Quadratic hedging for asset derivatives with discrete stochastic dividends

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Abstract

In this paper we analyze the effect of discrete stochastic dividends on the pricing and hedging of contingent claims, formulating the No Arbitrage condition without requiring the continuity of the implied gain process. We allow the stock jump dependence on an additional random source, still preserving the positivity of the stock value at the ex dividend date. We characterize all the equivalent martingale measures and analyze the quadratic hedging approaches, as local risk minimizing and mean variance hedging.
1 Introduction

The aim of this paper is to analyze the effects of discrete stochastic dividends on the market. We suppose that the investors know the dividend payment dates, but ignore their exact amount. An approach widely used in practice assumes that at dividend payment dates the value of the stock drops down exactly by the corresponding amount of the dividend. In such a way the gain process (implied by the stock) is continuous. This behavior satisfies a simple standard no-arbitrage requirement when the dividend is deterministic (compare [1]), but it is slightly too severe when the dividend is stochastic, depending on an additional randomness’ source. In practice there are some financial concrete circumstances that can modify the relationship between price drop and dividend yield (as, for example, when transaction costs and different tax rates (between dividends and capital gains) affect the market, see e.g. [2]). However the complexity of factors and rumors affecting the stock behavior around dividends payment dates doesn’t allow any rule of thumb to predict the exact relationship between dividends and stock behavior. Therefore, admitting in our market model the possibility of a discrepancy between stock price drop and dividend amount, we achieve adherence to empirical evidence in a No Arbitrage consistent way. Mathematically, this is due to the usual predictability requirement of the trading strategies, which restricts the possibilities of free lunch (see [7]).

In [4] the authors focus on the stock price drop, modelling it as a random variable and characterizing it via No Arbitrage arguments. However, under their approach it is not so easy to guarantee the positivity of the (ex-dividend) stock price. In [1] discrete dividends are represented by a percentage of the stock value. Allowing this percentage to be a bounded random variable instead of a deterministic function of the
stock price, one could have another model for stochastic dividends, taking care of the
ex-dividend stock price sign in a more natural way, though financially this framework
could appear more artificial. In this paper we study discrete stochastic dividends via
No Arbitrage Techniques in a general settlement, analyzing then particular features
due to additional specifications of the model (inspired by [4]) and their consequences
on the pricing of related derivatives. Being the market incomplete, we use typical
hedging techniques as *mean variance hedging* and *local risk minimizing*.

2 The No Arbitrage Market Model

We focus on a market constituted by an asset $S(t)$ and a riskless bond $B(t)$ for
$t \in [0, T]$. The asset $S$ pays a dividend $D$ at a fixed date $t = T_D^-$, with $T_D \in [0, T]$, 
jumping immediately after, i.e. at the instant $t = T_D$, at the level $S(T_D) = S_{T_D^-} + \Delta S$;
both the dividend $D$ and the jump of the stock $J = \Delta S$ can be affected in different
ways by rumors and other factors independent of the usual random sources driving
the stock behaviour. Therefore beyond an usual probability space $(\Omega^o, P^o, (F^o_t)_t)$,
(see for example [8]) representing the randomness’ source of the market for $t \neq 
T_D$, we consider the additional probability space $(\Omega^x, P^x, A^x)$ carrying an exogenous
uncertainty of the variables $J$ and $D$ in $t = T_D$. To be more precise, we consider
the product space $\Omega = \Omega^o \times \Omega^x$ with the completed right continuous version of the
filtration $\mathcal{F} = \mathcal{F}^o \otimes \mathcal{F}^x$, where $\mathcal{F}^x_t = \{\emptyset, \Omega^x\}$ for $0 \leq t < T_D$ and $\mathcal{F}^x_t = A^x$ for $T_D \leq t \leq T$; denoting with $P = P^o \otimes P^x$, the probability space $(\Omega, P, (\mathcal{F}_t)_t)$ can now
describe the dynamic random evolution of the market: we assume that the process
$S$ is $\mathcal{F}$–adapted and the random variables $D$ and $J$ are measurable with respect
to $\mathcal{F}_{T_D}$. In this settlement, the No Arbitrage condition (see [4] or [3]) requires the
existence of an equivalent probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that the implied discounted gain process $\tilde{G}(t) = \tilde{S}(t) + \tilde{D}(t)$ (where the $\tilde{}$ denotes the actualized values and $\tilde{D}(t) = \int_0^t \frac{dD}{B}$) is a martingale. Since the discounted cumulative dividend process $\tilde{D}(t) = 0$ for $0 \leq t < T_D$ and $\tilde{D}(t) = D/B(T_D)$ for $T_D \leq t \leq T$, we will focus especially on $t = T_D$, where the NA condition requires

$$\mathbb{E}^\mathbb{Q}\left[\Delta \tilde{G}(T_D) \mid \mathcal{F}_{T_D^-}\right] = 0$$

(1)

according to [4]. Without additional specification of the model, we cannot achieve more information on the structure of $\mathbb{Q}$. Condition (1) can only be formulated in terms of the stock and the dividend:

$$\mathbb{E}^\mathbb{Q}\left[\Delta \tilde{S}(T_D) + D/B(T_D) \mid \mathcal{F}_{T_D^-}\right] = 0$$

(2)

