Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo

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The Black-Scholes formula

\[ C = SN(d_1) - PV(X)N(d_2) \]

\[ d_1 = \frac{\ln(S/PV(X)) + \frac{1}{2} \sigma \sqrt{T}}{\sigma \sqrt{T}} \]  and  \[ d_2 = d_1 - \sigma \sqrt{T} \]

Objectives: to understand

• The Black-Scholes formula in terms of risk-neutral valuation

• How to use the risk-neutral approach to value assets using Monte Carlo (next week: the binomial method)
Valuing Options using Risk Neutral Probabilities

- In a complete market we can calculate:
  - unique risk neutral probabilities (and A-D prices)
  - the no-arbitrage price of an option as the risk-neutral expected pay-off, discounted at the riskless rate

\[
C = \frac{1}{1 + r_f} \hat{E}(\text{payoff})
\]

\[
= \frac{1}{1 + r_f} \sum_{s=1}^{S} \hat{\pi}_s \text{Max}(S_T - X, 0), \quad \text{for a call}
\]

\( \hat{\pi}_s \): RN probability of state \( s \)

- Black-Scholes formula can be obtained in exactly this way

Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo

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Definition: Lognormal Distribution

- If a random variable \( x \) has a normal distribution with mean \( \mu \) and standard deviation \( \sigma \) then:

\[ e^x \] has a lognormal distribution with mean \( e^{\mu + \frac{1}{2} \sigma^2} \)

<table>
<thead>
<tr>
<th>Returns</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean (mu)</td>
<td>10%</td>
</tr>
<tr>
<td>SD (sig)</td>
<td>40%</td>
</tr>
<tr>
<td>Current value S_0</td>
<td>100%</td>
</tr>
<tr>
<td>Time Period (years)</td>
<td>4</td>
</tr>
<tr>
<td>Mean S_0*Exp(mu + .5 * sig^2)</td>
<td>205.44</td>
</tr>
<tr>
<td>Current value * Exp(mu)</td>
<td>149.18</td>
</tr>
</tbody>
</table>

- A lognormal distribution (like the normal) has two parameters – can take these as mean \( \mu \) and standard deviation \( \sigma \) (of “x”)
Assumptions of the Black-Scholes Model

- The *continuously compounded rate of return* (CCR) on the underlying stock over a length of time $T$ has a *normal distribution* (if the expected return is constant over time):
  
  $$R_T^C = \ln \left( \frac{S_T}{S_0} \right) = \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} u$$

  - $\mu$ is the expected value of the CCR calculated from the expected stock price (the expected return over a very short period – “$dt$”)
  - $\sigma$ is the volatility of the CCR per year, so $\sigma \sqrt{T}$ is the volatility of CCR over the period of length $T$

- “$u$” is a normally distributed random variable with a mean of zero and standard deviation equal to one.

Assumptions of the Black-Scholes Model

- Since the continuously compounded rate of return (CCR) has a normal distribution the *stock price* itself ($S_T$) is lognormal:

  $$S_T = S_0 e^{R_T^C} = S_0 e^{(\mu - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} u}$$

  and the mean and variance of $R_T^C$ are:

  $$E(R_T^C) = \left( \mu - \frac{1}{2} \sigma^2 \right) T \quad \text{and} \quad \text{var}(R_T^C) = \sigma^2 T$$

- From the properties of the lognormal distribution this means that the expected future value of the stock price is:

  $$E(S_T) = S_0 e^{E(R_T^C) + \frac{1}{2} \text{var}(R_T^C)} = S_0 e^{(\mu - \frac{1}{2} \sigma^2)T + \frac{1}{2} \sigma^2 T} = S_0 e^{\mu T}$$

  - So the expected stock price just grows at the rate $\mu$ – **CORRECT!**
Why does setting the expected value of the continuously compounded return equal to the true (continuously compounded) expected return give the wrong answer?

