Pricing of Implied Volatility Derivatives:  
a Risk Neutral Model for Market Implied Volatility

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Abstract
In this paper we construct a risk neutral dynamics of the at the money implied volatility to price implied volatility futures and forward starting compound options. These are exotic derivatives contracts whose payoff depends on the future implied volatility. Starting from the description of the real mechanism on the basis of which implied volatilities are actually quoted by option traders in option markets, we derive the risk neutral dynamics for the stochastic implied volatility. In particular we obtain the risk neutral drift restriction that must be satisfied by each single stochastic implied volatility, individually considered, on the volatility surface invariant both to time to maturity and to relative futures prices with the same time to maturity. We show the risk neutral process of the instantaneous spot volatility towards which the risk neutral at the money market implied volatility converges by the absence of maturity arbitrage.

Introduction
Aim of the present paper is to construct a risk-neutral dynamics for the implied volatility in order to price exotic implied volatility derivatives. These derivatives, differently for example from covariance swaps whose payoff depends on the realised variance of the asset, have a payoff that depends on the implied volatility of an option written on the reference asset. Some examples of such derivatives are the VIX volatility index calculated from ATM short-term volatilities on S&P100, the VOLAX future contract listed at the Deutsche Terminborse (DTB) since January 1998 and forward starting at the money (ATM) compound option that have been recently treated in the OTC market. The underlying of the VOLAX future is the implied volatility of a synthetic at-the-money (ATM) straddle with three months to expiry. The settlement price of the volatility futures contract is based on the implied volatilities of the DAX index options. A (forward starting ATM) compound option is an implied volatility option because basically it is an option on the implied volatility of the underlying option. At the expiry, the holder of the option will compare the ATM implied volatility,
that makes the Black-Scholes price equal to the future option market price, with the implied
volatility that makes the Black-Scholes price equal to the strike of the compound option and will
decide about the convenience of the exercise.

This kind of derivatives allows the investors to hedge against changes in volatility. Indeed as
reported in Grunbichler and Longstaff (1996) “investors would now be able to manage their risk in
two dimensions, price risk and volatility risk, an opportunity previously out of reach for all but the
large and sophisticated options users”.

A simple but incorrect approach to price implied volatility derivatives is to use the implied
forward volatility computed from the current term structure of implied volatilities or to try to model
the future realized volatility or even to use a stochastic volatility model. In this way, we are
forgetting that: a) today we do not know which option will be ATM at the expiry of the compound
option and then we do not know which term structure to use, b) to use the forward volatility is
equivalent to assume that the realized variance is simply the square of the implied volatility and this
is true only when we have a flat smile, compare Britten-Jones and Neuberger (2000), c) to use the
instantaneous volatility as underlying variable in order to price these contracts, as e.g. in
Grunbichler and Longstaff (1996), contradicts the nature itself of the contract that is instead based
on the implied volatility. Therefore the necessity of building a model directly for the implied
volatility is required by the nature itself of the products we try to price.

This fact however should be viewed as natural, as we will notice in the first section, option
traders quote implied volatilities in moneyness (with respect to the futures price) and time to
expiration terms and not option prices directly. Implied volatility is transformed in a price by the
Black-Scholes and Black formulae. For this reason to try to construct a model directly for the
dynamics of the implied volatility is similar to the idea pursued for interest rates with the market
model by Brace et al.

