Do we need a **model** of the default process?

- **Assuming:**
  - ✓ *default* can occur only on premium *payment dates*,
  - ✓ a *CDS contract* for every payment date
  - ✓ a *riskless bond* for every payment date
  - ✓ future *riskless rates* are *known*,

then there is a *complete market* in claims that pay off depending on whether or not the underlying credit has survived.

- **Implication:** we can calculate *unique risk-neutral survival probabilities* and use these to calculate the *no-arbitrage value* of payments that depend on survival or non-survival of the underlying credit
Do we need a **model**, continued.

- **Under these assumptions**: for valuing financial instruments where the *cash flows* occur only on *premium payment dates* and depend only on whether a *given credit* has *defaulted* or *not*, there is therefore no need for any more specific assumptions about the process generating default (because we have the RN-probabilities).

- **Examples**:
  - a *CDS contract* with a non-current premium rate
  - (in principal) a *bond* issued by the underlying credit

  **Then …. no need for a model**

---

**When do we need a model?**

1. To deal with the possibility of *default between premium payment dates*. (Relatively *minor issue*)
   - in effect this is *interpolating* the RN *survival probability curve*
   - not really legitimate because if default occurs between payment dates then *market is not complete* .. but probably not too bad

2. For *basket credit derivatives*: (*Major issue*)
   - need to model the distribution of the *number of defaults* in a portfolio of different credits
   - default intensity approach, combined with *assumptions about default correlation*, provide one way of doing this.
Default Intensity Model

- Default intensity approach treats default as *stochastic hazard-rate process* with *risk-neutral default intensity* $\lambda_t$ – the risk-neutral probability of default per unit time.

- This means: if there has been no default up to time $t$, the *risk-neutral probability of default* over the next short interval $\Delta t$ is $\lambda_t \Delta t$.

- With a *constant* value of $\lambda$, default is the first jump of a *Poisson* process with intensity $\lambda$.
  
  ✓ risk neutral probability of *survival* up to $t$, $Q(t) = e^{-\lambda t}$
  
  ✓ and so .. the risk neutral probability of *default* up to time $t$ is $1 - Q(t) = 1 - e^{-\lambda t}$

**Effect of Default Intensity on Risk Neutral Survival Probability – $Q(t)$**

![Graph showing the effect of default intensity on risk neutral survival probability $Q(t)$ with two lines for $\lambda = 1\%$ and $\lambda = 5\%$.](chart.png)
Example 1: Zero Coupon Bond with Zero Recovery and Constant Default Intensity:

- Suppose riskless interest rate is constant and equal to \( r \)
- If risk-neutral intensity \( \lambda \) is also constant then we can write the price of a defaultable zero-coupon bond promising £1 at maturity \( T \) and with zero recovery as:

\[
B = e^{-rT} \left( \text{RNP} \left( \text{Survival} \right) \times 1 + \text{RNP} \left( \text{Default} \right) \times R \right) \\
= e^{-rT} \left( e^{-\lambda T} \times 1 + \left[ 1 - e^{-\lambda T} \right] \times 0 \right) = e^{-(r+\lambda)T} \times 1 = D(T)Q(T)
\]

in other words we just discount the promised payment at a rate that is equal to the riskless rate plus the RN default intensity

- In this case, the yield on the bond is \( r + \lambda \) and so the spread is just equal to \( \lambda \), the RN default intensity

Pause …..

- This is quite a neat trick .. it says that if we know the risk neutral default intensity we can value credit risky bonds just as we value default-free bonds simply by adjusting the discount rate (from \( r \) to \( r + \lambda \))

- Three problems:
  - this “neat” result depends on the form of recovery (discussed below)
  - from where do we get \( \lambda \)? (Answer: from CDS, see below)
  - the risk neutral default intensity may vary over time and depend on other economic variables
Example 2: Zero Coupon Bond with Non-Zero Recovery – Not Quite so Neat

• Suppose recovery is \( R \) (fraction of face value), then bond price is:

\[
B = e^{-rT} \left( \text{RNP}(\text{Survival}) \times 1 + \text{RNP}(\text{Default}) \times R \right)
\]

\[
= e^{-rT} \left( e^{-\lambda T} \times 1 + \left[ 1 - e^{-\lambda T} \right] \times R \right)
\]

\[
= e^{-rT} R + e^{-(r+\lambda)T} (1 - R)
\]

which is not so neat (no simple formula for yield)

