Resource Allocation via the Median Rule: Theory and Simulations in the Discrete Case

Klaus Nehring* Clemens Puppe†

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Abstract

We study the properties of the *median rule* for the collective choice of resource allocation under a budget constraint: Each individual submits a proposal and the feasible allocations are ranked according to the sum of their distances in the l_1-metric to the individual proposals. One of the allocations with minimal aggregate distance is chosen. This rule is shown to be remarkably robust against strategic manipulation. In particular, it is strategy-proof if all individuals have 'metrically single-peaked' preferences. The theoretical analysis of the median rule is complemented by simulation studies.

1 Introduction

We study voting over allocations of several public goods. It is well-known that in this context, every reasonable aggregation rule is in generically strategically manipulable, see Zhou (1991) extending the negative conclusion of Gibbard (1973) and Satterthwaite (1975) in the general abstract setting. Only in one-dimensional situations do there exist non-degenerate strategy-proof social choice mechanisms provided that all individual preferences are single-peaked, see Moulin (1980). This case in fact arises naturally when there are two public goods but not with a larger number of public goods.

*Department of Economics, University of California at Davis, kdnehring@ucdavis.edu.
†Department of Economics, Karlsruhe Institute of Technology, clemens.puppe@kit.edu.

In Nehring and Puppe (2007), we have shown that the notion of single-peakedness can be considerably generalized and that non-degenerate strategy-proof social choice functions exist on economically interesting and relevant 'generalized' single-peaked preference domains. Unfortunately, in the specific public goods allocation problem even the assumption of generalized single-peaked preferences does not enlarge the class of strategy-proof social choice functions beyond the dictatorial ones, as proved in Nehring and Puppe (2010).

In this paper we introduce the following 'median rule.' Each individual submits a proposal (his or her 'preference peak') and the feasible allocations are ranked according to the sum of their distances (in a specific but natural metric) to the individual proposals. One of the allocations with minimal aggregate distance is chosen. This rule refines the notion of the 'Condorcet set' introduced in Nehring, Pivato and Puppe (2014). We prove that the median rule is strategy-proof if all individuals have 'metric' single-peaked preferences. But even under the more general assumption of (generalized) single-peakedness the possibilities to manipulate the outcome of the median rule are quite restricted: a voter can neither move the closest nor the farthest median allocation closer to his/her own peak.

The median rule allows for ties in the final outcome so that, formally, it is a mapping from profiles of individual preferences to sets of allocations. On the other hand, it satisfies the 'tops-only' condition, that is, the only information needed for determining the set of winners is the most preferred allocation of each voter. It is remarkable that sensible rules satisfying the 'tops-only' condition can be formulated at all in this context.

We compare the behavior of the median rule in terms of strategic manipulation with the simple 'arithmetic mean' rule. Our simulation results indicate that in comparison with the mean rule, the median rule is much more robust against strategic manipulation; not necessarily in the sense that the outcome under strategic manipulation is close to the truthful outcome (that may be true under the mean rule as well), but in the sense that the announced peaks deviate much less from the individuals' true peaks under the median rule than under the mean rule.

2 The median rule for resource allocation

2.1 The resource allocation problem

A society can spend an amount M on the provision of L different public goods in discrete non-negative quantities. Throughout, we will assume that individuals have monotone preferences; moreover, we assume that the public goods are measured in money terms, i.e. we assume fixed prices. Together, these assumptions allow us to model the allocation problem as the choice of an element of the discrete $(L-1)$-dimensional simplex

$$X := \{x \in \mathbf{N}_0^L : \sum_{j=1}^{L} x_j = M\},$$

where $x = (x_1, ..., x_L)$ and x_j is expenditure for public good j.

By $d(x, y) := \frac{1}{2}\sum_j |x_j - y_j|$ we denote the (normalized) l_1-distance between $x \in X$ and $y \in X$. The normalization ensures that a transfer of one unit of money from one public good to another yields an allocation with unit distance from the original allocation. The set X is thus naturally endowed with a graph structure such that two elements x and y are connected by an edge if and only if $d(x, y) = 1$ (see Figure 1 for the case $L = 3$). The metric d is the natural graph distance given by the minimal number of edges needed to connect two elements by a path. We write xNy if x and y are neighbors in the graph, i.e. $xNy :\Leftrightarrow d(x, y) = 1$. Furthermore, we write $xN_{jk}y$ if $x_j = y_j + 1$, $x_k = y_k - 1$ and $x_l = y_l$ for all $l \neq j, k$, and we say that x and y are 'neighbors in jk-direction.' Note that while the binary relation N is symmetric, the relations N_{jk} are asymmetric and satisfy $xN_{jk}y \Leftrightarrow yN_{kj}x$.

A path connecting two allocations with a minimal number of edges is called a *shortest path*; note that shortest paths need not be unique. If y lies on some shortest path connecting x and z we say that y is *between x and z*. The set of all allocations between x and z is denoted by $[x, z]$. As is easily verified, we have

$$\begin{aligned} [x, z] &= \{w \in X \ : \ d(x, z) = d(x, w) + d(w, z)\} \\ &= \{w \in X \ : \ w_j \in [x_j, z_j] \text{ for all } j = 1, ..., L\} \end{aligned}$$

(see Fig. 1 where $[x, z] = \{x, y, y', z\}$, for instance). The set of all allocations which are closer to x than to y, i.e. the set of all $w \in X$ such that $d(w, x) < d(w, y)$, will be denoted by $\rangle x, y\langle$. Note that, by the triangle inequality, for all $x \neq y$ and all $w \in X$,

$$x \in [w, y] \ \Rightarrow \ w \in \ \rangle x, y\langle\, .$$

For neighbors, the set $\rangle x, y\langle$ admits a particularly simple description, as stated in the following result that will be eminently useful for our later analysis.