Assuming the continuity of the bond $B(t)$, if both the dividend $D$ and the stock jump $J = \Delta S(T_D)$ are known a moment before the dividend date $T_D$, we can require mathematically that $D$ and $J$ are $\mathcal{F}_{T_D^-}$-measurable. In this case equation (2) reduces to $\Delta \tilde{S}(T_D) + D/B(T_D) = 0$, implying the continuity of $\tilde{G}$ in $t = T_D$, criticized in the introduction as a too strict NA formulation. It is clear that we cannot assume the $\mathcal{F}_{T_D^-}$-measurability of both $D$ and $J$, since this would vanify the description of the empirical discrepancy between stock price drop and dividend, but some measurability property could provide a more explicit and useful characterization of $\mathbb{Q}$. Since dividends are usually announced before the payment date, it seems reasonable to assume (following [4])

$$D \text{ is } \mathcal{F}_{T_D^-}\text{-measurable}$$

(3)

Due to the particular structure of $\mathcal{F}$, hypothesis (3) implies that $D$ is constant with respect to events $x \in \Omega^x$, that is $D = D(\omega)$ for $\omega \in \Omega$. In the next section we use
assumption (3) introducing a market model, inspired by [4], modified to control more easily the sign of the stock after dividend payment; our attention will focus on the stock jump rather than on the dividend.

3 A stock jump model

Inspired by [4], where $D$ is a fixed quantity, we focus on the unpredictable (with respect to $\mathcal{F}^o$) jump $J$. Under the measurability hypothesis (3), we require moreover that

$$\tilde{S}(T_D^-) > \tilde{D}$$

Assumption (4) is rather a reasonable financial requirement than an additional mathematical restriction. Denoting with $X$ a random variable concentrated on $[\alpha, \beta] \subset ]-1, 1[$, we define

$$\tilde{J} = -\tilde{D} + X(\tilde{S}(T_D^-) - \tilde{D})$$

The second addend on the right-side of (5) tells us that the stock has a jump in $T_D$ equal to the amount paid out by the dividend plus a noise term $X(\tilde{S}(T_D^-) - \tilde{D})$ that can be positive or negative depending only on the $X$’s sign, thanks to (4). Equation (5) can be equivalently stated as

$$\tilde{S}(T_D) = (\tilde{S}(T_D^-) - \tilde{D})(1 + X)$$

where we immediately see that $\tilde{S}(T_D) > 0$ under our assumptions. The No Arbitrage requirement (2) becomes (if the bond is continuous) $-\tilde{D} = \mathbb{E}^Q[\tilde{J} | \mathcal{F}_{T_D^-}] = -\tilde{D} + (\tilde{S}(T_D^-) - \tilde{D})\mathbb{E}^Q[X | \mathcal{F}_{T_D^-}]$ and, being $\tilde{S}(T_D^-) - \tilde{D} > 0$ and $\mathcal{F}_{T_D^-}$-measurable, this simply means

$$\mathbb{E}^Q[X | \mathcal{F}_{T_D^-}] = 0$$
Therefore the restriction on the support of $X$ doesn’t matter: in order to guarantee that the NoArbitrage condition (6), formulated already in [4], is nonempty, we have only to assume that both the events $X > 0$ and $X < 0$ have strictly positive probability:

$$
P[X > 0] > 0 \quad P[X < 0] > 0 \quad (7)$$

Condition (6) is going to play a central role in the characterization of the equivalent martingale measures $Q$ in the next section. Now we specify further our market model. Denote with $B(t) = B(0)e^{\int_0^t r(s)ds}$ (where $r(s)$ is the instantaneous interest rate) the riskless bond; consider the implied discounted gain process, $\tilde{G}$, consisting of the discounted cumulative dividend process and the discounted stock value:

$$
\tilde{G}(t) = \begin{cases} 
\tilde{S}(t) & 0 \leq t < T_D \\
\tilde{S}(t) + D/B(T_D) & T_D \leq t \leq T 
\end{cases} \quad (8)
$$

We are looking for all equivalent probability measures $Q$ such that $\tilde{G}$ as in (8) is a martingale and, in particular, such that (6) holds true. Assume that under $P^o$ the process $S$ follows

$$
\begin{align*}
\begin{cases}
\tilde{S}(0) = S_0 \\
\tilde{S}(T_D) = (\tilde{S}(T_D^-) - \tilde{D})(1 + X) \\
d\tilde{S}(t) = ((\mu - r)dt + \sigma dW(t))\tilde{S}(t) \quad t \neq T_D
\end{cases}
\end{align*} \quad (9)
$$

where $W$ is a $P^o$–Brownian motion w.r.t. $\mathcal{F}^o$ and with usual assumptions on $\mu$ and $\sigma$ to guarantee existence and uniqueness of the solution of problem (9), given $S_0$. The randomness due to the dividend presence affects the stock behaviour only in $t = T_D$ perturbing the value of $\tilde{S}(T_D)$; before and after the dividend date, the stock follows a diffusion type equation. What we require is therefore a strong efficiency of the
market. The implied discounted gain process is consequently driven by:

\[
\begin{align*}
\tilde{G}(0) &= S_0 \\
\tilde{G}(T_D) &= (\tilde{G}(T_D^-) - \tilde{D})X + \tilde{G}(T_D^-) \\
d\tilde{G}(t) &= \tilde{\mu}dt + \tilde{\sigma}dW(t) \\
\end{align*}
\]

(10)

where \( \tilde{\mu}(t) = (\mu - r)\tilde{G}(t) \) for \( 0 \leq t < T_D \) and \( \tilde{\mu}(t) = (\mu - r)(\tilde{G}(t) - \tilde{D}) \) for \( T_D < t \leq T \) and, similarly, \( \tilde{\sigma}(t) = \sigma\tilde{G}(t) \) for \( 0 \leq t < T_D \) and \( \tilde{\sigma}(t) = \sigma(\tilde{G}(t) - \tilde{D}) \) for \( T_D < t \leq T \).

