Understanding Risk-Neutral Valuation

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March, 2012

Outline

• The no-arbitrage principle
• Arrow-Debreu (A-D) securities and market completeness
• Valuing options with one period to maturity via
  ✓ replication using underlying asset and borrowing / lending
  ✓ replication using A-D securities
  ✓ risk neutral probabilities
• Valuing options with several periods to maturity
No-arbitrage pricing

Arbitrage (Definition)

- An arbitrage opportunity is one which:
  a. Requires *no invested capital*
  b. Provides a *positive profit with 100% probability*
- Or (slightly more generally)
  a. Requires no invested capital,
  b. Provides a positive profit with a positive probability and has a zero probability of a loss.
- Anyone who prefers more to less would engage in arbitrage because it represents “*something for nothing*”
- **Therefore:** in any *competitive market* there should be *no arbitrage*
“No-Arbitrage Pricing”

- Although absence of arbitrage is simply a necessary requirement for equilibrium it is in some cases sufficient to allow us to price one security in terms of another.

- Idea: assets / portfolios with the same cash flows in each state must have the same price.

- Pricing via no-arbitrage is relative pricing: we calculate the price on one security in terms of the prices of others.

Example: Valuing a Call option
The next steps …

- We will use **three approaches** to value the same call option that matures in **one period** where the price of the underlying stock follows a **binomial process**
  1. Replication with stock and bond
  2. Replication with Arrow-Debreu (A-D) securities
  3. “Risk-Neutral” valuation

- Later we see how to deal with an option that matures in **more than one period**
  - in that case we will have to **revise the replicating portfolio** over time
  - this “**dynamic replication**” (or “**dynamic hedging**”) strategy is the key feature of option pricing

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**Example – a Call Option**

- Example: **TWO states** (as before) and **THREE assets**: a bond, the underlying stock and a call option

<table>
<thead>
<tr>
<th></th>
<th>Bond CF</th>
<th>Stock CF</th>
<th>Call Option ($E=100$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1</td>
<td>120</td>
<td>20</td>
</tr>
<tr>
<td>Down</td>
<td>1</td>
<td>90</td>
<td>0</td>
</tr>
<tr>
<td>Price</td>
<td>0.75 =&gt; $r_f = 33.33%$</td>
<td>75</td>
<td>?</td>
</tr>
</tbody>
</table>
The Set-up

<table>
<thead>
<tr>
<th>Stock</th>
<th>Bond</th>
<th>Call (E = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>75</td>
<td>0.75</td>
<td>?</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ B = 0.75 = \frac{1}{1+r_f} \Rightarrow r_f = 33.33\% \]

Understanding Risk Neutral Valuation

Call Option Value: Replication using the Underlying Stock and the Bond

<table>
<thead>
<tr>
<th></th>
<th>Bond CF</th>
<th>Stock CF</th>
<th>Stock Return</th>
<th>Call Option (E=100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1</td>
<td>120</td>
<td>+60%</td>
<td>20</td>
</tr>
<tr>
<td>Down</td>
<td>1</td>
<td>90</td>
<td>+20%</td>
<td>0</td>
</tr>
<tr>
<td>Price</td>
<td>0.75</td>
<td>75</td>
<td></td>
<td>?</td>
</tr>
</tbody>
</table>

- Replicating portfolio containing \( B \) units of the bond and \( \Delta \) units of the stock:
  
  "up state" \( B + \Delta \cdot 120 = 20 \)
  
  "down state" \( B + \Delta \cdot 90 = 0 \)

- Solution: \( B = -60 \) and \( \Delta = 2/3 \)  
  cost of replicating portfolio – i.e. price of option – is:

\[ P(call) = -60 \cdot 0.75 + \frac{2}{3} \cdot 75 = 5 \]
Call Option Value: *Replication* using the Underlying Stock and the Bond

- **Note**: the portfolio that replicates a call option contains:
  - ✓ the underlying stock
  - ✓ borrowing
- the *call value* is then just the *cost of this portfolio*

- This fact will be useful when we come to look at the Black-Scholes model.