Lemma 1 *Let $x, y \in X$ and suppose that $xN_{jk}y$, then*

$$\rangle x, y\langle \ = \ \{w \in X \ : \ w_j \geq x_j \text{ and } w_k < y_k\}. \tag{1}$$

Proof. If $x_j = y_j + 1$, $y_k = x_k + 1$, and $x_l = y_l$ for all $l \neq j, k$, then the following four sets partition X,

$$\begin{aligned} Y_1 &= \{w \in X \,:\, w_j \geq x_j \text{ and } w_k < y_k\}, \\ Y_2 &= \{w \in X \,:\, w_j < x_j \text{ and } w_k \geq y_k\}, \\ Y_3 &= \{w \in X \,:\, w_j \geq x_j \text{ and } w_k \geq y_k\}, \\ Y_4 &= \{w \in X \,:\, w_j < x_j \text{ and } w_k < y_k\}. \end{aligned}$$

Straightforward computation shows that $d(w, x) = d(w, y) - 1$ for all $w \in Y_1$, $d(w, x) = d(w, y) + 1$ for all $w \in Y_2$, and $d(w, x) = d(w, y)$ for all $w \in Y_3 \cup Y_4$. This implies that $\rangle x, y\langle = Y_1$ and $\rangle y, x\langle = Y_2$, as desired. For an illustration, see Fig. 1 where $xN_{31}y$ and, for instance, $z \in Y_2 = \rangle y, x\langle$ and $y' \in Y_4$. $\square$

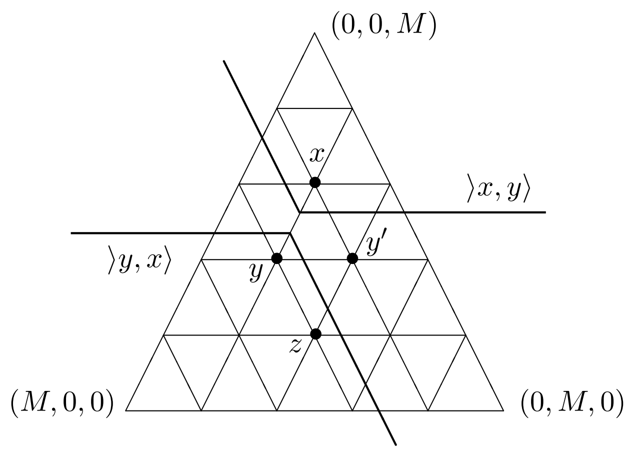

Fig.1: The sets $[x, z]$, $\rangle x, y\langle$ *and* $\rangle y, x\langle$

2.2 The median rule

In our context, the *median rule* can be described as follows: each individual submits a proposal of how the money should be spent on the set of public goods (i.e. a feasible allocation), and allocations are ranked according to the sum of their respective distances to the individual proposals. In the general judgement aggregation framework, the median rule has been proposed and axiomatized in a series of papers by Nehring and Pivato (2014a,b,c). The resource allocation problem is a special case of the general judgement aggregation problem. Another special case is the preference aggregation problem in the context of which the median rule is known as the Kemény-Young rule (cf. Kemény, 1959; Young and Levenglick, 1978).

Formally, denote by p the distribution of the proposals on X, i.e. for each $w \in X$, p_w is the number of individuals who proposed w. For all subsets $Y \subseteq X$, denote by $p(Y) := \sum_{w \in Y} p_w$ the popular support of Y. Furthermore, for all $x \in X$, denote by

$$R_p(x) := \sum_{w \in X} p_w d(x, w)$$

the *remoteness* of the feasible allocation x given the distribution p. An allocation is a *median allocation* if it solves

$$\arg\min_{x \in X} R_p(x) \;=\; \arg\min_{x \in X} \sum_{w \in X} p_w d(x, w),$$

i.e. a median allocation is one with minimal remoteness. Let $\mathrm{Med}(p)$ denote the set of median allocations given the distribution p. Evidently, $\mathrm{Med}(p)$ need not be a singleton but it is always non-empty since the median allocations are obtained as the solution to a minimization problem on a finite set.

If $L = 2$, i.e. if there are only two public goods, the space X is one-dimensional and the set of median allocations coincides with the standard median. In particular, if there is an odd number of individuals, the set $\mathrm{Med}(p)$ consists of the unique median proposal.

3 The structure of the set of median allocations

In this section, we prove two important properties of the set of median allocations. First, for all distributions p, the set $\mathrm{Med}(p)$ is convex, i.e. all allocations between two median allocations are also median allocations. Secondly, the set of median allocations is 'locally determined,' i.e. whether an allocation is among the set of median allocations can be decided by comparing it only to all of its neighbors. Before we prove these results, however, we completely describe the median rule in the case of three voters.

3.1 The case of three agents

In the following, we describe all possible constellations of the set of median allocations for up to three voters. This subsection is mainly pedagogical and aims at providing a first understanding of the remarkably complex structure of the optimization problem underlying the median rule in the present context.

First note that, evidently, for two voters with proposals x and y, respectively, the set of median allocations is given by $[x, y]$. Indeed, every allocation $w \in [x, y]$ lies on a shortest path between x and y, and therefore has remoteness equal to $d(x, y)$, while every allocation outside $[x, y]$ has strictly larger remoteness.

Now consider the case of three agents with proposals x, y and z, respectively. For simplicity, we consider the case $L = 3$; the general case follows analogously. If (at least) two agents propose the same allocation, that allocation is clearly the unique median allocation. Thus, suppose henceforth that x, y and z are pairwise different. In the following, we fix two proposals, say $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$, in generic position and describe how the set of median allocations changes when the location of x relative to y and z changes. Without loss of generality assume that $y_1 \geq z_1$, $y_2 \leq z_2$ and $y_3 \leq z_3$ (see Figure 2). Assume in all what follows that $x_2 \geq y_2$ and let us move x clockwise 'around' the interval $[y, z]$ as shown in Fig. 2. (The cases in which $x_2 < y_2$ can be treated in a completely symmetric manner.) In the top-left panel, we have $x_3 \leq y_3$. This implies that y is between x and z, hence y is the unique median allocation (marked in red in the top-left panel of Fig. 2).