4 The Martingale Measures

In this section we characterize the \( P \)-martingales and the equivalent martingale measures for \( \tilde{G} \) among all the martingale measures for the process \( \tilde{G} \). We recall that

**Definition 1** A signed martingale measure \( Q \) is a (signed) measure with \( Q \ll P \) such that \( Q(\Omega) = 1 \), the density \( \frac{dQ}{dP} \in L^2(\Omega) \) and \( \tilde{G} \) is a \( Q \)-martingale.

The measure \( Q \) is an equivalent martingale probability measure if \( \frac{dQ}{dP} > 0 \).

For quadratic hedging purposes it is sufficient to look at the equivalent martingale measures if the gain process is continuous (cfr [9]). Since in our model the process \( \tilde{G} \) may jump at the dividend payment date, the optimal related martingale measure may be a signed martingale measure (as in fact we will see in the last section) and therefore we need the characterization of the related larger class of densities. To this aim, we describe the structure of the \( \mathcal{F} \)-predictable processes and the Doob decomposition of the implied discounted gain process, \( \tilde{G} \), before stating Theorems 1 and 2, in the following propositions:
Proposition 1 With respect to the assumptions and notations of previous sections, the implied gain process $\tilde{G}$ admits under $P$ the Doob decomposition:

$$\tilde{G} = \tilde{G}_0 + A + M$$

with $A$ a finite variation process and $M$ a $\mathcal{F}$-martingale under $P$. In particular, denoting with $(\cdot)^c$ the continuous part of the processes, we have that

$$M^c = \int_0^t \tilde{\sigma}(s)dW(s)$$

and

$$A^c = \int_0^t \tilde{\mu}(s)ds$$

(where $(\cdot)^c$ denotes the continuous part of the processes). The jump part $\Delta \tilde{G}(T_D)$ splits into

$$\Delta M(T_D) = (\tilde{S}(T_D^-) - \tilde{D})(X - m_x)$$

where $m_x = \mathbb{E}^{P_x}[X]$ and

$$\Delta A = m_x(\tilde{S}(T_D^-) - \tilde{D}).$$

Proof. For the continuous part of the processes, it is enough to recall the behaviour of $\tilde{G}$ in (10). For the jump part, $\Delta \tilde{G}_{T_D} = X(\tilde{S}(T_D^-) - \tilde{D})$, we note that

$$\Delta M = \Delta \tilde{G}(T_D) = \mathbb{E}^P \left[ \Delta \tilde{G}(T_D) \big| \mathcal{F}_{T_D^-} \right]$$

and

$$\Delta A = \mathbb{E}^P \left[ \Delta \tilde{G}(T_D) \big| \mathcal{F}_{T_D^-} \right].$$

Thesis follows recalling again (10). □

Proposition 2 A predictable process $H(\omega, x, t)$ on $(\Omega, \mathcal{F}, P)$ is of the type

$$H(\omega, x, t) = \begin{cases} H_1(\omega, t) & 0 \leq t \leq T_D \\ H_2(\omega, x, t) & T_D < t \leq T \end{cases}$$

(11)
with \( H_1 \) predictable with respect to \( \mathcal{F}^o \) and \( H_2 \) measurable with respect to \( \mathcal{A}^x \otimes \mathcal{P}|_{\Omega^o \times [T_D, T]} \) where \( \mathcal{P}|_{\Omega^o \times [T_D, T]} \) is the sigma algebra generated by the predictable processes on \( \Omega^o \times [T_D, T] \).

Now we are able to prove that a martingale \( M \) on \( (\Omega, \mathcal{F}, P) \) decomposes in a sum of two strictly orthogonal martingales: a continuous one, with predictable representation with respect to the Brownian motion \( W \), and a pure jump martingale (which jumps exactly at time \( T_D \)). More precisely,

**Theorem 1** **Martingale Representation** Let \( M \) be a square integrable martingale on \((\Omega, \mathcal{F}, P)\). Then there exist a predictable process \( H \) (as previously defined), such that

\[
\mathbb{E}^P \left[ \int_0^T H_s^2 ds \right] = \int_\Omega dP^o(\omega) \int_{\Omega^x} dP^x(x) \int_{[0,T]} H^2(s; \omega, x) ds < \infty
\]

and a random variable \( N \), \( \mathcal{F}_{T_D} \)-measurable, with \( \int_{\Omega^x} N(\omega, x) dP^x(x) = 0 \) for \( P^o \)-almost every \( \omega \in \Omega^o \), such that the process

\[
N_t(\omega, x) = \begin{cases} 
0 & 0 \leq t < T_D \\
N(\omega, x) & T_D \leq t \leq T 
\end{cases}
\]  

(12)

is a pure jump martingale and

\[
M_t = M_0 + \int_0^t H_s(\omega, x) dW_s(\omega) + N_t
\]

**Proof.** Done in the appendix. \( \square \)

Theorem 1 gives us a representation of the martingales on the space \((\Omega, \mathcal{F}, P)\). Using this result, we can now characterize all the martingale measures for \( \tilde{G} \) as in Definition 1. This is done in the next theorem:
Theorem 2  Martingale Measures With respect to previous assumptions and notations, every signed martingale measure \( Q \ll P \) for \( \tilde{G} \) as in Definition 1 has density given by

\[
Z_t = \mathcal{E} \left( \int_0^t \tilde{\theta}_s dW_s \right) (1 + N_t)
\]  

(13)

where \( \mathcal{E}(\cdot) \) denotes the stochastic exponential and \( N_t \) is a pure jump \( P \)-martingale: \( N_t = 0 \) for \( 0 \leq t < T_D \), \( N_t = N_{T_D} \) for \( t \geq T_D \) and \( \mathbb{E}^P \left[ N(T_D) | \mathcal{F}_{T_D^-} \right] = 0 \).