Arrow-Debreu Securities and the fundamental pricing relation in finance
Arrow-Debreu [A-D] Securities

- In the binomial case the number of different “states” – i.e.,
  economic outcomes where the payoff on the stock is different –
  is TWO.
- In general we might have $S$ states (in a finite state model) or an
  infinite number of states (as in Black-Scholes)
- An A-D security is one that pays $1$ in one particular state ($s$)
  and zero in all other states

<table>
<thead>
<tr>
<th>State</th>
<th>Stock</th>
<th>A-D 1</th>
<th>A-D 2</th>
<th>A-D 3</th>
<th>A-D 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Understanding Risk Neutral Valuation

**Question:**
When can we use no-arbitrage pricing?

- **Answer:** when it is possible to replicate the cash
  flows on one asset (that we wish to price) with
  other assets for which we know the prices

- **Question:** when is replication (always) possible?
- **Answer:** When there is a full set of “Arrow-
  Debreu” [A-D] securities – a “complete” market
Complete markets and A-D Securities

- A market in which there is one A-D security for each state is called complete.
- In a complete market it is always possible to replicate any pattern of cash flows across states with a portfolio of A-D securities.
- The replication strategy is simple: to achieve a cash flow of $X_s$ in state $s$ we simply purchase a $X_s$ UNITS (a quantity) of state-s A-D securities.

<table>
<thead>
<tr>
<th>State</th>
<th>Required CF</th>
<th>A-D 1</th>
<th>A-D 2</th>
<th>A-D 3</th>
<th>A-D 4</th>
<th>Port CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>75</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

Understanding Risk Neutral Valuation

The Cost of the Replicating Portfolio

<table>
<thead>
<tr>
<th>State</th>
<th>Required CF</th>
<th>A-D 1</th>
<th>A-D 2</th>
<th>A-D 3</th>
<th>A-D 4</th>
<th>Port CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>75</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Price</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_4$</th>
</tr>
</thead>
</table>

- The cost of the replicating portfolio is: $\text{Cost} = 100q_1 + 90q_2 + 75q_3 + 40q_4$
- Unless the cash flows $\{100, 90, 75, 40\}$ sell for this price then there is arbitrage.
“Implicit” A-D securities

• A-D securities may also be created from combinations (portfolios) of conventional securities.
  ✓ example below.

• If we are able to “create” a full set of A-D securities then the market is also complete.

No Arbitrage Pricing:
The General Case

\[ P = X_1 q_1 + X_2 q_2 + \ldots X_s q_s = \sum_{s=1}^{S} X_s q_s \]

• this no-arbitrage condition is the fundamental pricing relation in finance. From it we can obtain:
  ✓ option pricing
  ✓ the CAPM
  ✓ multi-factor models of the cost of capital

• The risk-neutral valuation method comes directly from this relation
### No Arbitrage Pricing of a Bicycle

- The no-arbitrage relation essentially says that the **cost of a package** (the complex security) must be **consistent** with the **cost of the components** (the A-D securities)
- Arbitrage = **inconsistency**

<table>
<thead>
<tr>
<th>Component (s)</th>
<th>Qty (X_s)</th>
<th>Price (q_s)</th>
<th>Qty * Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gear-change</td>
<td>1</td>
<td>q_g</td>
<td>1 * q_g</td>
</tr>
<tr>
<td>Brakes</td>
<td>2</td>
<td>q_b</td>
<td>2 * q_b</td>
</tr>
<tr>
<td>Wheels</td>
<td>2</td>
<td>q_w</td>
<td>2 * q_w</td>
</tr>
<tr>
<td>.....</td>
<td>.....</td>
<td>.....</td>
<td>.....</td>
</tr>
<tr>
<td><strong>Total Price</strong></td>
<td></td>
<td></td>
<td>$\sum_{s=1}^{S} X_s q_s$</td>
</tr>
</tbody>
</table>

### Valuing an Option via the Replicating Portfolio

- Option pricing theory almost always relies on **complete markets** and so we are able to find replicating portfolios:
  - … **not** in the sense that an A-D security exists for each **detailed state** of economy
  - .. but rather, that there is an A-D security for each **possible value** of the price of the **underlying security**
  - … and so we can replicate all derivatives with cash flows that are defined by the price of the underlying security
- Because the **market is complete** we can value options using
  - a. replication using the underlying security
  - b. or …. replication using **A-D securities**
  - c. or (as we’ll see) …. **risk neutral probabilities**

and all these approaches are **exactly equivalent**
Calculating the A-D Prices

- To *value the option* using *A-D securities* we first have to calculate the A-D prices
- We do this via *replication* – exactly as we did for the call option itself
- We then *use* the *A-D prices* to *value the option*