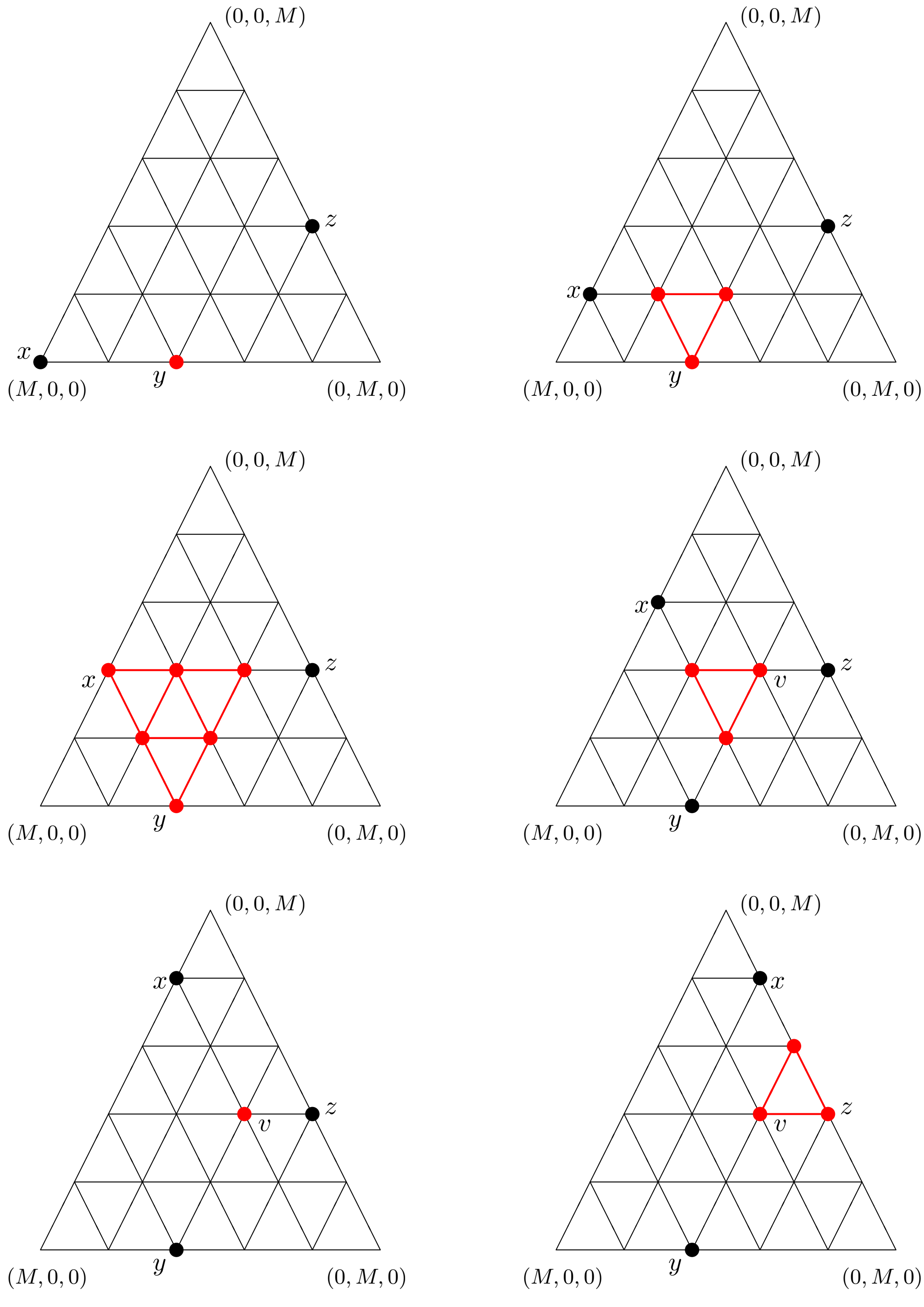

Fig. 2: The set of median allocations with three agents

In the top-right panel, we have $y_3 < x_3 < z_3$; in this case, straightforward computation yields as the set of median allocations the red marked triangle spanned by y and the two projections of x to the lines $\{w \in X \ : \ w_1 = y_1\}$ and $\{w \in X \ : \ w_2 = y_2\}$, respectively, holding x_3 constant. Exactly the same logic yields the red marked triangle in the middle-left panel of Fig. 2 in which we have $x_3 = z_3$.[1]

If x moves 'further up,' i.e. if $x_3 > z_3$, there are two cases. If $x_1 \geq y_1$ (which may in general be feasible even though it is not possible in Fig. 2) nothing changes relative to the previous case: the set of median allocations is still the one shown in the middle-left panel of Fig. 2. On the other hand, if $x_1 < y_1$ but still $x_1 \geq z_1$, the set of median allocations is given by the triangle spanned by the two projections of x to the lines $\{w \in X \ : \ w_3 = z_3\}$ and $\{w \in X \ : \ w_2 = y_2\}$, respectively, holding x_1 constant, and the allocation $v = (v_1, v_2, v_3)$ such that $v_2 = y_2$ and $v_3 = z_3$ (see Fig. 2); this holds for the three remaining panels in Fig. 2, the middle-right panel and the two bottom panels. In the left-bottom panel, all three allocations coincide with v which in fact represents the usual coordinate-wise median of x, y and z in this case; in the bottom-right panel, the two projections of x, holding x_1 fixed, have exchanged their position relative to the situation depicted in the middle-right panel: the projection to the line $\{w \in X \ : \ w_2 = y_2\}$ is now closer to x while the projection to the line $\{w \in X \ : \ w_3 = z_2\}$ now coincides with z. In concluding the analysis of the special case of three voters, we note that the set of median allocations always has triangular shape (with a unique allocation as limiting case).

3.2 Two fundamental properties: convexity and local determinacy

From the preceding analysis it is immediate that the set of median allocations is convex if the set of agents is less than, or equal to three. We now prove that this holds generally (Proposition 1). Moreover, whether or not an allocation is a median allocation can be decided by local comparison, i.e. by a 'majority vote' between the given allocation and each of its neighbors ('local determinacy,' Proposition 2).

In order to prove these results, we need several auxiliary lemmata. For two neighbors x and y and a given distribution p, we write xM_py if $p(\rangle x, y\rangle) > p(\rangle y, x\rangle)$, i.e. if the set of allocations which are closer to x than to y has more mass than the set of allocations which are closer to y than to x. Moreover, we write xI_py if neither xM_py nor yM_px. The first lemma shows that the ranking among neighbors induced by the median rule simply corresponds to majority voting, where an individual is construed as 'voting' for x in a binary comparison with y if the individual's proposal is closer to x than to y.