The process \( N \) and \( \tilde{\theta} \) are constrained by the following relation

\[
\begin{align*}
\tilde{\theta}_t &= -\tilde{\mu}_t = -\frac{\mu - r}{\sigma} \\
\int_{\Omega^o} N_{T_D}(\omega, x)X(x)d\mathbb{P}^x(x) &= -m_x
\end{align*}
\]

(14)

for \( \mathbb{P}^o \text{a.e.}\ \omega \in \Omega^o \).

Moreover, \( Z_t \) defines an Equivalent Martingale Probability Measure iff

\( N_{T_D} > -1 \)

Proof. Done in the Appendix. □

Remark The Martingale Measure densities in equation (13) are the product of the continuous factor

\[
Z_t^o = \mathcal{E} \left( \int_0^t \tilde{\theta}_s dW_s \right)
\]

(15)

and of the jump factor

\[
U_t = 1 + N_t
\]

(16)

We see immediately that \( Z^o \) is the density of a well known probability measure, since \( Z_T^o = \frac{dQ^o}{dP^o} \), where \( Q^o \) is the usual risk neutral measure implied by Girsanov theorem, when no dividends and rumors affect the market. The continuous factor \( Z^o \) is even in our case uniquely determined by the structure of the market model. On the contrary, the jump part \( U \) is constrained by the second equation in (14), but not uniquely determined. □
5 Quadratic Hedging Techniques

We have pointed out in the concluding remark of previous section that the martingale measure is not uniquely determined. This is due to the uncompleteness of the market, since an exogenous source of risk represented by \((\Omega^x, \mathcal{F}^x, P^x)\) affects dividend and stock behaviour. Therefore, given an option whose discounted payoff is \(\psi(\tilde{S}(T))\), we find an interval of No Arbitrage prices, as \(Q\) varies among all equivalent martingale measures as in Theorem 2:

\[
V_Q = \mathbb{E}^Q[\psi(\tilde{S}(T))]
\]

We look for those prices related to quadratic hedging techniques (in case these prices are implied by equivalent martingale measures): Mean Variance Hedging and Local Risk Minimizing.

The aim of the mean variance hedging is to minimize the variance of the trade-off between the option we have to replicate and the value at time \(T\) of the self-financing hedging strategy we decided to follow. The dual problem is

\[
(MVH) \text{ Find the minimal variance martingale measure } Q^* \text{ such that } \min_Q \mathbb{E}^P \left[ \left( \frac{dQ}{dP} \right)^2 \right]
\]

is obtained for \(Q = Q^*\)

On the contrary, the local risk minimizing technique pursues the minimization of risk of high costs for the hedging strategies. The related optimal measure among all the martingale measures is the minimal martingale measure, defined in this way (compare [9]). Consider the canonical decomposition of the special semimartingale \(\tilde{G}\) under \(P\) given in Lemma 1: \(\tilde{G} = \tilde{G}_0 + A + M\), with \(A\) finite variation process and \(M\) a \(\mathcal{F}\)-martingale under \(P\). Then

\[
(LRM) \text{ Find the minimal martingale measure } \tilde{Q} \text{ such that every } P\text{-martingale } Y
\]
strongly orthogonal to $M$, i.e. $< Y, M >= 0$, is a $\hat{Q}$-martingale

In our model the two optimal measures coincide, having the same density, as emphasized in the following

**Theorem 3** The minimal variance martingale measure and the minimal martingale measure have the same density

$$\frac{dQ^*}{dP} = Z^o U^*$$

where, using the notations of Theorem 2, $Z^o_i = E \left( \int_0^t \tilde{\theta}_s dW_s \right)$ and $U^*$ is given by

$$U^* = 1 - m_x(v_x)^{-1}(x - m_x)$$

where $m_x$ and $v_x$ are, resp., the mean and the variance of $X$ under $P^x$.

**Proof.** We compute at first the density for the minimal variance martingale measure $Q^*$, solving problem [MVH]. Recalling Theorem 2, we have to find the $\mathcal{F}_{TD}$-measurable $U$ that solves

$$\begin{cases}
\min_U E_P \left[ E^2 \left( \int_0^T \tilde{\theta}_s dW_s \right) U^2 \right] \\
\int_{[\alpha, \beta]} U(., x) dP^x(x) = 1 \quad P^o - a.e. \quad \text{in} \quad \Omega^o \\
\int_{[\alpha, \beta]} U(., x)X(x) dP^x(x) = 0 \quad P^o - a.e. \quad \text{in} \quad \Omega^o
\end{cases}$$

Problem (18) has an immediate solution in case $m_x = 0$ thanks to Minkowski inequality. Indeed, define the scalar product

$$(X|Y) := \frac{E^P \left[ E^2 \left( \int_0^T \tilde{\theta}_s dW_s \right) XY \right]}{E^P \left[ E^2 \left( \int_0^T \tilde{\theta}_s dW_s \right) \right]}$$

with $X, Y$ measurable with respect to $\mathcal{F}_{TD}$ and such that $E^P \left[ E^2 \left( \int_0^T \tilde{\theta}_s dW_s \right) X^2 \right] < \infty$. The first constraint in (18) implies that $(U|1) = 1$ and therefore $(U|U) \geq \frac{||U||^2}{||1||^2} = \ldots$
1. Since $U^* = 1$ P-a.e. satisfies even the second constraint in (18) when $m_x = 0$ and reaches the minimum, being $(U^*|U^*) = 1$, we achieve the thesis.