<table>
<thead>
<tr>
<th></th>
<th>Bond CF</th>
<th>Stock CF</th>
<th>Pure “up” security</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1</td>
<td>120</td>
<td>1</td>
</tr>
<tr>
<td>Down</td>
<td>1</td>
<td>90</td>
<td>0</td>
</tr>
<tr>
<td>Price</td>
<td>0.75</td>
<td>75</td>
<td>?</td>
</tr>
</tbody>
</table>

- **Replicating portfolio** containing $B_u$ units of the bond and $\Delta_u$ units of the stock:

  - "up state" $B_u + \Delta_u \cdot 120 = 1$
  - "down state" $B_u + \Delta_u \cdot 90 = 0$

- Solution: $B_u = -3$ and $\Delta_u = 1/30 \Rightarrow$ cost of replicating portfolio – i.e. price of pure “up” security – is:

  $$ q_{up} = -3 \cdot 0.75 + \frac{1}{30} \cdot 75 = -2.25 + 2.5 = 0.25 $$
Value of “Down” A-D Security Using Replication

<table>
<thead>
<tr>
<th></th>
<th>Bond CF</th>
<th>Stock CF</th>
<th>Pure “down” security</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Up</strong></td>
<td>1</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td><strong>Down</strong></td>
<td>1</td>
<td>90</td>
<td>1</td>
</tr>
<tr>
<td><strong>Price</strong></td>
<td>0.75</td>
<td>75</td>
<td>?</td>
</tr>
</tbody>
</table>

- **Replicating portfolio** containing \( B_d \) units of the bond and \( \Delta_d \) units of the stock:
  - "up state" \( B_d + \Delta_d \cdot 120 = 0 \)
  - "down state" \( B_d + \Delta_d \cdot 90 = 1 \)

- Solution: \( B_d = 4 \) and \( \Delta_d = -1/30 \Rightarrow \) cost of replicating portfolio – i.e. price of pure “down” security – is:

\[
q_{down} = 4 \cdot 0.75 - \frac{1}{30} \cdot 75 = 3 - 2.5 = 0.5
\]

Option Value: Replication Using A-D Securities

- from *states model*, replicating portfolio for call option contains 20 “up” securities and zero “down” securities:

<table>
<thead>
<tr>
<th>State ((s))</th>
<th>Pure Securities ((q_s))</th>
<th>Option Payoff ((X_s))</th>
<th>(q_sX_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>0.25</td>
<td>20</td>
<td>5.0</td>
</tr>
<tr>
<td>Down</td>
<td>0.50</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>Total</td>
<td>0.75</td>
<td>--</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Option Price \(= 20 \cdot 0.25 + 0 \cdot 0.50 = 5\)
Risk-Neutral Valuation

The States Model and Risk Neutral Valuation

- **Risk neutral valuation** is a very important concept in option pricing.
- In fact it is nothing more than a simple *redefinition of the variables* in our basic pricing relation using A-D security prices:

\[ p_X = \sum_{s=1}^{S} X_s q_s \]

- The *price of a bond* paying \(£1\) at maturity is just the sum of the A-D prices:

\[ B \equiv \frac{1}{1+r_f} = \sum_{s=1}^{S} q_s \]
The States Model and Risk Neutral Valuation

- We just rescale the A-D prices by their sum (i.e., by the bond price)
  \[ \hat{\pi}_s = \frac{q_s}{S} = \frac{q_s}{B} \Rightarrow q_s = B\hat{\pi}_s. \]

- Notice that the rescaled A-D prices add to one:
  \[ \sum_{s=1}^{S} \hat{\pi}_s = \sum_{s=1}^{S} \frac{q_s}{B} = \frac{1}{B} \sum_{s=1}^{S} q_s = \frac{1}{B} S = 1. \]

- Since the re-scaled A-D prices are not negative (because the A-D prices cannot be negative) and add to one we can call them “probabilities” – although REMEMBER they are just A-D prices in disguise

The States Model and Risk Neutral Valuation

- Calling the scaled A-D prices “probabilities”, the asset price can be written in terms of its “expected cash flow”:
  \[ P = \sum_{s=1}^{S} X s q_s = B \sum_{s=1}^{S} X s \hat{\pi}_s = \frac{1}{1+r_f} \sum_{s=1}^{S} X s \hat{\pi}_s = \frac{1}{1+r_f} \hat{E}[X_s] \]

  where \[ \hat{E}[X_s] \] means “expected value calculated using the re-scaled A-D prices as probabilities”

- This way of writing the pricing relation is called “risk neutral valuation” because it has the same form as the value of a risky asset in a market where investors are risk neutral: the “expected cash flow” discounted at the riskless interest rate.

