Market risks and coherent measures

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Artzner et al. (1999) and Delbaen (2000) introduced and developed the very successful concept of coherent risk measures.

An incomplete market and one period of uncertainty \((0; T)\) are considered. A supervisor that has to decide whether the future value of a held position is acceptable or not. The values in \(T\) of positions are represented by a.s. bounded random variables on a probability space \((\mathcal{F}; \mathbb{P})\).

A measure of risk is simply a mapping \(\frac{1}{2}: L^1 \to \mathbb{R}\). A position is acceptable to the supervisor if \(\frac{1}{2}(X) \leq 0\), unacceptable otherwise. Delbaen (2000) gives the following characterization of coherent risk measure (with the Fatou property):

A measure of risk \(\frac{1}{2}: L^1 \to \mathbb{R}\) is coherent if there exists an \(L^1(\mathbb{P})\)-closed, convex set \(\mathcal{P}\) of probability measures, all of them being absolutely continuous with respect to \(\mathbb{P}\), such that

\[
\frac{1}{2}(X) = \sup_{Q \in \mathcal{P}} E_Q [X]
\]

for all \(X \in L^1\).

The dependence of the measure of risk only on the distribution of the random variable seems, at least, of practical interest (see Artzner et al. (1999) and Wang et al. (1997) for examples). In this note we prove that, under mild assumptions, a measure of risk only depending on the distribution is coherent if it coincides with the expected loss of the position.

Theorem 1 Let \((\mathcal{F}; \mathbb{P})\) be an atomless probability space and \(\frac{1}{2} \in L^1\). If \(\frac{1}{2}\) is a coherent measure of risk that only depends on the distribution of the random variable. If there exists \(A \in \mathcal{F}\) such that \(\frac{1}{2}(1_A) < 0\) and \(\frac{1}{2}(1_A) = \frac{1}{2}(1_A)\), then

\[
\frac{1}{2}(X) = \int E_{\mathbb{P}} [X]
\]
for all $X \in L^1$.

In the case of comonotone risk measures (see Delbaen (2000) and Wang et al. (1997) for details), the assumptions of the Theorem can be further weakened.

**Corollary 2** Let $(\mathcal{F};\mathbb{P})$ be an atomless probability space and $\frac{1}{2} L^1 \to \mathbb{R}$ be a comonotone coherent measure of risk that only depends on the distribution of the random variable. If there exists a nonconstant $M \in L^1$ such

$$\frac{1}{2}(M) = \frac{1}{2}(M),$$

then

$$\frac{1}{2}(X) = \mathbb{E}_\mathbb{P}[X]$$

for all $X \in L^1$.

We can interpret the above results as follows. Assume there exists a frictionless and arbitrage free market for a financial security $M$ and let $\frac{1}{2}(M)$ be its (market) price. Since spending $\frac{1}{2}(M)$ allows to annihilate the risk of holding $M$, it seems natural to consider acceptable the risk $\frac{1}{2}(M)$, that is, $\frac{1}{2}(M) \cdot \frac{1}{2}(M)$. The same argument for $-M$ yields $\frac{1}{2}(M) \cdot \frac{1}{2}(M)$. Finally, subadditivity of $\frac{1}{2}$ implies $\frac{1}{2}(M) \cdot \frac{1}{2}(M) \cdot \frac{1}{2}(M) \cdot \frac{1}{2}(M)$,

$$1 = \frac{1}{2}(M) \cdot \frac{1}{2}(M) \cdot \frac{1}{2}(M) \cdot \frac{1}{2}(M); \tag{1}$$

(1) Theorem 1 applies if the traded security $M$ is a cash or nothing call, while for Corollary 2 any risky security will do. In both cases one single security is needed.

A particular case in which Eq. (1) holds is the one in which all the measures in $\mathbb{P}$ are risk neutral, meaning that the scenarios envisioned by the supervisor are a subset of the ones envisioned by the market, in other words that marketed securities are allowed to drive positions toward acceptability. A characterization of such measures is provided.

**References**