Lemma 2 *For any distribution p and any two neighbors x and y, $p(\rangle x, y\rangle) - p(\rangle y, x\rangle) = R_p(y) - R_p(x)$. In particular, xM_py if and only if $R_p(x) < R_p(y)$.*

Proof. As in the proof of Lemma 1, we obtain that, for all $w \in \rangle x, y\rangle$, $d(w, x) - d(w, y) = -1$, for all $w \in \rangle y, x\rangle$, $d(w, x) - d(w, y) = 1$, and for all other $w \in X$, $d(w, x) - d(w, y) = 0$. From

[1] In the latter two cases, we have assumed that $x_1 \geq y_1$, in particular that x is not an element of the triangle spanned by y, the allocation v with $v_2 = y_2$ and $v_3 = z_3$, and the allocation t with $t_1 = y_1$ and $t_3 = z_3$; if it is, the set of median allocation is given by the smaller triangle spanned by x and its two projections to the line $\{w \in X \ : \ w_2 = y_2\}$: one holding x_1 constant, the other holding x_3 constant.

this the assertions of Lemma 2 are immediate. $\square$

As an immediate corollary of Lemma 2, we obtain that a neighbor y of a median allocation x is itself a median allocation if and only if xIy.

The next lemma establishes a specific relationship in the respective outcomes of the majority vote among neighbors 'in the same direction' (see Figure 3).

Lemma 3 *Let x, y be such that $x_j > y_j$ and $y_k > x_k$. If $xN_{jk}x'$ and $y'N_{jk}y$, then $\rangle y', y\rangle \supseteq \rangle x, x'\rangle$ and $\rangle y, y'\rangle \subseteq \rangle x', x\rangle$. Moreover, $xM_px' \Rightarrow y'M_py$ and $yM_py' \Rightarrow x'M_px$.*

Proof. By (1) we have

$$\begin{array}{rcll} \rangle y', y\rangle & = & \{w \in X \ : \ w_j \geq y_j + 1 & \text{and} \quad w_k \leq y_k - 1\}, \\ \rangle x, x'\rangle & = & \{w \in X \ : \ w_j \geq x_j & \text{and} \quad w_k \leq x_k\}, \\ \rangle y, y'\rangle & = & \{w \in X \ : \ w_j \leq y_j & \text{and} \quad w_k \geq y_k\}, \\ \rangle x', x\rangle & = & \{w \in X \ : \ w_j \leq x_j - 1 & \text{and} \quad w_k \geq x_k + 1\}, \end{array}$$

which gives the first assertion. From this, we obtain $p(\rangle x, x'\rangle) \leq p(\rangle y', y\rangle)$ and $p(\rangle x', x\rangle) \geq p(\rangle y, y'\rangle)$. Thus, if $p(\rangle x, x'\rangle) > p(\rangle x', x\rangle)$, then $p(\rangle y', y\rangle) > p(\rangle y, y'\rangle)$, and if $p(\rangle y, y'\rangle) > p(\rangle y', y\rangle)$, then $p(\rangle x', x\rangle) > p(\rangle x, x'\rangle)$, which proves the second claim. $\square$

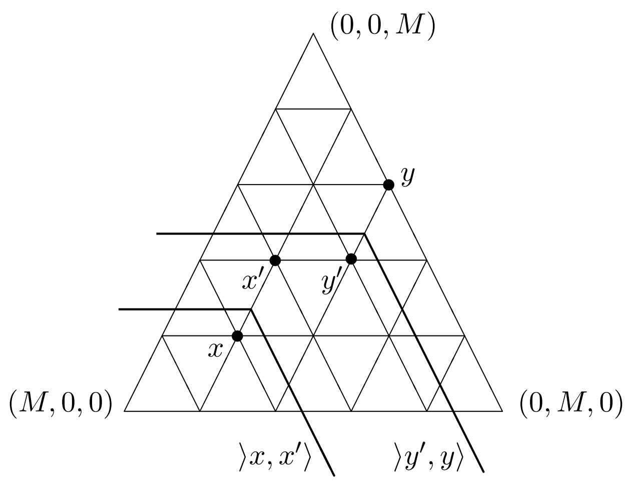

Fig. 3: The sets $\rangle y', y\rangle$ and $\rangle x, x'\rangle$ are ordered by set inclusion

Lemma 3 has the immediate and important consequence that the remoteness function is *quasi-convex* in all directions. Specifically, we obtain the following corollary.

Corollary 1 *Suppose that x and y differ only in the amount allocated to goods j and k, i.e. suppose that $x_l = y_l$ for all $l \neq j, k$, then*

$$z \in [x, y] \ \Rightarrow \ R(z) \leq \max\{R(x), R(y)\}; \tag{2}$$

moreover, if $R(x) \neq R(y)$, then

$$z \in]x,y[\;\Rightarrow\; R(z) < \max\{R(x), R(y)\},$$

where we denote $]x,y[:= [x,y] \setminus \{x,y\}$.

Another noteworthy consequence of the results derived so far is that no median allocation can be on the 'loosing side' of two neighbors, as follows.

Lemma 4 *Let $x, y \in X$ be neighbors and suppose that xM_py. Then,*

$$\mathrm{Med}(p) \;\cap\; \rangle y, x\rangle = \emptyset$$

Proof. Let $xN_{jk}y$ and consider any $z \in \rangle y, x\rangle$. Let z' be the neighbor of z in jk-direction, i.e. $z'N_{jk}z$. (Note that such allocation z' always exists.) By Lemma 3, xM_py implies $z'M_pz$, thus by Lemma 2, $R_p(z') < R_p(z)$. In particular, z does not admit the minimal remoteness, and hence, $z \notin \mathrm{Med}(p)$ which proves Lemma 4. □

Proposition 1 *The set of median allocations is convex, i.e. for all p and all $x, y \in \mathrm{Med}(p)$, one has $[x, y] \subseteq \mathrm{Med}(p)$.*

Proof. Let $x, y \in \mathrm{Med}(p)$ be two distinct allocations, and consider any shortest path between them. By contradiction, suppose that there exists an allocation on this shortest path that is not a median allocation, i.e. an allocation with strictly larger remoteness than either x and y. Then, there must exist neighbors z, z' along this path such that $R_p(z') < R_p(z)$, hence by Lemma 2, $z'M_pz$. But we have either $x \in \rangle z, z'\rangle$ or $x \in \rangle z, z'\rangle$ contradicting Lemma 4. □

The next result shows that the set of median allocations is in fact determined by 'local' majority voting, i.e. by a pairwise comparison of any given allocation with all its neighbors.