- The \[ \hat{\pi}_s \] are called “risk-neutral probabilities” (RNP’s)
So what is a “risk-neutral” probability?

- An A-D security is one that pays $1 in one state (e.g., state $s$) and zero in all other states.
- The spot price of an A-D security is $q_s$.
- The risk-neutral probability for state $s$ is:
  \[ \hat{\pi}_s = \frac{q_s}{B} = q_s \times (1 + r_f) \]

- This means that the risk-neutral probability for state $s$ is simply the forward price of the state $s$ A-D security.

Risk Neutral Valuation and A-D Prices

- Valuation using risk neutral probabilities / A-D prices
  - is exactly equivalent to replication: if we can value an asset using RNP’s, we COULD always use replication
  - can be used only when we can replicate cash flows (i.e., in a complete market or if cash flows are linear dependent)
  - does NOT assume investors actually are risk neutral: the risk premium adjustments are reflected in the differences between the risk-neutral probabilities and the actual – or “natural” – probabilities
- So why is it so often used in option pricing in preference to valuation using replicating portfolios?
  - simply: because the calculations are easier.
Valuation using Risk-Neutral Probabilities

- By definition risk neutral probabilities are just A-D prices divided by bond price (0.75)

<table>
<thead>
<tr>
<th></th>
<th>State Prices</th>
<th>Risk Neutral Probs. (RNP)</th>
<th>Option Payoff (E=100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>0.25</td>
<td>0.25 / 0.75 = 0.3333</td>
<td>20</td>
</tr>
<tr>
<td>Down</td>
<td>0.50</td>
<td>0.50 / 0.75 = 0.6667</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>0.75</td>
<td>1</td>
<td>---</td>
</tr>
</tbody>
</table>

- Option value using risk neutral pricing is therefore

\[
P(\text{call}) = \frac{\hat{E}(CF)}{1 + r_f} = \left(\frac{1}{1 + 0.3333}\right) (20 \times 0.3333 + 0 \times 0.6667) = 5
\]

Pause to reflect …

- Why are we using the \textit{binomial assumption}?
  - is it \textit{realistic}? ........ No!
  - is it \textit{used in Black and Scholes}? ........ No!
  - then, \textit{why}?

- Because:
  - It is \textit{simple} … AND
  - It \textit{shares} with the \textit{Black-Scholes} model the property that, under this model, the \textit{market is complete (see later)}. 

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Default Risk and the Binomial Assumption

• In the case of default risk (and unlike the Black-Scholes setup) the outcomes actually are binomial.
• So (ignoring interest rate risk) our binomial analysis – complete markets etc. – applies directly to claims where payoffs differ only between default and non-default states (CDS, risky bonds etc.)

Three Exactly Equivalent Methods of Valuing Derivatives

Replication

• Step 1: look at the contract – what are the cash flows?
• Step 2: construct a portfolio that replicates these cash flows
• Step 3: calculate cost of this portfolio
• This is the value of the derivative

A-D Prices

• Step 1: look at the contract – what are the cash flows?
• Step 2: work out the A-D prices for the underlying asset.
• Step 3: Value the cash flows on the contract using the A-D prices.
• This is the value of the derivative

RNPs

• Step 1: look at the contract – what are the cash flows?
• Step 2: work out the RNP’s for the underlying asset.
• Step 3 Calculate the risk neutral expected value of the cash flows on the contract and discount at the riskless rate
• This is the value of the derivative
Why Risk Neutral Valuation can be easier than Replication

- Replicating stock-bond portfolio $\Delta = 2/3$ and $B = -60$
- Replicating stock-bond portfolio depends on payoffs on call
Risk-neutral probabilities (and A-D prices) independent of payoff on option

- But, for any underlying asset, the RNP’s are the same no matter what option we are trying to value
- …. and, we can calculate the RNP’s from the price and cash flows on the underlying asset itself and the bond
- So, for a given underlying asset, we calculate ONE set of RNP’s and we can then value any option on that asset.