• The reason is that the future stock price is a non-linear (and convex) function of the continuously compounded return.
• This means that the expected stock price increases with the variance of the return.
• Therefore, if we set the expected value of the continuously compounded return equal to the correct expected return (e.g., given by the CAPM – 20% say) the expected value of the stock price gives a continuously compounded return that is higher than 20% (point A versus point B is the figure)
• To make the expected price of the stock equal the point B (and therefore give an expected return equal to the correct value) we must reduce the expected value of the continuously compounded return by an amount that depends on the variance. This is what the term \(-\frac{1}{2} \sigma^2 T\) does in the formulae given on the previous slide.

The Black-Scholes Theory

• A lognormal distribution is a continuous distribution
  ✓ the “number” of possible states is infinite (in the binomial case it was just two per period)

  • In the Black-Scholes model the number of assets is just two (the underlying stock and borrowing / lending) exactly as in the binomial example

• Black and Scholes’ big, surprising, deep, Nobel-prize-winning result is that, under their assumptions, the market is complete
Risk Neutral Probabilities and A-D Prices in Black-Scholes

Risk Neutral Distribution in the Black-Scholes Model

- In Black-Scholes, *risk neutral distribution*:
  - is also lognormal
  - has same *volatility* parameter ($\sigma$)
  - BUT *mean* parameter ($\mu$) is changed so that the expected return on the stock is the riskless *interest rate* (exactly as in the binomial case)
  - $v$ (like $u$) is also a normally distributed random variable

- If underlying asset has *positive risk premium* this means that the risk-neutral distribution is *shifted to the left*

$$R_T^{C(RN)} = (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}v$$
A-D prices in Black-Scholes

- In the binomial case the A-D price is just the corresponding risk-neutral probability discounted at the riskless rate:

\[
q_s = \frac{1}{(1 + r_f)} \hat{\pi}_s
\]

\(\hat{\pi}_s\): RN probability of state \(s\)  \(q_s\): A-D price for state \(s\)

- In B-S, because the distribution of the asset price is continuous, we have a “distribution” of A-D prices
- To calculate the distribution of A-D prices in the B-S case we just “discount” the risk-neutral distribution at the riskless interest rate (as in the binomial case).

A-D Prices in Black-Scholes

- “density” means: price of A-D security that pays £1 if stock price is between 122 and 124 (say) is equal to area under curve between 122 and 124

\(\text{A-D Price} = \text{RND} / (1 + r_f)\)
The Black-Scholes “theory” vs. the Black-Scholes “formula”

• The *Black-Scholes theory* – their key result – is that (under their assumptions) the market is *complete* and that we can calculate the *risk-neutral distribution* of the underlying asset.

• The *Black-Scholes formula* is the result we get when we apply the theory to the particular problem of valuing European puts and calls. This is much narrower.

• There are many, many cases when we can apply the theory without being able to use the formula.

Calculating Black-Scholes price as risk neutral expected payoff discounted at riskless rate
Calculating Option Values in the Black-Scholes Model using Risk Neutral Probabilities

- To value an option (or any asset) in **EITHER** the binomial model **OR** the Black-Scholes model we:
  - calculate the expected cash flow using the risk-neutral probabilities
  - discount at the riskless rate of interest
- E.g., for a European call option with exercise price $X = 120$ we calculate the expected value of the cash flow at maturity using the R-N distribution and discount at the riskless rate.

![Risk-Neutral Distribution of Stock Price](image)

**Call Option Payoff $X = 120$**

**Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo**

Calculating the Black-Scholes value of a call

- The payoff at maturity is zero for $S < X$ and $(S-X)$ for $S \geq X$
- Using the RN distribution the discounted expected payoff is*: 

$$C = e^{-rT} \int_{0}^{\infty} Max(S_T - X, 0) \hat{f}(S_T) dS_T$$

$$= 0 + e^{-rT} \int_{X}^{\infty} (S_T - X) \hat{f}(S_T) dS_T$$

$$= \left[ e^{-rT} \int_{X}^{\infty} S_T \hat{f}(S_T) dS_T \right] - e^{-rT} X \left[ \int_{X}^{\infty} \hat{f}(S_T) dS_T \right]$$

$$= \left[ e^{-rT} \int_{X}^{\infty} S_T \hat{f}(S_T) dS_T \right] - PV(X) \text{prob}_{RN}(S \geq X)$$

Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo
Calculating the Black-Scholes value of a call