The well-known formulas by Black and Scholes (1973) and Black (1976) are nowadays
universally the common market convention recipes in pricing options. These two formulas have
been extremely successful despite the numerous flaws and limitations of assumptions on which their
derivation relies. In particular the assumption of constant volatility has always been questionable
and from an empirical point of view it is systematically contradicted. In order to produce more
realistic option pricing formulas in the financial literature models with stochastic volatility were
initially derived, such as for example the ones of Hull and White (1987), Stein and Stein (1991) and
Heston (1993). All these models try to calculate exact option prices assuming different equilibrium
mean reverting stochastic processes for the instantaneous variance of the spot price and then
deriving the option formula with two state variables, the stochastic spot price and the stochastic spot
volatility. The main drawback of the approach followed by the previous authors lies in the fact that
a) these models cannot in most of the cases be fitted to really quoted market option prices for
different strikes and different maturities, b) they require the specification of a functional form for
the volatility risk premium and usually this specification is arbitrary and it is chosen mainly on the
basis of analytical considerations. For this reason pursuing the same fashion followed to derive the
risk neutral model of the term structure of forward interest rates by Heath et al. (1992), Derman and
Kani (1998) derived a risk neutral model (i.e. consistent with market option prices) for both the term
structure and the strike structure of local volatilities. The great improvement of the framework
developed by Derman and Kani (1998) relies on the virtual possibility of exactly calibrating all
option prices quoted in the market during every point in time, but it is extremely difficult to be
implemented owing to the non-markovianity of the modelled local volatilities, which prevents the
model itself from being of immediate practical use. To overcome the intrinsic complexity of the
model developed by Derman and Kani (1998), Schonbucher (1998) and Ledoit and SantaClara
(1999) derived much more practical implied volatility market models, which represent the
counterpart of what Brace et al. (1997) developed when modelling directly risk neutral market
dynamics for libor and swap rates. The great breakthrough of the approach followed by
Schonbucher (1998) and Ledoit and SantaClara (1999) is that for the first time in the financial
literature the implied volatility, i.e. the volatility that plugged into the Black-Scholes formula gives
an option price matching an actual market option price, is modelled as an input rather than as an
output. This original way of modelling is justified by the fact that implied volatility is a different
way to quote an option price, using as market convention the Black-Scholes formula. Then to quote
different implied volatilities for different strikes and maturities is a simple way to cope with the
fallacies of the Black-Scholes assumptions. Given that we accept the view that the presence of a
smile helps to take into account deviations of real markets from model assumptions, the implied
volatilities assume therefore the dignity of independent variables.

In other words it can be said that according to this view the option price depends on two
underlying state variables, the spot price of the underlying security and the implied volatility. In
particular Schonbucher (1998) models the absolute implied volatility of which he finds the risk
neutral drift restriction. On the other side, closer to the market practice as we will see, Ledoit and
SantaClara (1999) model indirectly the relative (in terms of maturities and moneyness of the
options) implied volatilities with respect to absolute option prices in order to find similar risk neutral drift restrictions to the ones calculated by Schonbucher (1998).

In this paper we derive directly the risk neutral restriction on the relative implied volatilities by making option prices relative in terms of maturities and moneyness of the options themselves. In this way our state variables are the real implied volatilities that are actually quoted by the option market makers, who ignore spot levels and expiration dates when making quotes. For this reason our model is a real market model of implied volatilities, because we model just the variables that are quoted in the market.

In the first section of this paper we explain how implied volatilities are quoted in option markets by traders and we show how futures option prices, which are invariant to time to maturity and relative strike, can be seen as re-scaled time to maturity invariant futures prices of which we derive their risk neutral drift terms. Then in the second section we determine the functional form that needs to be taken by the drift of the implied volatility coherent with risk neutral invariant futures option prices. We make in the third section the time to maturity approach to zero in order to see what happens to the at the money implied volatility, which should converge by the absence of arbitrage to the instantaneous spot volatility. In this we are able to find the risk neutral dynamics of the instantaneous spot volatility. Also we show that in our model set-up the previous convergence is not implied by the volatility bubble paradox for instantaneous implied volatilities introduced by Schonbucher (1998). Finally in section four we show how our methodology can be practically applied to price exotic derivatives such as implied volatility futures and forward starting compound options in comparison with the prices obtained for the same derivatives when not coherently with the absence of arbitrage that the at the money implied volatility multiplied by the time to maturity is a proxy of the total future variance of the underlying security price.