In case $m_x \neq 0$ then the density $U^* = 1$ doesn’t match the second constraint in (18). However, since both constraints are P$^o$-a.e. pointwise defined with respect to $\omega \in \Omega^o$, we can formulate from (18) a minimum norm problem in the Hilbert space $L^2([\alpha, \beta], P^x)$ with the usual scalar product $(X|Y) = \int_{[\alpha, \beta]} X(x)Y(x)dP^x$. Denoting with $P_0(x) = 1$ and $P_1(x) = v_x^{-1/2}(x - m_x)$, where $v_x$ is the variance of $X$ under $P^x$, it is easy to check that $P_0$ and $P_1$ are orthonormal in $L^2([\alpha, \beta], P^x)$. Now consider the problem of finding $U^* \in L^2([\alpha, \beta], P^x)$ such that

$$
\begin{cases}
\min_{U^*}(U^*|U^*) \\
(U^*|P_0) = 1 \\
(U^*|P_1) = -m_x(v_x)^{-1/2}
\end{cases}
$$

We notice that (19) is a minimum norm problem on an affine subspace of $L^2([\alpha, \beta], P^x)$ with the solution

$$U^* = 1 \cdot P_0 - m_x(v_x)^{-1/2} \cdot P_1$$

For P$^o$-a.e. $\omega \in \Omega^o$ a generic $U = U(\omega, x)$ satisfying the constraints in (18) is such that: $\mathbb{E}^P[\mathcal{E}^2(\cdot)U^2(\cdot, \cdot)] = \int_{\Omega^o} \mathcal{E}^2(\cdot)(\int_{[\alpha, \beta]} U^2(\cdot, \cdot)dP^x)dP^o \geq \int_{\Omega^o} \mathcal{E}^2(\cdot)(\int_{[\alpha, \beta]} (U^*)^2dP^x)dP^o = (U^*|U^*) \int_{\Omega^o} \mathcal{E}^2(\cdot)dP^o$, since $\mathcal{E}^2(\cdot) > 0$ and for P$^o$-a.e. $\omega \in \Omega^o$, being $U(\omega, \cdot) \in L^2([\alpha, \beta], P^x)$, we have that $\int_{[\alpha, \beta]} U^2(\omega, \cdot)dP^x \geq \int_{[\alpha, \beta]} (U^*)^2(\cdot)dP^x$. Therefore, $U^*$ is precisely the minimizing density in (18) we were looking for.

Consider now the local risk minimizing problem [LRM]. Recalling Theorem 1, let $Y = Y_0 + Y_c + Y_d$ where $Y_t^c = \int_0^t h_s^c dW_s$ and $Y_t^d = I_{\{t \geq T_D\}} \Delta Y$ with $\int_{[\alpha, \beta]} \Delta Y dP^x = 0$. 


The orthogonality requirement implies that:
\[
\begin{align*}
< Y^c, M^c > &= 0 \\
< Y^d, M^d > &= 0
\end{align*}
\]
(20)
where \((\cdot)^c\) denotes the continuous part of the processes and \((\cdot)^d\) the discontinuous one. From the first equation in (20) we simply obtain that \(h^Y = 0\). The second one rewrites as
\[
0 = < Y^d, M^d > = I_{\{t \geq T_D\}} \int_{[\alpha, \beta]} \Delta Y \Delta M dP^x
\]
Recalling the structure of \(\Delta M\) from Lemma 1, since \((\tilde{S}(T_D^-) - \tilde{D}) > 0\) is \(\mathcal{F}_{T_D^-}\)-measurable and thanks to the properties of \(\Delta Y\), the second orthogonality condition in (20) can be stated as
\[
\int_{[\alpha, \beta]} \Delta Y X dP^x = 0
\]
(21)
Now looking among the densities of the martingale measures \(Z = Z^o U\), with \(U\) constrained as in Theorem 2, if \(m_x = 0\) we have that \(\hat{Z} = Z^o\) reaches our goal. In fact, \(Y Z^o\) is a \(P\)-martingale for every \(Y_t = I_{\{t \geq T_D\}} \Delta Y\) as in equation (21) and \(U = 1\) is admissible, satisfying the constraints implied by Theorem 2:
\[
(U|1) = 1 \\
(U|X) = 0
\]
(22)
On the contrary, if \(m_x \neq 0\) then \(U = 1\) is not admissible anymore, and we have to look for \(\hat{Z} = Z^o \hat{U}\), with \(\hat{U}\) satisfying (22) and such that
\[
\int_{[\alpha, \beta]} \Delta Y \hat{U} dP^x = 0
\]
for all \(\Delta Y\) (\(\Delta Y(\omega, \cdot) \in \mathcal{L}^2([\alpha, \beta], P^x)\) for \(P^o\)a.e. \(\omega \in \Omega^o\) ) s.t.
\[
\begin{align*}
(\Delta Y|1) &= 0 \\
(\Delta Y|X) &= 0
\end{align*}
\]
(23)
In terms of the orthonormal basis of polynomials of $L^2(\alpha, \beta, P^x)$ obtained via Gram-Schmidt procedure, $(P_0, P_1, P_2, \ldots, P_n, \ldots)$, we have to find $\hat{U}$ such that

$$(\Delta Y|\hat{U}) = 0$$

for all $\Delta Y$ ($\Delta Y(\omega, \cdot) \in L^2(\alpha, \beta, P^x)$ for $P^a$ a.e. $\omega \in \Omega^o$) s.t.