Proposition 2 *Let $x \in X$. Then, $x \in \mathrm{Med}(p)$ if and only if, for all neighbors $y \in X$ of x, not (yM_px).*

Proof. By definition, a median allocation cannot lose in pairwise comparison against any of its neighbors. Conversely, let $x \in X$ be such that xM_px' or xI_px' for all x' with xNx'. We will show that $x \in \mathrm{Med}(p)$ by contradiction. Thus, suppose that there is some other point $z \in X$ with a strictly lower remoteness. Choose two neighbors y, y' on a shortest path from x to z such that yM_py' and $y' \in [x, y]$, and assume without loss of generality that $y'N_{jk}y$. Since y' is between x and y we must have $x_j > y_j$ and $x_k < y_k$. Now consider x' with $xN_{jk}x'$. By Lemma 3, yM_py' implies $x'M_px$ which contradicts the assumption that x does not lose against any of its neighbors. □

3.3 Further properties of the set of median allocations

The set of median allocations has further remarkable properties, two of them are collected in this subsection. First, we show that allocations that lie strictly between two other median allocations can have no mass themselves, i.e. no individual can have proposed them (although, by Proposition 1 above, they must be median allocations as well).

Proposition 3 *Suppose that $x, y \in \text{Med}(p)$ and $z \in]x, y[$, then $p_z = 0$.*

Proof. Without loss of generality assume that $xNzNy$, $x_j > y_j$, $y_k > x_k$ and $xN_{jk}z$. Let y' be the element in X such that $y'N_{jk}y$ (y' may be equal to z). By Proposition 1, $y' \in \text{Med}(p)$. By Lemma 3, we obtain $\rangle x, y'\rangle \subseteq \rangle z, y\rangle$ and $\rangle y', x\rangle \supseteq \rangle y, z\rangle$. In fact, we have $\rangle x, y'\rangle \subsetneq \rangle z, y\rangle$ because $z \in \rangle z, y\rangle$ but $z \notin \rangle x, y'\rangle$. Suppose now, by contradiction, that $p_z > 0$. Then $p(\rangle z, y\rangle) > p(\rangle x, y'\rangle)$ and $p(\rangle y', x\rangle) \geq p(\rangle y, z\rangle)$. Since y' and x are both median allocations, they must have the same (minimal) remoteness, hence $p(\rangle x, y'\rangle) = p(\rangle y', x\rangle)$. But then, $p(\rangle z, y\rangle) > p(\rangle z, y\rangle)$, i.e. zM_py, contradicting the assumption that y is a median allocation, using Lemma 2.

□

Note that it is in fact possible that *no* median allocation has positive mass, see e.g. the situation in the middle-right panel of Fig. 2 above.

The analysis of the three voter case shows that the set of median allocations can be large if the agents' proposals are far away from each other, see e.g. the middle-left panel in Fig. 2. However, this can happen only if there is a voter, or a group of voters, whose proposals are far from the proposals of *all* other voters. Indeed, it turns out that the set of median allocations is 'small' whenever each voter's proposal is connected to any other voter's proposal by a path of proposals. Specifically, denote by $\Delta_{\text{Med}(p)} := \max_{x,y \in \text{Med}(p)} d(x, y)$ the *diameter* of the set of median allocations, by $\text{supp}(p)$ the support of the distribution p, and say that a subset $Y \subseteq X$ is *connected* if, for each pair $x, y \in Y$, there is a path connecting x and y that lies entirely in Y.

Proposition 4 *If* $\text{supp}(p)$ *is connected, then* $\Delta_{\text{Med}(p)} \leq 1$.

Proof. If $|\text{Med}(p)| = 1$, then clearly $\Delta_{\text{Med}(p)} \leq 1$. Thus, suppose that $|\text{Med}(p)| > 1$. Since $\text{Med}(p)$ is convex, there exist at least to neighbors $x, y \in \text{Med}(p)$. First, we show that for any two neighbors $x, y \in \text{Med}(p)$, we have

$$p(\rangle x, y\rangle) = p(\rangle y, x\rangle) > 0. \tag{3}$$

Indeed, since $x, y \in \text{Med}(p)$, we have xIy, i.e. $p(\rangle x, y\rangle) = p(\rangle y, x\rangle)$. Suppose that $p(\rangle x, y\rangle) = p(\rangle y, x\rangle) = 0$, i.e. suppose that the sets $\rangle x, y\rangle$ and $\rangle y, x\rangle$ have no mass, and consider the set $Y := X \setminus (\rangle x, y\rangle \cup \rangle y, x\rangle)$. If $xN_{jk}y$, then by Lemma 1, we have $Y = Y^+ \cup Y^-$ where $Y^+ = \{w : w_j \geq x_j \text{ and } w_k \geq y_k\}$ and $Y^- = \{w : w_j < x_j \text{ and } w_k < y_k\}$. In particular, the intersection $Y^+ \cap Y^-$ is empty, and in fact no two allocations w^+, w^- such that $w^+ \in Y^+$ and $w^- \in Y^-$ can be neighbors of each other. The connectedness of $\text{supp}(p)$ thus implies that $\text{supp}(p)$ lies entirely in Y^+, or entirely in Y^-. But this is not possible, since in that case, the common neighbor of x and y in direction of $\text{supp}(p)$ would have strictly smaller remoteness than either x and y. This shows (3).

Now assume that there exist two median allocations $x, z \in \text{Med}(p)$ with $d(x, z) > 1$. By the convexity of $\text{Med}(p)$, we can in fact assume without loss of generality that $d(x, z) = 2$, and hence that there exists a common neighbor $y \in \text{Med}(p)$ of x and z. There are now two cases: either (i) the three allocations are 'in a row,' i.e. $xN_{jk}yN_{jk}z$ for some j and k, or (ii) there exists another common neighbor y' of x and z such that $xN_{jk}y$ and $y'N_{jk}z$, for some j

and k. Note that, by the convexity of $\mathrm{Med}(p)$, we have $y \in \mathrm{Med}(p)$ in both cases, and also $y' \in \mathrm{Med}(p)$ in the second case. We will now show that neither case is possible.