• But … there is an alternative assumption – with recovery – that restores the “neat” result

Example 3: “Market Value” Recovery – Neat Again

• Suppose recovery is a constant fraction \( R \) of the amount the bond was worth the instant before it defaulted

• There is no evidence that this is what happens in practice (the constant recovery of par assumption is probably best – recall Enron) – but:

  ✓ this assumption produces prices that are quite close to those with constant recovery of par

  ✓ it makes the calculations much easier

• In this case we can show that the bond price is:

\[
B = e^{-(r+(1-R)\lambda)T} \times 1
\]

i.e., the same as the zero recovery value but with an adjusted intensity equal to the risk-neutral expected loss rate \((1-R)\lambda\).
Calculating $\lambda$ from the CDS Spread

- With *market value recovery*, if the interest rate and the intensity are constant, then the bond spread (and therefore the CDS spread) will be equal to the risk neutral expected loss rate.
- In this case, if $\lambda$ is constant over time, it is easy to back out the default intensity from the CDS spread:

\[
\text{CDS Spread} = (1 - R)\lambda
\]

\[\Rightarrow \lambda = \frac{\text{CDS Spread}}{(1 - R)}\]

**Non-Constant RN Default Intensity**

- The term structure of *RN survival probabilities* (the $Q$’s from CDS) may be inconsistent with a constant RN default intensity.
- In this case we can calculate the RN default intensity for the period $(t-1)$ to $t$, $\lambda_t$, as:

\[
\frac{Q_t}{Q_{t-1}} = e^{-\lambda_t \Delta}
\]

where $\Delta$ is the interval of time between $(t-1)$ and $t$.
- Here $\lambda_t$ is the *conditional RN default intensity* – conditional on no default having occurred before time $(t-1)$.

- Actual *credit spreads* (especially for investment grade) are much larger than *natural expected loss rates*

- For given recovery rate this means *risk-neutral default intensity is much larger than natural default intensity* (implies significant risk premia)

Source: Berndt, Douglas, Duffie, Ferguson and Schranz, "Measuring Default Risk Premia from Default Swap Rates and EDFs", 2004

Modelling Correlated Defaults using the Default Intensity Model combined with the Copula Approach
Modelling Correlation:

- **Default** is a binomial event: it happens or it doesn’t
- But **difficult** to include default **correlation** directly into standard binomial framework

- Two common approaches:
  - ✓ **copula approach**: widely used in pricing – but needs caution
  - ✓ **structural approach**: ...used in risk management (will discuss later in course)

Caution over Copulas

- A “copula” allows us to **separate** the **dependence structure** between two or more random variables from their **unconditional** (or marginal) **distributions**.
- Sounds very powerful **BUT** problem is that often very **little guidance** available in how to choose copula
- **Gaussian copula** (method described here) **widely used** in practice but quite possibly a **poor description** of reality.
Simulating Default Times

- The starting point is the intensity model with constant intensity $\lambda$.
- Under this model the risk neutral probability of default ($\tau$ is the time of default) up to time $t$ is:

$$p(\tau \leq t) = 1 - \exp(-\lambda t)$$

- As with any cumulative distribution, if we were to make a random drawing from the risk neutral distribution of default times, $\tau$, the cumulative RN probability $p(\tau \leq t)$ would be equally likely to be anywhere within the range zero to one (see further intuition below).
- We can reverse this process and, by drawing a random variable that is equally likely to come from anywhere between zero and one, we can look up a corresponding drawing from the risk neutral distribution of default times.

The Copula Approach to Valuing Correlation Products 17

Simulating default time in Intensity Model

![Diagram of Simulating default time in Intensity Model]

The Copula Approach to Valuing Correlation Products 18
Inverse Cumulative Method for Random Numbers: Intuition

*See diagram on next but one slide*

- We wish to make a random drawing from the risk neutral distribution of default times.
- The diagram shows both the probability density and the cumulative distribution.
- The total area under the density is one: suppose we divide up this area into 10 equal regions (marked by the vertical dotted lines).
- A default time drawn at random would be equally likely to fall into any of these 10 intervals.
- We now use the following rule:
  - randomly draw a number between 1 and 10
  - use this to choose one of the 10 intervals
  - our random number is the value of the default time in the middle (say) of the interval.