$$\begin{cases}
    (\Delta Y|P_0) = 0 \\
    (\Delta Y|P_1) = 0
\end{cases}$$

(24)

Being

$$\hat{U} = P_0 - m_x(v_x)^{-1/2}P_1 + \sum_{n \geq 2} a_n^U P_n$$

we have to find coefficients $a_n^U$ with $n = 2, 3, \ldots$ such that for all $\Delta Y = \sum_{n \geq 0} a_n^Y P_n$ satisfying (24), hence $\Delta Y = \sum_{n \geq 2} a_n^Y P_n$, we must have

$$0 = (\hat{U}|\Delta Y) = \sum_{n \geq 2} a_n^U a_n^Y \cdot 1$$

Being $a_n^Y$ coefficients free to vary, we obtain

$$a_n^U = 0 \quad n = 2, 3, \ldots$$

Hence, $\hat{U} = P_0 - m_x(v_x)^{-1/2}P_1$ as for the minimal variance measure. Therefore the two optimal measures coincide $\hat{Q} = Q^*$. □

The minimal variance (minimal martingale) measure with density $\frac{dQ^*}{dP} = Z^U$ is in general a signed martingale measure, but if $U^*$ satisfies the positivity constraint:

$$U > 0$$

(25)

then $Q^*$ is in fact an equivalent probability measure. There are some cases where the constraint (25) is automatically satisfied by $U^*$ defined in (17), Theorem 3. A sufficient condition is stated in the following
Proposition 3  Suppose that alternatively one of the following two assumptions holds true:

\[
\begin{cases}
    m_x < 0 \\
    s_x > m_x \alpha
\end{cases}
\]

(26)

where \( s_x = \int_{[\alpha, \beta]} X(x)^2 dP_x \), or

\[
\begin{cases}
    m_x > 0 \\
    s_x > m_x \beta
\end{cases}
\]

(27)

In both cases, the solution \( U^* \) in Theorem 3 is strictly positive.

To clarify these conditions on the first two moments of the random variable \( X \), assume that \( X \) has a uniform distribution on \([\alpha, \beta]\). Since \( m_x = \frac{\alpha + \beta}{2} \), if \( \beta = -\alpha \) then \( U^* = 1 \) in Theorem 3 and the related \( Q^* \) is, of course, a probability measure. Consider now the case \( \beta < -\alpha \): being \( s_x = \frac{\beta^2 + \alpha \beta + \alpha^2}{3} \) and \( m_x < 0 \) it is easy to see that condition (26) in proposition 3 is satisfied if \( \beta > -\frac{\alpha}{2} \). On the contrary, if \( \beta > -\alpha \) then \( m_x > 0 \) and condition (27) in proposition 3 is satisfied, whenever \( \beta < -2\alpha \). Summing up, we can equivalently express (26) as \( -\frac{\alpha}{2} < \beta < -\alpha \) and (27) as \((-\alpha) < \beta < (-2\alpha)\). Therefore, it is possible to compensate a negative (in mean) stock discrepancy \((\beta < -\alpha)\) if this discrepancy takes even significant positive values \((\beta > -\frac{\alpha}{2})\). Similarly a positive (in mean) stock discrepancy \((\beta > -\alpha)\) is acceptable from a quadratic hedging point of view if the positive distortion is not too heavy \((\beta < -2\alpha)\).

5.1 Examples and Conclusions

We have seen that the risk neutral measure is not unique, but the optimal measures (i.e. the minimal martingale measure and the variance optimal measure) coincide. Therefore, provided that this optimal measure \( Q^* \) is a martingale probability measure (as guaranteed by proposition 3), we can compute the related NA price of a given
claim as an expectation under \( Q^* \). If \( m_x = 0 \), this price is simply the expectation of the discounted gain process under the product measure \( Q^o \times P^x \). However, as soon as the payoff has a nonlinear relation with the underlying value, this expectation differs from the usual price under \( Q^o \). To quantify this difference we perform explicit computations for a plain vanilla call option on \( S \) with strike \( K \) and maturity \( T \).

According to [1] we assume that the dividend is a known percentage of the stock value at the dividend date, i.e. \( D = \delta \cdot S(T_D^-) \) where \( \delta \) is a fixed number in \((0, 1)\). We have to determine

\[
V^* = \mathbb{E}^*[\tilde{S}(T) - Ke^{-rT}]^+
\]

(28)

where \( \mathbb{E}^*[] \) denotes the expectation under \( Q^* \). Let \( P^\xi \) be a process driven from the third equation in (9) for all \( t \in [0, T] \) with initial value \( P^\xi(0) = \xi \). Since under \( Q^* = Q^o \times P^x \) the conditional low of \( \tilde{S}(T) \) is captured by

\[
\tilde{S}(T) \mid X = x = \tilde{S}(0)(1 - \delta)(1 + x)P^1(T)
\]

we have that

\[
V^* = \mathbb{E}^*[(\tilde{S}(T) - Ke^{-rT})^+] = \int_{[\alpha, \beta]} \mathbb{E}^*[(\tilde{S}(T) - Ke^{-rT})^+ \mid \mathcal{F}_T]dQ^o(\omega) = \int_{[\alpha, \beta]} c(\tilde{S}(0)(1 - \delta)(1 + x))dP^x(x)
\]

where \( c(\xi) \) denotes the price of a call option written on the asset \( P^\xi \) with same strike and maturity of our original option (cfr [1]).