Case (i). Since $x, y, z \in \mathrm{Med}(p)$, we have $xI_p yI_p z$, thus $p(\rangle x, y\rangle) = p(\rangle y, x\rangle)$ and $p(\rangle y, z\rangle) = p(\rangle z, y\rangle)$. By Lemma 3, we have $\rangle y, z\rangle \supsetneq \rangle x, y\rangle$ and $\rangle y, x\rangle \supsetneq \rangle z, y\rangle$. By the inequality (3), we have $p(\rangle x, y\rangle) > 0$ and $p(\rangle z, y\rangle) > 0$. Since the support of p is connected, the sets $\rangle y, z\rangle \setminus \rangle x, y\rangle$ and $\rangle y, x\rangle \setminus \rangle z, y\rangle$ must therefore both have positive mass. But this implies $p(\rangle y, z\rangle) > p(\rangle x, y\rangle)$ and $p(\rangle y, x\rangle) > p(\rangle z, y\rangle)$ in contradiction to $p(\rangle y, z\rangle) = p(\rangle z, y\rangle)$.

Case (ii). Since $x, y, y', z \in \mathrm{Med}(p)$, we have $xI_p y$ and $y'I_p z$, thus $p(\rangle x, y\rangle) = p(\rangle y, x\rangle)$ and $p(\rangle y', z\rangle) = p(\rangle z, y'\rangle)$. By Lemma 3, we have $\rangle y', z\rangle \supsetneq \rangle x, y\rangle$ and $\rangle y, x\rangle \supsetneq \rangle z, y'\rangle$. By the inequality (3), we have $p(\rangle x, y\rangle) > 0$ and $p(\rangle z, y'\rangle) > 0$. Since the support of p is connected, this implies that at least one of the sets $\rangle y', z\rangle \setminus \rangle x, y\rangle$ or $\rangle y, x\rangle \setminus \rangle z, y'\rangle$ must have positive mass. Assume without loss of generality that it is $\rangle y', z\rangle \setminus \rangle x, y\rangle$. Then we obtain

$$p(\rangle y', z\rangle) > p(\rangle x, y\rangle) = p(\rangle y, x\rangle) \geq p(\rangle z, y'\rangle),$$

in contradiction to the assumption that $y'I_p z$. $\square$

4 Limited manipulability of the median rule

Can an individual by submitting an appropriate non-truthful proposal influence the outcome of the median rule to his or her advantage? This question turns out to have a subtle answer. While the general answer is, yes, even under generalized single-peaked preferences, there is an important sense in which the median rule is robust against strategic manipulation. This is detailed in this subsection. The following introductory example is instructive.

Consider the case $L = 3$, and suppose that under the distribution p two individuals propose the allocation y, one individual proposes the allocation z, one individual z', and yet another individual, say individual i, the allocation x. The resulting set of median allocations is the diamond-shaped set $[y, z']$ (see Figure 4 below); all allocations $w \in [y, z']$ have the (minimal) remoteness of $R_p(w) = 6$. By non-truthfully reporting $\tilde{x}$ instead of x, individual i removes the allocation z' from the set of median allocation, changing it to the red triangle consisting of y and its two neighbors in direction of z', and reducing the minimal remoteness to 5, see Fig. 4 on the left-hand side. However, it is not clear that individual i would indeed want to do so since this removes one of the two median allocations that are closest to the allocation x, assumed to be individual i's most preferred allocation in this example. To describe further possibilities, observe that by reporting $\tilde{x} = z'$ instead of x, individual i can shrink the set of median allocations to the two allocations marked in red in the right-hand side of Fig. 4 (reducing the minimal remoteness to 5 as well). (Again, it is not evident whether an individual with preferred allocation x would want to manipulate in this way.) Finally, note that by reporting $\tilde{x} = y$ instead of x, individual i can shrink the set of median allocations to the single allocation y (with remoteness 4).

Whether or not an individual would want to manipulate the outcome in the manner just described depends on her or his preferences over outcomes and, in fact, over *sets of outcomes.* However, the possibility to influence the outcome turns out to be remarkably limited in the sense that an individual can reduce neither the minimal nor the maximal distance of a median allocation to her or his most preferred allocation; this is easily verified in all cases just described, and indeed also in all possible scenarios in the three voter case (see Fig. 2 above) and can be proved in general as follows.

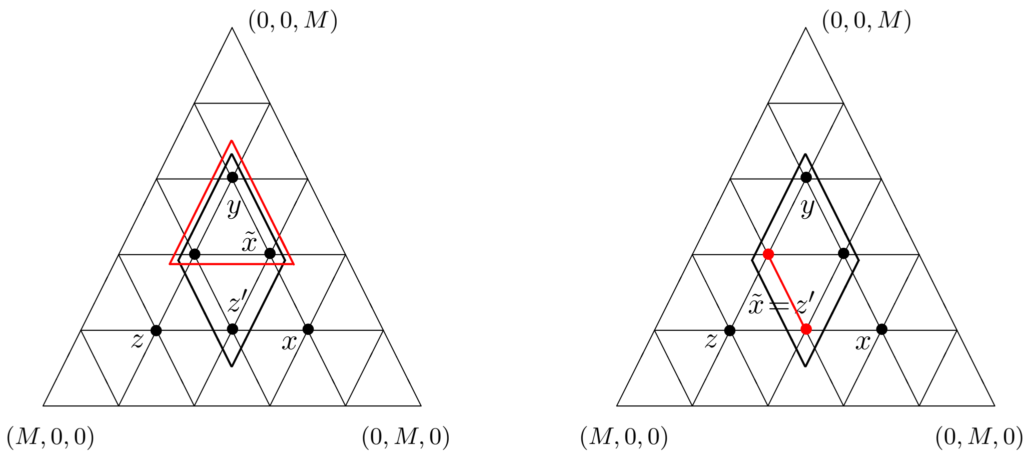

Fig. 4: Manipulating the set of median allocations

As in the above examples fix an individual, say individual i with most preferred allocation x, and denote by $\mathrm{Med}(p)$ the set of median allocations given the distribution p of proposals in which individual i submits x. Furthermore, denote by $\tilde{p}$ the distribution of proposals that differs from p only in the proposal of individual i who proposes, say $\tilde{x} \neq x$. Then, the allocation in $\mathrm{Med}(p)$ that is closest to x is at least as close to x than the closest allocation in $\mathrm{Med}(\tilde{p})$; similarly, the allocation of $\mathrm{Med}(p)$ that is farthest away from x is at least as close to x than the farthest allocation in $\mathrm{Med}(\tilde{p})$. Formally, for $x \in X$ and non-empty $Y \subseteq X$, denote by

$$\begin{aligned} d_{\min}(x,Y) &:= \min_{w\in Y} d(x,w) \\ d_{\max}(x,Y) &:= \max_{w\in Y} d(x,w). \end{aligned}$$

Theorem 1 *Suppose that an individual submits the allocation x in the distribution p, and let $\tilde{p}$ differ from p only in the proposal of this individual who submits an allocation different from x in $\tilde{p}$. Then,*

$$d_{\min}(x, \mathrm{Med}(p)) \leq d_{\min}(x, \mathrm{Med}(\tilde{p})) \tag{4}$$

i.e. a unilateral deviation from the truthful proposal cannot move the closest median allocation closer to one's true peak, and

$$d_{\max}(x, \mathrm{Med}(p)) \leq d_{\max}(x, \mathrm{Med}(\tilde{p})) \tag{5}$$

i.e. a unilateral deviation from the truthful proposal cannot move the farthest median allocation closer to one's true peak.