- The risk-neutral expected payoff on the call discounted at the riskless rate, i.e., the call price is therefore:

\[
C = \left[ e^{-rT} \int_{X}^{\infty} S_T \hat{f}(S_T) dS_T \right] - PV(X) \text{prob}_{RN}(S \geq X)
\]

- Evaluating this expression using the lognormal risk-neutral distribution for the Black-Scholes model we obtain the Black-Scholes formula

\[
C = SN(d_1) - PV(X)N(d_2)
\]

\[d_1 = \frac{\ln(S / PV(X))}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T}\]

Calculating Black-Scholes Value by Adding up payoff x RN probability

- Working out RN probs. and average payoff on call for stock price intervals of 0.1 and then calculating RN expected payoff as sum of av. Payoff x prob. we find call value of 1.77799 vs. 1.77792 from Black-Scholes formula:

<table>
<thead>
<tr>
<th>j</th>
<th>S_j</th>
<th>u_j</th>
<th>RN_prob</th>
<th>RN Prob</th>
<th>Payoff</th>
<th>Av_Payoff</th>
<th>RNP * Av Payoff</th>
</tr>
</thead>
<tbody>
<tr>
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<td>50.00</td>
<td>0.127027</td>
<td>0.550541</td>
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<td>0.000</td>
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</tr>
<tr>
<td>2</td>
<td>50.10</td>
<td>0.147007</td>
<td>0.558437</td>
<td>0.00789628</td>
<td>0.100</td>
<td>0.050</td>
<td>0.000395</td>
</tr>
<tr>
<td>3</td>
<td>50.20</td>
<td>0.166947</td>
<td>0.566294</td>
<td>0.00785744</td>
<td>0.200</td>
<td>0.150</td>
<td>0.001179</td>
</tr>
<tr>
<td>4</td>
<td>50.30</td>
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<td>0.574110</td>
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<td>0.300</td>
<td>0.250</td>
<td>0.001954</td>
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<tr>
<td>5</td>
<td>50.40</td>
<td>0.206709</td>
<td>0.581881</td>
<td>0.00777133</td>
<td>0.400</td>
<td>0.350</td>
<td>0.002720</td>
</tr>
<tr>
<td>6</td>
<td>50.50</td>
<td>0.226530</td>
<td>0.589606</td>
<td>0.00772419</td>
<td>0.500</td>
<td>0.450</td>
<td>0.003476</td>
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<tr>
<td>7</td>
<td>50.60</td>
<td>0.246313</td>
<td>0.597280</td>
<td>0.00767441</td>
<td>0.600</td>
<td>0.550</td>
<td>0.004221</td>
</tr>
<tr>
<td>8</td>
<td>50.70</td>
<td>0.266056</td>
<td>0.604902</td>
<td>0.00762207</td>
<td>0.700</td>
<td>0.650</td>
<td>0.004954</td>
</tr>
<tr>
<td>9</td>
<td>50.80</td>
<td>0.285761</td>
<td>0.612469</td>
<td>0.00756724</td>
<td>0.800</td>
<td>0.750</td>
<td>0.005675</td>
</tr>
<tr>
<td>10</td>
<td>50.90</td>
<td>0.305426</td>
<td>0.619979</td>
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<td>0.900</td>
<td>0.850</td>
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<tr>
<td>11</td>
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<td>0.00745042</td>
<td>1.000</td>
<td>0.950</td>
<td>0.007078</td>
</tr>
</tbody>
</table>
Interpreting the Black-Scholes Formula

\[ C = \frac{SN(d_1)}{RN} - \frac{PV(X) \times N(d_2)}{RN} \]

Interpreting the Merton Formula for the Value of Credit Risky Debt

- In the same way, the Merton formula for the value of credit risky debt can be interpreted as the sum of:
  - the value of the payment \((V)\) in default
  - The value of the payment of the face value \((B)\) in no-default

\[ D = \frac{VN(-d_1)}{value \ in \ default} + \frac{PV(B)}{Riskless \ PV} \times \frac{N(d_2)}{Risk-neutral \ prob \ of \ no-default} \]
We need numerical methods when we cannot find a formula for the option value

