Structural Models II:
The distance-to-default and estimating default probabilities

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Credit Risk Elective
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Outline of Session

• Using the structural approach to estimate the probability of default

• The relation between risk-neutral probabilities and the credit spread

• The relation between the natural and risk-neutral probability of default.
Using the Structural Approach to Estimate the Probability of Default

The Distance-to-Default

- In the Merton model default occurs when the value of the assets at the maturity of the bond \( V_T \) is below the face value of the debt \( B \).

- The rate of return on the assets over the life of the bond is \( \ln\left(\frac{V_T}{V}\right) \) (continuously compounded) – where \( V \) is the current value of the assets – so default occurs when the rate of return on the assets is \( \ln\left(\frac{B}{V}\right) \) or worse.

- This occurs when there is a sufficiently large (and almost always negative) “surprise” in the rate of return on the assets.
  - “Surprise” means the difference between the return that actually occurs and the expected value of the return.

- Question: how do we use this default threshold in the rate of return \( \ln\left(\frac{B}{V}\right) \) to estimate the probability of default?
Distance-to-Default

\[ \text{Distance-to-default} = \left( \frac{\ln(B/V) - (\mu - \frac{1}{2} \sigma_v^2)T}{\sigma_v \sqrt{T}} \right) \]

The Distance-to-Default

- In the Merton (Black-Scholes) setup the \textit{expected value} of the continuously compounded return on the assets up to maturity is \((\mu - \frac{\sigma^2}{2})T\), where \(\mu\) is the expected rate of return on the assets, \(\sigma_v\) is the annualised volatility of the return on the assets and \(T\) is the time-to-maturity.
- So the \textit{surprise in the rate of return} that will lead to default is:

\[ "\text{Surprise}" = \text{Actual Return} - \text{Expected Return} = \ln(B/V) - (\mu - \frac{1}{2} \sigma_v^2)T \]
The Distance-to-Default.

• In the Merton (Black-Scholes) setup the continuously compounded return has a normal distribution and so the probability of default depends on how many standard deviations the “surprise” in the return represents.

• The standard deviation of the rate of return on the assets is $\sigma \sqrt{T}$ and so the number of standard deviations represented by the default threshold surprise in the rate of return is:

$$ \frac{\ln(B/V) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}} $$

• This will usually be a negative number

The Distance-to-Default and the Probability of Default.

• The distance-to-default (D2D) is the negative of the ratio on the previous slide, i.e., the ratio of (minus) the surprise in the rate of return necessary to trigger default and the standard deviation of this number.

$$ \text{Distance-to-Default (D2D)} = \frac{\ln(V/B) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}} $$

• Since the surprise in the return divided by its standard deviation has (i) a normal distribution, (ii) a zero mean and (iii) a standard deviation of one, the probability that the surprise is worse than the default threshold level – i.e., the probability of default – is simply:

$$ \text{Default Prob.} = N(-D2D) $$

where $N(.)$ is the cumulative normal distribution
Example

- Continuing the example used earlier:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Expected return on assets ($\mu$)</th>
<th>12%</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of the assets ($V$)</td>
<td>€9 M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Face value of debt ($B$)</td>
<td>€5 M</td>
<td>Vol. of asset returns ($\sigma$)</td>
<td>30%</td>
</tr>
<tr>
<td>Time to maturity ($T$)</td>
<td>5 years</td>
<td>Riskless rate ($r$)</td>
<td>5%</td>
</tr>
</tbody>
</table>

Over the 5 years to maturity

\[
\text{Cont. comp. return on asset to trigger default} = \ln \left( \frac{B}{V} \right) = \ln \left( \frac{5}{9} \right) = -58.8\% \quad (A)
\]

\[
\text{Expected value of cont. comp. return} = (\mu - \frac{1}{2} \sigma^2)T = \left(12\% - \frac{1}{2} \times 0.30^2\right) \times 5 = 37.5\% \quad (B)
\]

Threshold in surprise necessary to produce default = $A - B$

\[
A - B = -58.8\% - 37.5\% = -96.3\%
\]

Example, contd.

<table>
<thead>
<tr>
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<td>5 years</td>
<td>Riskless rate ($r$)</td>
<td>5%</td>
</tr>
</tbody>
</table>

Over 5 years the volatility of the return is $= 30\% \times \sqrt{5} = 67.1\%$

No. of standard deviations represented by surprise is $= \frac{-96.3\%}{67.1\%} = -1.44$

so default probability $= N(-1.44) = 7.56\%$
Distance to Default and Default Probabilities

- In general, distance to default is smaller (and default probability higher) when volatility is higher and maturity is longer.