Proof. We start with the following observation. Let z be any allocation different from x and consider any neighbor z' of z in direction of x, i.e. suppose that $x_j > z_j$, $x_k < z_k$ and $z' N_{jk} z$. Then,

$$zM_p z' \;\Rightarrow\; zM_{\tilde{p}} z' \quad \text{and} \quad z' M_{\tilde{p}} z \;\Rightarrow\; z' M_p z, \tag{6}$$

i.e. if z wins against z' if the individual reports x, then z wins against z' also if the individual reports any allocation different from x, and if z loses against z' under $\tilde{p}$ then it loses against z' also if the individual reports the allocation x. Indeed, this follows immediately from Lemma 2 and the fact that $x \in \rangle z', z\langle$.

We now prove the first inequality (4). Let $z \in \mathrm{Med}(p)$ be a median allocation with minimal distance to x, say $d(x, z) = r$. If $r = 0$ there is nothing to show, thus assume $r \geq 1$. For any j, k such that $x_j > z_j$ and $x_k < z_k$, let z_{jk} be the neighbor of z in direction of x, i.e. $z_{jk} N_{jk} z$. Moreover, denote by Z_x^- the set of all such neighbors, i.e.

$$Z_x^- \;=\; \{y \in X : yNz \text{ and } d(x, y) = r - 1\}.$$

Since z is a median allocation closest to x, we have $zM_p y$ for all $y \in Z_x^-$ by Lemma 2. Thus by (6), also $zM_{\tilde{p}} y$. By Lemma 4, for any $y \in Z_x^-$, no element of $\rangle y, z\langle$ can be a median allocation under $\tilde{p}$. The proof of the inequality (4) is therefore completed by noting that

$$\{y \in X : d(x, y) < r\} \subseteq \bigcup_{y \in Z_x^-} \rangle y, z\langle\,.$$

To prove inequality (5), let individual i report x under the distribution p and $\tilde{x}$ under the distribution $\tilde{p}$. We proceed by induction on the distance $d(x, \tilde{x})$. Thus, assume first that $d(x, \tilde{x}) = 1$, i.e. that $\tilde{x}$ is a neighbor of x. Let $z \in \mathrm{Med}(p)$ be a median allocation under p with maximal distance to x; without loss of generality, we may assume $d(x, z) \geq 1$. By Proposition 2, $z \notin \mathrm{Med}(\tilde{p})$ holds only if z looses against one of its neighbors under the distribution $\tilde{p}$. Partition the set of neighbors of z as follows,

$$\begin{aligned}
Z_x^- &= \{y \in X : yNz \text{ and } d(x, y) = d(x, z) - 1\},\\
Z_x^0 &= \{y \in X : yNz \text{ and } d(x, y) = d(x, z)\},\\
Z_x^+ &= \{y \in X : yNz \text{ and } d(x, y) = d(x, z) + 1\}.
\end{aligned}$$

Since the report of x already supports any element in Z_x^- in pairwise comparison against z under p, z cannot loose against such element under $\tilde{p}$. Moreover, since z has maximal distance to x among all median allocation under p, z wins (strictly) against all elements in Z_x^+. Since $\tilde{x} \notin \rangle z', z\langle$ for all $z' \in Z_x^+$, z cannot loose against any element in Z_x^+ under $\tilde{p}$. Thus, either $z \in \mathrm{Med}(\tilde{p})$, or z looses against one element of Z_x^0 under $\tilde{p}$, say $\tilde{z} \in Z_x^0$. Since $x \notin \rangle \tilde{z}, z\langle$ the latter is only possible if $\tilde{x} \in \rangle \tilde{z}, z\langle$ and $\tilde{z} I_p z$, hence $\tilde{z} \in \mathrm{Med}(p)$, i.e. $\tilde{z}$ was a median allocation under p. Furthermore, $\tilde{x} \in \rangle \tilde{z}, z\langle$ can hold only if x and z differ in exactly two coordinates. This, in turn, implies that reporting $\tilde{x}$ instead of x does not reduce the support of $\tilde{z}$ against any of its neighbors, i.e. $\tilde{z} \in \mathrm{Med}(\tilde{p})$. This proves that reporting a neighbor $\tilde{x}$ instead of x cannot reduce the maximal distance to a median allocation: either z itself remains a median allocation under $\tilde{p}$, or a neighbor $\tilde{z}$ just as distant as z from x was already a median allocation under p and remains one under $\tilde{p}$.

Now consider, for any $x' \in X$, the distribution p' which differs from p only in that individual i reports x' instead of x, and let z be any element in $\mathrm{Med}(p')$ with maximal distance to x. First, we assume that $x' \neq z$. Suppose that $d(x, z) \geq 1$ and consider any neighbor of $\tilde{x}$ with $d(x, \tilde{x}) = d(x, x') + 1$, i.e. $\tilde{x}$ is one of the neighbors of x' 'away from' x. Let $\tilde{p}$ be the distribution that differs from p' only in that individual i reports $\tilde{x}$ instead of x'. We will show that $d_{\max}(x, \mathrm{Med}(\tilde{p})) \geq d_{\max}(x, \mathrm{Med}(p'))$. This is immediate if $z \in \mathrm{Med}(\tilde{p})$, i.e. if z remains a median allocation under $\tilde{p}$. Thus, assume that $z \notin \mathrm{Med}(\tilde{p})$. As above, by Proposition 2, this implies that z looses against one of its neighbors under $\tilde{p}$. Partition the set of neighbors of z as above in Z_x^-, Z_x^0 and Z_x^+. Since $d(x, \tilde{x}) > d(x, x')$ the allocation z cannot loose against any element of Z_x^- under $\tilde{p}$. Moreover, since z has maximal distance to x among all elements of $\mathrm{Med}(p')$, z wins against any element of Z_x^+ under p'. Because the move from x' to its neighbor $\tilde{x}$ can change the difference in the support of z vis-á-vis any of its neighbors by at most one, z cannot loose against any element of Z_x^+ under $\tilde{p}$. Thus, if z looses against one of its neighbors under $\tilde{p}$, that neighbor, say $\tilde{z}$, must be an element of Z_x^0, and in fact a median allocation under the distribution p' by Lemma 2 (since $\tilde{z} I_{p'} z$). Since the move from x to $\tilde{x}$ does not decrease the difference in support of $\tilde{z}$ vis-á-vis any of its neighbors, we obtain $\tilde{z} \in \mathrm{Med}(\tilde{p})$. Since $d(x, \tilde{z}) = d(x, z)$, the desired conclusion follows.