*continued next slide*

Inverse Cumulative Method: Contd.

- All that is required to implement this method is to know where the boundaries of the intervals lie.
- With 10 intervals each interval accounts for 10% of the probability and so the cumulative probability at the first boundary is 10%, at the second it is 20% and so forth.
- *We can simply look up these values on the cumulative distribution:* notice that the default time boundaries for the probability density (horizontal axis) correspond to the 10%, 20% etc. points on the cumulative distribution.
- We could therefore implement the method as follows:
  - Choose a number \((k)\) from one to 10
  - Look up the value of the default times that corresponds to cumulative probabilities of \((k-1)*10\%\) and \(k*10\%\) (the left and right hand boundaries for the \(k^{th}\) interval) and choose the number in the middle.
- The actual method we use (choosing \(U\) from a uniform distribution on \([0,1]\)) is equivalent to doing this with an infinite number of intervals.
Simulating Default Times

- In summary, therefore, to simulate a default time $\tau$ in the intensity model we:
  1. choose a random number, $U$, so that it is equally likely to be anywhere in the range $[0,1]$ – i.e., from a uniform distribution on $[0,1]$.
  2. Solve for $\tau$ as follows:

\[ U = 1 - \exp(-\lambda \tau) \rightarrow \tau = -\frac{1}{\lambda} \ln(1 - U) \]

and the value of $\tau$ we obtain is a random drawing from the risk neutral distribution of default times.
The Gaussian Copula Method for Default-Time Correlation and FTD Valuation

- To simulate **correlated default times** for FTD and CDO valuation an approach known as the *Gaussian Copula Method* is often used
- Correlation is modelled either through dependence on a **single common factor** or (sometimes) from a general correlation matrix
- Using the single common factor approach: if the correlation between each pair of names is \( \rho \) then for \( N \) names we calculate \( N \) correlated random variables \( \varepsilon_1, \ldots, \varepsilon_N \) as:

\[
\varepsilon_i = \sqrt{\rho} m + (\sqrt{1-\rho}) v_i, \quad i = 1, \ldots, N, \quad m \sim N(0,1) \text{ and } v_i \sim N(0,1)
\]

Note: For any pair of names, \( i \) and \( j \), \( m, v_i \) and \( v_j \) are independent and therefore \( \text{corr}(\varepsilon_i, \varepsilon_j) = \text{cov}(\varepsilon_i, \varepsilon_j) = \rho \) (since \( \sigma(\varepsilon_i) = \sigma(\varepsilon_j) = 1 \))

---

**Generating Correlated Default Times**

- For each trial in the simulation:
  - generate \( N \) correlated values of \( \varepsilon \) (as on previous slide) – one for each name/credit
  - for each of the \( \varepsilon \)'s, calculate the corresponding default time as:

\[
\tau_i = -\frac{1}{\lambda} \ln(1 - U_i) \quad \text{where} \quad U_i = N(\varepsilon_i)
\]

and \( N(.) \) represents the cumulative normal distribution
- For an FTD, calculate the **minimum** time-to-default and, if this is less than the contract maturity record a **default**
Generating Correlated Default Times

- Simulate correlated $U_1$, $U_2$ etc. And use these to generate correlated default times

Valuing an FTD – The Basic Idea

- Using simulation
  1. value loss leg up to time of default or end of contract, which ever comes first
  2. value premium leg for 1 b.p. – again, up to time of default or end of contract, which ever comes first
  3. find premium that equates value of loss and premium legs
Valuing an FTD

- Value of the **loss leg** of the FTD
  - risk neutral expected discounted value of the loss leg
- Value of the **premium leg** (for a 1 b.p. fee, for example)
  - risk neutral expected discounted value of the 1 b.p. fee stream to default or maturity, whichever is shorter
- *Dividing* the value of the loss leg by the value of a 1 b.p. per year premium leg, gives the FTD premium.

Takeaways

- What the **default intensity approach** is.
- Why we *don’t need it for analysis of CDS in simple case* (complete market)
- Why *we do need it* (or something) to deal with case where
  - *default* occurs *between* CDS premium *payment dates* (*interpolation* scheme)
  - we are dealing with *basket derivatives* and need to model the distribution of the *number of defaults*.
- Copula approach