### Distance-to-Default*

<table>
<thead>
<tr>
<th>Vol</th>
<th>T</th>
<th>150</th>
<th>100</th>
<th>80</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>1</td>
<td>5.89</td>
<td>3.87</td>
<td>2.75</td>
<td>1.31</td>
</tr>
<tr>
<td>20%</td>
<td>20</td>
<td>3.02</td>
<td>2.56</td>
<td>2.31</td>
<td>1.99</td>
</tr>
<tr>
<td>40%</td>
<td>1</td>
<td>2.80</td>
<td>1.78</td>
<td>1.23</td>
<td>0.51</td>
</tr>
<tr>
<td>40%</td>
<td>20</td>
<td>0.84</td>
<td>0.61</td>
<td>0.49</td>
<td>0.33</td>
</tr>
</tbody>
</table>

### Default Probabilities*

<table>
<thead>
<tr>
<th>Vol</th>
<th>T</th>
<th>150</th>
<th>100</th>
<th>80</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>1</td>
<td>0.00%</td>
<td>0.01%</td>
<td>0.30%</td>
<td>9.48%</td>
</tr>
<tr>
<td>20%</td>
<td>20</td>
<td>0.13%</td>
<td>0.52%</td>
<td>1.03%</td>
<td>2.31%</td>
</tr>
<tr>
<td>40%</td>
<td>1</td>
<td>0.26%</td>
<td>3.73%</td>
<td>11.03%</td>
<td>30.65%</td>
</tr>
<tr>
<td>40%</td>
<td>20</td>
<td>20.11%</td>
<td>27.06%</td>
<td>31.34%</td>
<td>37.24%</td>
</tr>
</tbody>
</table>

*Note: Assumptions - expected return on assets = 10%; face value of debt = 50

The MKMV Approach
MKMV: using option related methods to estimate the probability of default

Default Probability = \( N \left( \frac{\ln(B/V) - (\mu - \frac{1}{2} \sigma^2 V)T}{\sigma V \sqrt{T}} \right) = N(-\text{distance-to-default}) \)

- Black-Scholes-Merton model predicts that the probability of default will depend on just the “distance to default” which depends on:
  1. **Leverage**: the ratio of the amount of debt to the current (market) value of the firm’s assets (\( B/V \))
  2. **volatility** of the firm’s assets over the period up to the maturity of the debt (\( \sigma \sqrt{T} \))
  3. **expected rate of return** on the firm’s assets (\( \mu \))
- Moody’s – KMV estimates these three parameters and uses them to estimate the probability of default

**Moody’s KMV: Input Parameters**

- **Value of \( B \) that triggers default:**
  - MKMV estimate **default point** (\( B \)) from short and long term debt
  - **smaller weight** on long-term debt since less likely to trigger default
- **Current value of assets** (\( V \)):
  - based on **market value of equity**
  - for most (even large) firms only part of debt will be marketed so some reliance on book values
Moody’s KMV: Input Parameters, contd.

- **Asset Volatility** ($\sigma$):
  - based on time series volatility of equity returns
- Since *return on assets* is weighted average of return on equity and return on debt:
  \[ R_V = \frac{E}{V} R_E + \frac{D}{V} R_D \quad \text{where} \quad V = E + D \]
- It follows that:
  \[ \sigma_V^2 = \left( \frac{E}{V} \right)^2 \sigma_E^2 + 2 \left( \frac{E}{V} \right) \left( \frac{D}{V} \right) \sigma_{E,D} + \left( \frac{D}{V} \right)^2 \sigma_D^2 \]
- And, since debt volatility is typically much smaller than equity vol.:
  \[ \sigma_V \approx \left( \frac{E}{V} \right) \sigma_E \]

Calculating Default Probability from D2D

- MKMV *do not* compute default probability from cumulative normal distribution
  - they find that using the normal distribution underestimates the number of defaults
- Instead, they *calibrate* distance-to-default to actual (historical) default frequencies using proprietary database of defaults.
- MKMV call their estimates of default probabilities *expected default frequencies* or “EDF”
Compaq vs. Anheuser-Busch

Market Net Worth

FIGURE 2  Evolution of asset values and default points for Compaq and Anheuser-Busch

Source: Moody’s-KMV
Olympus Corporation is a Japan-based manufacturer of optics, photography and reprography products.

In early November 2011, the company admitted a cover-up over nearly two decades of investment losses and the use of several acquisitions to hide the losses. Seven of its leaders, including the former chairman and executive vice president were arrested in connection with a USD 1.7 billion accounting scandal.
Olympus – EDF

Source: Moody’s-KMV

Olympus – EDF & Bond Spread

Source: Moody’s-KMV
What do we learn from MKMV?