Next, consider the case in which $x' = z$, and assume without loss of generality that z is in fact the single element of $\mathrm{Med}(p')$ with maximal distance to x (otherwise repeat the argument just given replacing z by another element of $\mathrm{Med}(p')$ with maximal distance to x). In this case, z wins against all elements of Z_x^0, hence $z \in \mathrm{Med}(\tilde{p})$, or z looses against $\tilde{x}$ (which must be an element of Z_x^+). In the latter case, we obtain $\mathrm{Med}(\tilde{p}) \cap \rangle z, \tilde{x} \langle = \emptyset$ by Lemma 4, and hence $d_{\max}(x, \mathrm{Med}(\tilde{p})) < d_{\max}(x, \mathrm{Med}(p'))$, as desired.

Inequality (5) now follows in full generality by induction. Indeed, consider any misreport $\tilde{x}$ and a shortest path from x to $\tilde{x}$. Travelling along this path increases the distance to x by one at each step. By the arguments just given, the maximal distance of a median allocation to x is not reduced at any single step, hence it is not reduced in total. $\square$

We conclude this subsection with the following observation.

Proposition 5 *Let $y \in \mathrm{Med}(p)$ for some distribution p with $x \in \mathrm{supp}(p)$. If the distribution p_{xy} differs from p only in that one individual submits y instead of x, then*

$$y \in \mathrm{Med}(p_{xy}) \quad \textit{and} \quad \mathrm{Med}(p_{xy}) \subseteq \mathrm{Med}(p).$$

Proof. Clearly, if y does not loose against any of its neighbors under the distribution p, it cannot loose against any of its neighbors under p_{xy} either; hence, by Proposition 2, $y \in \mathrm{Med}(p_{xy})$. Next, consider any $z \notin \mathrm{Med}(p)$ and any shortest path from y to z. For some neighbors w', w along this path we must have $w' M_p w$, i.e. w' wins against w in pairwise comparison. Suppose that $w' N_{jk} w$ and consider the neighbor z' of z in jk-direction, i.e. $z' N_{jk} z$. By Lemma 3, z' wins against z under p, i.e. $z' M_p z$. Moreover, we have $y \in \rangle z', z \langle$, hence also $z' M_{p_{xy}} z$. This implies $z \notin \mathrm{Med}(p_{xy})$, and thus by contraposition $\mathrm{Med}(p_{xy}) \subseteq \mathrm{Med}(p)$, as claimed. $\square$

5 Strategy-Proofness

The result of the previous section has the immediate implication that the median rule is strategy-proof provided that all individuals have 'metric' single-peaked preferences of the form $y \succeq z \Leftrightarrow d(x,y) \leq d(x,z)$ for some $x \in X$, and provided that strategy-proofness is defined with respect to any extended preference relation $\succeq^*$ over subsets of X satisfying

$$M \succeq^* M' \text{ whenever } [\min M \succeq \min M' \text{ and } \max M \succeq \max M'].$$

6 Results from simulation studies

We have shown that even though the median rule is not generally strategy-proof, the possibilities to influence the outcome are restricted such that an individual can move neither the closest nor the farthest median allocation closer to his or her own peak. To what extent is the median rule robust to manipulation under generalized single-peaked preferences? Analytical answers to this question are likely to be hard to obtain. Therefore, we study the behavior of the voters and its influence on the set of median allocations by simulation. We developed a simulation program using Java and examined several variables concerning the announced peaks and the resulting median allocations, see Lindner (2011, in german).

In a first step, the (true) peaks of the voters are drawn by a pseudo-random number generator. We assume that the voters have Cobb-Douglas preferences and parameters are set such that a voter's peak maximizes his or her utility on the simplex. Then, voters determine their best response, i.e. their utility-maximizing announced peak, given the announced peaks of the other voters. This is simulated both simultaneously and sequentially for several iterations. We say that a voting situation converges, if no voter changes his or her announced peak after a finite number of iterations. We also distinguish between scenarios where voters are optimizing (i) their best element in the midpoint set, (ii) simultaneously their best and their worst midpoint (dominance relation), (iii) the expected utility of the midpoint set (under a uniform distribution), and finally also (iv) using a Hurwicz criterion.

The behavior under the median rule is compared with the *mean rule*, i.e. where the mean of the announced peaks is chosen. Variables under consideration, for example, are the number of iterations, the number of manipulating voters and the sum of deviations of the announced peaks from the true ones. We also determine the Hausdorff distance (using the natural metric) between the resulting set of median alllocations and the set of median allocations under truthful voting. All of this is done for several combinations of voters, publics goods and budgets.

	convergence	number of iterations	agents manipulating	sum of deviations	(Hausdorff) distance
Midp.	76.73%	4.71	2.04	5.51	2.87
Mean	43.31%	11.07	4.98	83.52	5.57

Table 1: Results for 5 voters, 3 public goods and a budget of 50

Table 1 shows results for a voting situation with 3 publics goods and a budget of 50. Five voters, all having Cobb-Douglas preferences, try to improve their best element in the set of median allocations simultaneously. We took a sample with a size of 10000 and stopped a run after 15 iterations assuming that the voting situation does not converge in that case.

The first results from our simulation studies indicate that the median rule is only little vulnerable against manipulation: the fraction of voters who manipulate under the median rule is by far less than the same fraction under the mean rule; the same holds for the sum of deviations from the true peaks to the announced ones. Furthermore, the influence on the outcome of the voting rule under the median rule is approximately 5 per cent of the budget while it is twice as high under the mean rule.