Understanding Risk Neutral Valuation

A Simpler Way to Work out Risk Neutral Probabilities

- If we use risk neutral valuation on the underlying asset itself:

\[
P = \left( \frac{1}{1 + r_f} \right) \left( X_{\text{up}} \cdot \hat{\pi}_{\text{up}} + X_{\text{down}} \cdot \hat{\pi}_{\text{down}} \right)
\]

- This means that the RNP’s are the probabilities that make the (risk neutral) expected rate of return equal to the interest rate:

\[
\hat{E}(\text{Return}) \equiv R_{\text{up}} \cdot \hat{\pi}_{\text{up}} + R_{\text{down}} \cdot \hat{\pi}_{\text{down}} = r_f
\]

\[
\Rightarrow \hat{\pi}_{\text{up}} = \frac{r_f - R_{\text{down}}}{R_{\text{up}} - R_{\text{down}}} \quad \text{and} \quad \hat{\pi}_{\text{down}} = \frac{R_{\text{up}} - r_f}{R_{\text{up}} - R_{\text{down}}}
\]

where \( R_{\text{up}} \) and \( R_{\text{down}} \) are the rates of return on the stock in the “up” and “down” states

Understanding Risk Neutral Valuation
RNPs in the Example

- In our example:

\[ R_{up} = 0.6, \quad R_{down} = 0.2 \quad \text{and} \quad r_f = 0.333 \]

\[ \hat{\pi}_{up} = \frac{r_f - R_{down}}{R_{up} - R_{down}} = \frac{0.333 - 0.20}{0.6 - 0.2} = \frac{1}{3} \]

and

\[ \hat{\pi}_{down} = \frac{R_{up} - r_f}{R_{up} - R_{down}} = \frac{0.60 - 0.333}{0.6 - 0.2} = \frac{2}{3} \]

- **Question**: will the RNPs lie between zero and one?
- **Answer**: if not there is arbitrage
- **Why**?

**Intuition**: if the riskless rate lies between the “up” and “down returns”

- ✓ there is no arbitrage
- ✓ the RNP’s will lie between zero and one.

<table>
<thead>
<tr>
<th>No arbitrage: RNP’s between zero and one</th>
<th>Arbitrage RNP’s not between zero and one</th>
<th>Arbitrage RNP’s not between zero and one</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{up} )</td>
<td>( R_{up} )</td>
<td>( r_f )</td>
</tr>
<tr>
<td>( r_f )</td>
<td>( R_{down} )</td>
<td>( R_{down} )</td>
</tr>
<tr>
<td>( R_{down} )</td>
<td>( r_f )</td>
<td></td>
</tr>
</tbody>
</table>
Market Completeness and Trading

- Suppose that
  - the underlying stock follows a binomial process over two periods
  - The option matures at $t=2$
- There are now four states at the option maturity date but still only two assets:
  - The stock
  - The bond
- Is the market still complete?
  - If we can *trade* the stock and the bond at $t=1$ .. *YES*
  - If we *can’t trade* .. *NO*
Without trading: the market in the multi-period binomial model is **NOT** complete

- If we *cannot trade* at $t=1$ then it is clear that the market is **incomplete**
  - There are *FOUR states* at $t=2$ and only *TWO securities*
  - We cannot create A-D securities and we cannot price the option by replication

---

**Multi-period Binomial Model**

- However, *if we can trade* the stock and the bond at each stage, the *market is complete* in a multi-period binomial model

- This is a *very important insight*
If we can trade the market is complete

- To see why suppose we want to create an A-D security for state “uu” (at $t=2$)
- We can only arrive at state “uu” if we first arrive at state “u”
- But, from state “u”, the market is complete over the last period and we can create an A-D security for state “uu” using the stock and the bond .. Suppose the cost of doing this is $q(u,uu)$
- Now suppose we are at $t=0$ .. The problem is to make sure that when (and if) we arrive in state “u” at $t=1$ we can buy the portfolio that pays one in state “uu”
- In other words we need a portfolio that pays $q(u,uu)$ in state “u” and zero in state “d” (payoff from “d” is zero) …. This is just a quantity $q(u,uu)$ of the A-D security that pays 1 in state $u$ and zero in state $d$

