i - \frac{1}{2} \sigma_{V,i}^2 \right) T}{\sigma_{V,i} \sqrt{T}} \right) = N \left( \frac{R_{D,i} - \left( \mu_i - \frac{1}{2} \sigma_{V,i}^2 \right) T}{\sigma_i \sqrt{T}} \right) \]

- In this model we assume that the **default probability**, \( p \), is constant across firms

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Idea: Single Common Factor and Large Homogeneous Portfolio

- Working out the distribution of portfolio losses directly when the \( \varepsilon \)'s are correlated is not easy

- But, if we work out the **distribution conditional on the market shock**, \( m \), then we can exploit the fact that the remaining shocks are independent and work out the portfolio loss distribution
Structural Approach, contd.

- The shock to the return, $\varepsilon_i$, is related to the common and idiosyncratic shocks by:
  \[ \varepsilon_i = \sqrt{\rho m} + \sqrt{1 - \rho} v_i \]

- Default occurs when:
  \[ \varepsilon_i = \sqrt{\rho m} + \sqrt{1 - \rho} v_i < \frac{R_{D,i} - (\mu_i - \frac{1}{\tau} \sigma_{V,i}^2)}{\sigma_{V,i}\sqrt{T}} = N^{-1}(p) \]
  or
  \[ v_i < \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} \]

Vasicek and the Intensity Model

- We’ll see later that the Vasicek model is essentially the same as the intensity model when:
  - the intensity is the same for all the names; and
  - the number of names becomes large
  - equal investment in each name
  - we use the Gaussian copula
The Default Condition in Vasicek

\[ v_i < \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} \]

- A large value of \( m \) means a “good” shock to the market (high asset values)
- The larger the value of \( m \) – the market shock – the more negative the idiosyncratic shock, \( v_i \), has to be to trigger default
- The higher the correlation, \( \rho \), between the firm shocks, the larger the impact of \( m \) on the critical value of \( v_i \).

Conditional Default Probability

- Conditional on the realisation of the common shock, \( m \), the probability of default is therefore:

\[
\text{Prob}(\text{default} \mid m) = \text{Prob}\left( v_i < \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} \right) = N\left( \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} \right) = \theta(m), \text{ say}
\]

and therefore \( \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} = N^{-1}(\theta(m)) \)
The relation between $\theta(m)$ and $m$

- For a given market shock, $m$, $\theta(m)$ gives the *conditional* probability of default on an individual loan.

**Implications of Conditional Independence**

- For a given value of $m$, as the number of loans in the portfolio $\to \infty$, the *proportion* of loans in the portfolio that actually default converges to the probability $\theta(m)$ <<<< KEY IDEA

- In the chart, if the market shock is $m^*$ then the *actual* proportion of defaults in the portfolio converges to 15% as # loans $\to \infty$.
The critical value of \( m \)

- For a given actual frequency of loss, \( \theta \), we can calculate the corresponding value of the market shock, \( m(\theta) \) that will produce exactly that level of loss:

\[
\theta = N\left( \frac{N^{-1}(p) - \sqrt{\rho m(\theta)}}{\sqrt{1-\rho}} \right)
\]

\[
N^{-1}(\theta) = \frac{N^{-1}(p) - \sqrt{\rho m(\theta)}}{\sqrt{1-\rho}}
\]

\[
m(\theta) = \frac{N^{-1}(p) - \sqrt{1-\rho}N^{-1}(\theta)}{\sqrt{\rho}}
\]