- In some cases we can find a \textit{formula} for the value of an option (e.g., the Black-Scholes formula)

- \textbf{BUT} often, though we continue to use the Black-Scholes theory (and the Black-Scholes risk-neutral distribution) there is no formula for the option price

  ✓ \textbf{Example}: true for almost all American options (except in cases – such as call on non-dividend paying stock – where American and European options are worth the same)
Monte-Carlo

- Standard technique to calculate the expected value of some function $f(x)$ of a random variable $x$:

- How it works:
  1. Generate random numbers drawn from the distribution of the random variable $x$ ("drawings")
  2. For each drawing ($x_i$) calculate $f(x_i)$
  3. Then simply take the average value of the $f(x_i)$’s

- Wide variety of important problems (pricing, risk assessment etc.) can be solved using Monte Carlo.

Option Valuation with Monte-Carlo

- implementing Monte Carlo option pricing
  1. make random drawings from the risk-neutral distribution of the stock price at the maturity of the option
  2. for each stock price calculate the payoff on the option
  3. average these payoffs (this gives the risk neutral expected payoff on the option)
  4. discount the risk neutral expected payoff at the riskless rate.

Key step: will explain how we do this
How to calculate a random drawing of the stock price under the risk-neutral distribution

**Step 1:** For $j^{th}$ trial generate drawing from normally distribution with mean of zero and standard deviation of one ($v_j$).

**Step 2:** Calculate drawing from risk-neutral distribution of stock price as:

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma \sqrt{T} v_j}$$

- Excel or @Risk will give you the random variables (the $v$’s) and then, for each $S_j$ we simply work out the payoff on the option, average the payoffs and discount at the riskless rate.

### Monte-Carlo Example – first 4 samples: 3-month call

**Ex. Price = 100, $S_0 = 100, \sigma = 30\%, r_f = 10\%**

<table>
<thead>
<tr>
<th></th>
<th>Norm Dist (0,1) random variable $v_j$</th>
<th>Stock Price at Maturity $S_{j,T}$</th>
<th>Option Payoff Max($S_{j,T} - 100$, 0)</th>
<th>Cumulative average Option Payoff</th>
<th>Cumulative average discounted at $r_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7729</td>
<td>113.8468</td>
<td>13.8468</td>
<td>13.8468</td>
<td>13.5049</td>
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<tr>
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<td>98.2863</td>
<td>0.0000</td>
<td>6.9234</td>
<td>6.7524</td>
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<tr>
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<td>0.0000</td>
<td>4.6156</td>
<td>4.5016</td>
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<tr>
<td>4</td>
<td>2.1393</td>
<td>139.7447</td>
<td>39.7447</td>
<td>13.3979</td>
<td>13.0671</td>
</tr>
</tbody>
</table>

* Note: $\bar{S}_{j,T} = S_0 e^{\left(r_f - \frac{1}{2}\sigma^2\right)T + \sigma \sqrt{T} v_j} *$

Spreadsheet available on Portal
Approximating the Risk-Neutral Distribution with Monte-Carlo

Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo

Calculating the Option Price with Monte-Carlo

Risk Neutral Valuation, the Black-Scholes Model and Monte Carlo
Implementing Monte Carlo with @Risk

- The @Risk software package allows you to carry out Monte-Carlo in Excel even more simply.
- You will find @Risk essential in the exercises on basket credit derivatives later in the course and so it is worthwhile finding out now how to use it.

Summary

- In Black-Scholes theory market for options (and effectively all claims on underlying asset) is complete
- This means we can calculate unique A-D prices and risk neutral probabilities
- We can “read” the Black-Scholes formula as:
  - RN expected payoff on option discounted at riskless rate
- OR
  - Cost of replicating portfolio
- In many cases (e.g., most American options) there is no formula for the option price and we need to use a numerical approach
  - idea: calculate the RN expected payoff and discount at the riskless rate
Key Concepts

• Market completeness in B-S
• Lognormal distribution
• RN distribution in B-S
• A-D prices in B-S
• B-S formula as discounted RN expected payoff
• Delta (hedge ratio)
• Delta hedging strategy
• Monte Carlo valuation options