Understanding Risk Neutral Valuation

Complete Market with Trading: Example

<table>
<thead>
<tr>
<th>A-D security</th>
<th>$t=1$</th>
<th>$t=0$</th>
<th>$t=0$ $(x \cdot 28)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial state</td>
<td>$u$</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Pays in state</td>
<td>$uu$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>Qty: bond</td>
<td>-0.8</td>
<td>-1.8</td>
<td>-0.504</td>
</tr>
<tr>
<td>Qty: stock</td>
<td>0.007143</td>
<td>0.02</td>
<td>0.0056</td>
</tr>
<tr>
<td>Cost of A-D</td>
<td>0.28</td>
<td>0.38</td>
<td>0.1064</td>
</tr>
</tbody>
</table>

- “up” A-D security at $t = 0$ costs 0.38 and pays £1 in state “u” at $t = 1$
- To construct the “uu” security we need £0.28 in state “u” at $t=1$ and zero in state “d”.
- so at $t=0$ we purchase 0.28 units of the “u” security at a cost of 0.28*0.38 = £0.1064. This is the price at $t=0$ of the “uu” A-D security
**Key Point: At t=1 we need to trade (1)**

<table>
<thead>
<tr>
<th>Old Portfolio (from t = 0)</th>
<th>New Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
</tr>
<tr>
<td>Stock</td>
<td>140</td>
</tr>
<tr>
<td>Bond</td>
<td>1</td>
</tr>
<tr>
<td>Portfolio</td>
<td>---</td>
</tr>
</tbody>
</table>

- In state “u”:
  - increase stock holding from 0.0056 to 0.007143; and
  - repay “old” borrowing of 0.504 and take out new loan of 0.72 (by selling quantity 0.80 of bond paying 1 at \( t = 2 \); price per unit = 0.9)
- I.e., need to trade both stock and bond

**Key Point: At t=1 we need to trade (2)**

<table>
<thead>
<tr>
<th>Old Portfolio (from t = 0)</th>
<th>New Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
</tr>
<tr>
<td>Stock</td>
<td>90</td>
</tr>
<tr>
<td>Bond</td>
<td>1</td>
</tr>
<tr>
<td>Portfolio</td>
<td>---</td>
</tr>
</tbody>
</table>

- In state “d”:
  - sell stock holding of 0.0056 to realise 0.504 and use proceeds to …
  - repay “old” borrowing of 0.504.
- I.e., need to trade (liquidate) both stock and bond.
Complete Market with Trading: Example, contd.

- Example shows that when we can trade a long-lived security (e.g., a stock) a market may be complete even though the number of securities is smaller than the number of states
  - notice that we had to trade the stock and the bond at time \( t = 1 \) (we increase stock holding from 0.0056 to 0.007143 and borrow more)

- This insight is key to understanding the Black-Scholes model in which:
  - there are just two assets: the stock and the bond. (apart from options)
  - there is an infinite number of states
  - but … the market is nonetheless complete

- Example on previous slide is important but replication is usually a complicated way to do the calculations

- For multi-period options generally easier to use the risk-neutral approach and work out the RNPs from the “up” and “down” rates of return on the stock and the interest rate

Using Risk-Neutral Approach to Value “uu” A-D Security

\[
\hat{r}_{up} = \frac{r_f - R_{down}}{R_{up} - R_{down}} \quad \hat{r}_{down} = \frac{R_{up} - r_f}{R_{up} - R_{down}}
\]

\[r_f = 11.111\%\]

\[\hat{E}(\text{Payoff at } uu) = \frac{1 \times 0.311 \times 0.422}{(1 + 0.1111)^2} = 0.1064 \text{ as before}\]

At \( t = 0 \) risk-neutral probability of state “uu” is 0.422*0.311 = 0.131

The value of £1 in state “uu” is therefore:

\[
E(\text{Payoff at } uu) = \frac{1 \times 0.311 \times 0.422}{(1 + 0.1111)^2} = 0.1064 \text{ as before}
\]
A last word on completeness

- Notice that nothing about completeness depends on the *probability* of different states .. only on the number of *possible* states
- We have seen that when we can trade the underlying asset we can have complete set of A-D securities for some future date even when the number of securities is much smaller than the number of states
- However, in the discrete time case (i.e., not in the Black-Scholes framework) completeness requires that over any period when we can’t trade (e.g., between $t=0$ and $t=I$) the number of securities *IS* at least as large as the number of states

Key Concepts

- A-D securities
- market completeness
- valuing options using
  - replication
  - A-D prices
  - risk neutral probabilities
- calculating risk-neutral probabilities from “up” and “down” returns on stock
- relation between trading and market completeness in multi-period binomial case
- valuing options using risk neutral probabilities in multi-period trees
- valuing American options in multi-period trees