The distribution of portfolio loss

- Since the proportion of portfolio losses decreases with \( m \), the probability that the proportion of loans that default (\( L \)) is less than \( \theta \) is:

\[
\text{Prob}(L < \theta) = \text{prob}(m > m(\theta)) = \text{prob}\left( m > \frac{N^{-1}(p) - \sqrt{1-\rho}N^{-1}(\theta)}{\sqrt{\rho}} \right)
\]

\[
= N\left( -\left[ \frac{N^{-1}(p) - \sqrt{1-\rho}N^{-1}(\theta)}{\sqrt{\rho}} \right] \right)
\]

\[
\text{Prob}(L < \theta) = N\left( \frac{\sqrt{1-\rho}N^{-1}(\theta) - N^{-1}(p)}{\sqrt{\rho}} \right)
\]
Loan Loss Distribution

\[
\text{Prob}(L < \theta) = N \left( \frac{\sqrt{1 - \rho} N^{-1}(\theta) - N^{-1}(p)}{\sqrt{\rho}} \right)
\]

- This result gives the cumulative distribution of the fraction of loans that default in a well diversified homogeneous portfolio where the correlation in default comes from dependence on a common factor.

- **Homogeneity** means that each loan has:
  - the same default probability, \( p \)
  - (implicitly) the same loss-given-default
  - the same correlation, \( \rho \), across different loans

- The distribution has two parameters
  - default probability, \( p \)
  - correlation, \( \rho \)

Loan Loss Distribution with \( p = 1\% \) and \( \rho = 12\% \) and 0.6%
Example of Vasicek formula Applied to Bank Portfolio

Source: Vasicek

Merton-model Approach to Distribution of Portfolio Losses

Relationship between the Vasiceck model and the intensity model with the Gaussian Copula
Fundamentally, Vasicek model gives same results Intensity model and Gaussian copula (!)

- Default condition in Vasicek model:
  \[
  \epsilon_i = \sqrt{\rho m} + \sqrt{1 - \rho^2} \epsilon_i < \frac{R_{D,i} - \left(\mu_i - \frac{1}{T} \sigma_{v,i}^2\right)}{\sigma_{v,i} \sqrt{T}} = N^{-1}(p)
  \]

- In other words, whether a normally distributed \( N(0,1) \) variable is larger or smaller than a given fixed number, \( N^{-1}(p) \)

and .. in intensity model (with the Gaussian copula) .. the same (!)

- In the intensity model default occurs when
  \[
  \tau_i = -\frac{1}{\lambda} \ln(1 - U_i) \leq \tau^* \quad \text{where} \quad U_i = N(\epsilon_i)
  \]
  i.e., when the default time \( \tau_i \) is smaller than the “maturity” \( \tau^* \)

- Define
  \[
  \tau^* = -\frac{1}{\lambda} \ln(1 - U^*) \quad \text{and} \quad U^* = N(\epsilon^*)
  \]

- Then default occurs when
  \[
  \epsilon_i \leq \epsilon^*
  \]
Or.. in pictures ..

- **If** the value of $e$ that we draw is smaller than the critical value
  $$\varepsilon_i \leq \varepsilon^*$$

- **Then** $\tau_i$ is less than $\tau^*$ and we have a default
  $$\tau_i = -\frac{1}{\lambda} \ln(1 - U_i) \leq \tau^*$$
  where $U_i = N(\varepsilon_i)$

Merton-model Approach to Distribution of Portfolio Losses

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**Example**

- Intensity model with 1000 names and equal intensity and Vasicek model with equal default probability and correlation
The bottom line ..

- The Vasicek model is the same as the intensity model with a Gaussian copula, identical default probabilities and a large number of names.

Applications

- Vasicek’s obtains a formula for the distribution of losses with:
  - single common factor
  - homogeneous portfolio
  - large number of credits

- But the approach can be generalised to a much more realistic (multi-factor) correlation structure and granularity in the portfolio holdings
  - quite widely used in banking for management of risk of loan portfolio
Takeaways

• **Vasicek’s formula** gives useful quick method for generating *distribution* of *losses* in *large portfolio*
  - ✓ in *one-factor version* fundamentally the same as *Gaussian copula*
  - ✓ exploitation of *conditional independence* is useful idea

• **Applications** tend to be in *risk management* of actual loan losses (*natural distribution*) rather than pricing (*risk-neutral distribution*)
  - ✓ *less evidence* of *poor performance* in natural distribution (same story about structural model again)