Suppose that \( X \) is a random variable with uniform distribution on \([\alpha, \beta]\). Since we are analyzing the case \( m_x = 0 \), we take \( \alpha = -0.5 \) and \( \beta = -\alpha \). We suppose that the initial value of the stock is \( S(0) = 42 \), the strike of the option \( K = 40 \), riskless interest rate \( r = .10 \), volatility \( \sigma = .20 \) and \( T = 0.5 \). Suppose that the dividend is the 20\% of the stock value, i.e. \( \delta = .2 \). With these parameters, the value of the option under the optimal measure is \( V^* = 2.7010 \). It’s interesting to compare this value with the Black-Scholes price of a call option written on \( S \) under the continuity assumption.
of the implied gain process. Under this (too) severe NA requirement, $S(T)$ is equal in law to $P^\xi(T)$, with $\xi = \tilde{S}(0)(1 - \delta)$; therefore the price of the call option becomes $c(\tilde{S}(0)(1 - \delta)) = c(42 \times 0.8) = 0.5266$ that is less than $V^*$, as we expected thanks to the convexity of the payoff (Jensen inequality). The difference between the two values is not negligible! Obviously this difference decreases if the distortion factor $X$ is concentrated on a smaller interval. For example, if the support of the uniform random variable $X$ is given by $\alpha = -0.1$ and $\beta = 0.1$, then $V^* = 0.6387$.

There is a well documented empirical evidence ([2],[5]) that stock prices drop on average by less than the value of the dividend. In our model this means that $m_x > 0$. Assume therefore that $X$ has a uniform distribution on $[\alpha, \beta]$ with $\alpha = -0.2$ and $\beta = 0.3$. With these parameters (the others unchanged), condition (27) in proposition 3 is satisfied since $(-\alpha) < \beta < (-2\alpha)$. In this case the optimal measure $Q^*$ has density $Z^\omega U^*$ and therefore $V^*$ in (28) becomes

$$V^* = \int_{[\alpha, \beta]} c(\tilde{S}(0)(1 - \delta)(1 + x))(1 - (m_x/v_x)(x - m_x))dP^x(x) = 1.1350$$

What happens if we assume a fixed mean distortion of the stock jump? this approach has been heavily criticized in [4] as inconsistent from a theoretical point of view. Nevertheless, since it is used sometimes in practice to simplify computations, it seems interesting to compare this value with our optimal fair price. Assuming that $\tilde{S}(T_D)(\omega, x) = (1 - \delta)(1 + m_x)\tilde{S}(T_D^-)(\omega)$ for a.e. $x \in [\alpha, \beta]$ and $\omega \in \Omega^\omega$ we have to compare $V^*$ with $c((\tilde{S}(0)(1 - \delta)(1 + m_x)) = c(42 \times 0.8 \times 1.1) = 1.6140$ which is significantly higher.

These numerical examples cannot furnish an exhaustive analysis of the general pricing problem, however they give some useful suggestion. In case of small perturbation of the stock value (and with $m_x = 0$) we don’t lose much accuracy for
pricing purposes using the *traditional* Black-Scholes formulae and the usual Girsanov change of measure \( Q^o \): the simplifying continuity assumption of the gain process is reasonable and reduces the computational efforts. However, as soon as the stock discrepancy becomes relevant, it is necessary to deal with appropriate models for the behaviour of the stock price in order to compute one of the favorite NA prices for the derivative.

6 Appendix

We give in this section the proofs of both theorems of Section 4

**Proof of Theorem 1 (Martingale Representation)**

We have to prove for a s.i. martingale that it decomposes in

\[
M_t(\omega, x) = \begin{cases} 
M_0 + \int_0^t H_s(\omega, x) dW_s(\omega) & 0 \leq t < T_D \\
M_0 + \int_0^t H_s(\omega, x) dW_s(\omega) + N(\omega, x) & T_D \leq t \leq T 
\end{cases}
\]  

(29)

Since a s.i. martingale is determined by its terminal value \( M_T \) it is enough to prove the thesis for \( M_T \), i.e. to see that \( M_T = M_0 + \int_0^T H_s(\omega, x) dW_s(\omega) + N(\omega, x) \). Denote with \( \tilde{M}_T = M_T - M_0 = M_T - \mathbb{E}^P[M_T] \). Thanks to density arguments in \( L^2 \) spaces, we can reduce to the case where \( \tilde{M}_T \) is of the form \( \tilde{M}_T(\omega, x) = K(\omega)Y(x) \). Since \( \mathbb{E}^P[\tilde{M}_T] = 0 \), we can assume without loss of generality that \( \mathbb{E}^{P^*}[K] = 0 \). Setting \( m_Y = \mathbb{E}^{P^*}[Y] \), let \( \tilde{Y}(x) = Y(x) - m_Y \) and denote with \( J_t(x) = I_{(t \geq T_D)} \tilde{Y}(x) \) for \( 0 \leq t \leq T \). Consider the \( \mathcal{F}^o \)-predictable process \( k \) such that, denoting with \( Z_t(\omega) = \int_0^t k_s(\omega) dW_s(\omega) \), we have \( Z_T = K \). Then, integration by part formula implies

\[
J_T(x)K(\omega) = \int_0^T J_s \, dZ_s + \int_0^T Z_s \, dJ_s
\]
since \([Z, J] = 0\) Hence,

\[
\tilde{M}_T(\omega, x) = K(\omega)m_y + K(\omega)\tilde{Y}(x)
\]

\[
= m_y \int_0^T k_s(\omega)dW_s(\omega) + \int_0^T J_s^-(x)k_s(\omega)dW_s(\omega) + \left( \int_0^{T_D} k_s(\omega)dW_s(\omega) \right)J_T(x)
\]

Thesis follows with \(H_s(\omega, x) = (m_y + J^-_s(x))k_s(\omega)\) and \(N(\omega, x) = (\int_0^{T_D} k_s(\omega)dW_s(\omega))J_{TD}\).