• Leverage and the volatility of a firm’s assets are major determinants of its risk of default

• Volatility of the market value of equity provides a useful basis for estimating asset volatility

The relation between the natural and risk-neutral probability of default in the Merton Model.
Natural vs. Risk-Neutral Default Prob.

• The calculations above (including MKMV’s) are designed to calculate the natural probability of default and that is why the expected return on the assets ($\mu$) appears in the formula rather than the riskless interest rate ($r$).

• To calculate the risk-neutral probability of default we substitute the riskless rate ($r$) for the expected return ($\mu$):

$$D2D_{RN} = \frac{\ln(V/B)+(r-\frac{1}{2}\sigma^2_V)T}{\sigma_V\sqrt{T}}$$
while

$$D2D_{Nat} = \frac{\ln(V/B)+(\mu_V-\frac{1}{2}\sigma^2_V)T}{\sigma_V\sqrt{T}}$$

• If we now use the CAPM to calculate the risk premium on the assets

$$\mu_V = r + RP_V = r + \beta_V \sigma_M \mu_M - r + \frac{\text{Cov}(R_V, R_M)}{\text{Var}(R_M)} (\mu_M - r)$$

$$= r + \rho_{V,M} \sigma_V \frac{(\mu_M - r)}{\sigma_M} = r + \rho_{V,M} \sigma_V \lambda_M,$$

(Note: $\text{Cov}(R_V, R_M) = \rho_{V,M} \sigma_V \sigma_M$)

Here $\rho_{V,M}$ is the correlation between the firm’s assets and the market, $\sigma_M$ and $\sigma_V$ are the standard deviations of returns on the market and on the firm’s asset and $\lambda_M$ is the market price of risk, i.e., $(\mu_M - r) / \sigma_M$.
Natural vs. Risk-Neutral Default Prob.

• Substituting the expression for the risk premium on the asset into formula for $D2D_{Nat}$ we find:

$$D2D_{Nat} = \frac{\ln(V/B) + (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$$

$$= \frac{\ln(V/B) + (r + \rho_{V,M}\sigma_V\lambda_M - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$$

$$= D2D_{RN} + \rho_{V,M} \lambda_M \sqrt{T}$$

Natural vs. Risk-Neutral Default Prob., contd.

• The difference between the risk-neutral and natural D2D and probability of default depends on:
  ✓ the market price of risk
  ✓ the correlation between the firm’s assets and the market (if the correlation is zero then the beta would also be zero and the expected return on the firm’s assets would be equal to the riskless rate)
  ✓ the time to maturity
• If the risk premium on the asset is positive then
  ✓ the risk-neutral expected return is lower (than natural expected return)
  ✓ The size of the shock (using the risk-neutral distribution) that leads to default is smaller; and so
  ✓ the risk-neutral probability of default is higher.
Natural vs. Risk-Neutral Default Prob., contd.

\[ p_{RN} = N \left[ N^{-1} \left( p_{Nat} \right) + \rho_{V,M} \lambda_M \sqrt{T} \right] \]

The Relation between Credit Spreads and Risk-Neutral Probabilities
Risk Neutral Probabilities and Credit Spreads

• Adapting our simple analysis from earlier in course, we can show that the credit spread is (approximately) equal to the risk-neutral expected loss rate.

• Suppose a one-period defaultable bond pays 100 if no default and \(100x (1 – L)\) in the event of default where \(L\) is the percentage loss-given-default (LGD)

Risk-neutral expected payoff \(= 100(1 – \hat{\pi}) + 100(1 – L)\hat{\pi}\)
where \(\hat{\pi}\) is the risk-neutral probability of default

The price of the bond is the RN expected payoff discounted at the riskless rate \((R)\). Equating this to the face value discounted at the yield \((y)\) we have

\[
\text{Price} = \frac{100(1 – \hat{\pi}) + 100(1 – L)\hat{\pi}}{1 + R} = \frac{100}{1 + y} \Rightarrow \text{spread} \equiv y – R \approx L\hat{\pi}
\]
i.e., spread is (approx) equal to \(L\hat{\pi}\), the “risk-neutral expected loss rate”

Risk Neutral Probabilities and Credit Spreads

• If spread is equal to risk-neutral expected loss rate, the risk-neutral probability of default is simply the credit spread divided by the loss rate (LGD):

\[
\text{spread} \equiv y – R \approx L\hat{\pi}
\]

\[
\Rightarrow \hat{\pi} = \frac{\text{spread}}{L}
\]
Takeaways

- Concept of *distance-to-default*
- Use of distance-to-default in estimating *default probabilities*
- Relation between *credit spreads* and RN *probabilities* of default.
- KMV approach
- Relation between *natural* and *risk-neutral* distance to default