\(\square\)

**Proof of Theorem 2** (Martingale Measures)

By definition the density of a \(Q\) is given by \(Z_t = \mathbb{E}^P\left[ \frac{dQ}{dP} \middle| F_t \right]\). Being \(Z\) a \(P\)-martingale, it admits a representation (by Theorem 1) of the following form:

\[
Z_t = 1 + \int_0^t h_s dW_s + R_t
\]

with \(R_t\) pure jump martingale such that \(R(t) = 0\) for \(0 \leq t < T_D\), \(R(t) = \Delta R_{TD}\) for \(T_D \leq t \leq T\) and \(\mathbb{E}^P\left[ \Delta R_{TD} \middle| F_{TD}^- \right] = 0\).

The discounted gain process \(\tilde{G}\) is a \(Q\)-martingale if and only if \(\tilde{G}Z\) is a \(P\)-martingale. Applying Integration by Part Formula, we obtain

\[
d(\tilde{G}_tZ_t) = \tilde{G}_t^- dZ_t + Z_t^- d\tilde{G}_t + d[\tilde{G}_t, Z_t]
\]

and recalling (10)

\[
d(\tilde{G}_t^tZ_t) = \tilde{G}_t^- dZ_t + Z_t^- (\tilde{\mu}_t dt + \tilde{\sigma}_t dW_t + \Delta \tilde{G}(t)) + \tilde{\sigma}_t h_t dt + \Delta \tilde{G}(t) \Delta R(t)
\]

which must be the differential of a \(P\)-martingale. To this aim, we notice that the first term, \(\tilde{G}_t^- dZ_t\), defines already a (local) \(P\)-martingale, as well as \(Z_t^- \tilde{\sigma}_t dW_t\). On the contrary, the continuous bounded variation terms must cancel out:

\[
Z_t^- \tilde{\mu}_t + \tilde{\sigma}_t h_t = 0
\]

(31)
And the remaining pure jump part must be martingales:

\[ \mathbb{E}^P \left[ Z_{T_D}^- \Delta \tilde{G}(T_D) + \Delta \tilde{G}(T_D) \Delta R(T_D) \mid \mathcal{F}_{T_D^-} \right] = 0 \]

Recalling that \(\Delta \tilde{G}(T_D) = (\tilde{S}(T_D^-) - \tilde{D})X\), the last equality implies

\[ \mathbb{E}^P \left[ (Z_{T_D}^- + \Delta R(T_D))X \mid \mathcal{F}_{T_D^-} \right] = 0 \]

constraining \(R\) such that

\[ \mathbb{E}^P \left[ RX \mid \mathcal{F}_{T_D^-} \right] = -Z_{T_D}^- m_x \quad (32) \]

Hence, if \(before\) the dividend payment date, i.e. for \(t < T_D\), we simply have

\[ Z_t = \mathcal{E}(\int_0^t \tilde{\theta}_s dW_s) \]

with \(\tilde{\theta} = -\mu/\bar{\sigma} = -(\mu - r)/\sigma\), \(after\) the dividend payment date we have to take care of jumps. Define the pure jump martingale \(N\) as follows:

\[ N_t := \begin{cases} 0 & 0 \leq t < T_D \\ R(Z_{T_D^-})^{-1} & T_D \leq t \leq T \end{cases} \]

(Notice that \( \mathbb{E}^P \left[ N_{T_D} \mid \mathcal{F}_{T_D^-} \right] = (Z_{T_D^-})^{-1} \mathbb{E}^P \left[ R_{T_D} \mid \mathcal{F}_{T_D^-} \right] = 0 \).)

Since \( R_t = \int_0^t Z_s^- dN_s \), from (30) and (31) it turns out that

\[ Z_t = 1 + \int_0^t Z_s^- dX_s \]

where \(dX_t := \tilde{\theta} dW_t + dN_t\). Therefore \(Z\) is precisely the stochastic exponential of \(X\)

\[ Z_t = \mathcal{E}(X)_t \]

(compare [8]). Being \(N_t\) and \(\int_0^t \tilde{\theta} dW_s\) orthogonal, the stochastic exponential of their sum \(X\) reduces to:

\[ Z_t = \mathcal{E} \left( \int_0^t \tilde{\theta} dW_s \right) (1 + N_t) \]

\[ 21 \]
Moreover, the constraint (32) on $R_t$ is satisfied if and only if

$$
\mathbb{E}^P \left[ N_{T_D} X \big| \mathcal{F}_{T_D^-} \right] = (Z(T_D^-))^{-1} \mathbb{E}^P \left[ RX \big| \mathcal{F}_{T_D^-} \right] = -\mu_x
$$

Finally, the density $Z_t = \mathcal{E} \left( \int_0^t \tilde{\theta} dW_s \right) (1 + N_t)$ is strictly positive, meaning that the related $Q$ is an equivalent probability measure, if and only if

$$N_t > -1$$

\[\square\]

References


