Measuring Inequality with Ordinal data

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Outline

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Introduction

- Ordinal data issue widespread in inequality analysis
- Many applications proceed just as though cardinal:
  - life satisfaction / inequality of happiness: Oswald and Wu (2011), Stevenson and Wolfers (2008b), Yang (2008)
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  - but these have limitations
- Present approach based on Cowell and Flachaire (2014)

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Income Inequality

- 3 ingredients:
  - **"income":** family income, earnings, wealth $x \in X \subseteq \mathbb{R}$.
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- Obviously can’t do that here: μ is undefined

Utility

Cardinalisation and inequality

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  - Already pointed out in Atkinson (1970)
  - Dalton (1920) suggested inequality of (cardinal) utility
  - But if, for all i, you multiply u_i by $\lambda \in (0,1)$ and add $\delta = \mu[1-\lambda]$...
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- Is this just a technicality?
- Can we proceed just as with regular income?

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Categorical variable

Example: Access to Services

	Case 1	Case 2
	n_k	n_k
Both Gas and Electricity	25	0
Electricity only	25	50
Gas only	25	50
Neither	25	0

Categorical variable

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- It seems clear that Case 1 is more unequal than Case 2

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- Compare distributions across countries

SRH Results: four countries

	Austria	**UK**	**Mexico**	**Bangladesh**
	number of responses			
Very good	423	318	7193	494
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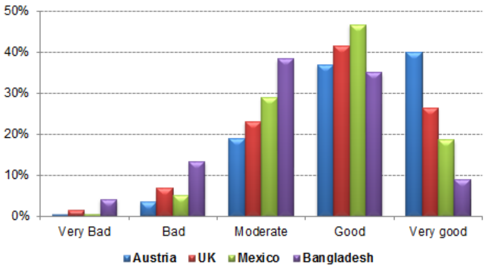
- For each country: compute freq distributions across categories
- How to evaluate inequality?

SRH Inequality: Gini



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(1,2,3,4,5)	0.111	0.130	0.116	0.154	(BD,UK,Mx,At)

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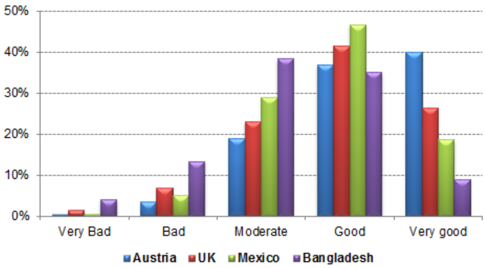

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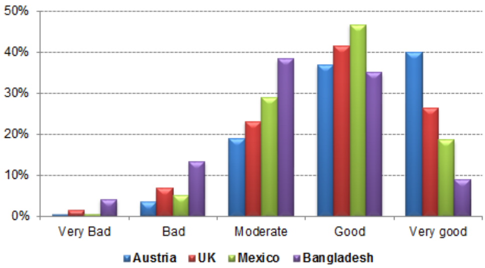

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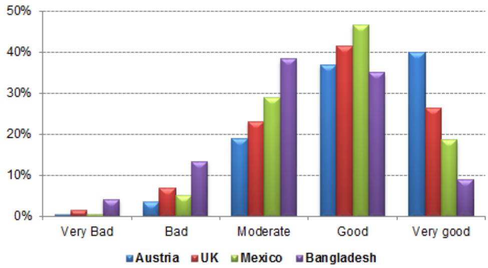
(-1000,2,3,4,5)	0.608	0.821	0.856	2.377	(BD,Mx,UK,At)

SRH Inequality: Coeff of Variation



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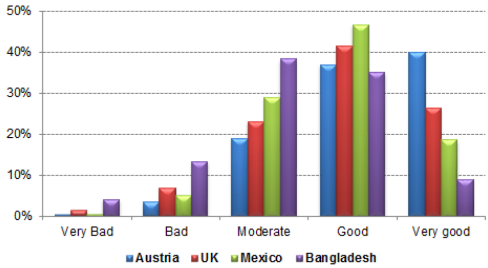

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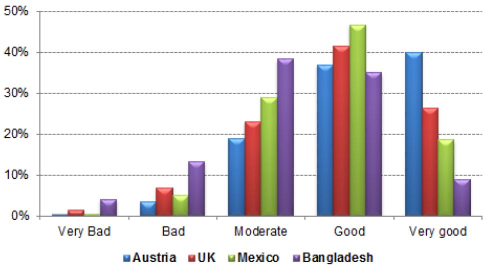

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(-1000,2,3,4,5)	187.5	11.43	40.45	5.264	(At,Mx,UK,BD)

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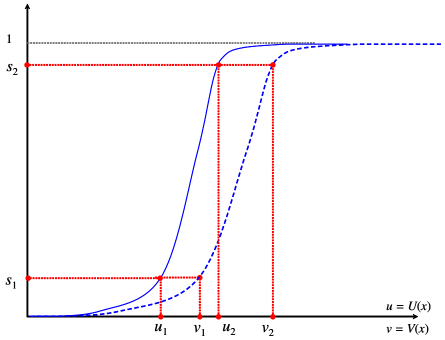
- In some cases defined given additional distribution-free information
  - example: if it is known that utility is $\log(x)$
- In some cases requires information on distribution
  - GRE, TOEFL
  - "opportunity" (de Barros et al. 2008)

Status and Distribution (1)

- i's status uniquely defined for a given distribution of u



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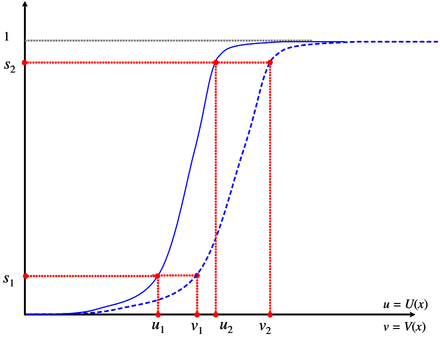

- disposes of the problem of cardinalisation
  - U and $V = \varphi(U)$ two cardinalisations of the utility of x
  - for each i:u_i and v_i map into s_i

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- Merger principle
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  - then this should leave i's status unaltered
- Principle implies that status should be additive in the n_k
  - downward-looking status: $\sum_{\ell=1}^{k(i)} n_\ell$
  - upward-looking status: $\sum_{\ell=k(i)}^{K} n_\ell$
  - see also Yitzhaki (1979)

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- Inequality: aggregate distance from e
  - don't need an explicit distance function
  - implicitly define through inequality ordering $\succeq$

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- **[Independence]** If $\mathbf{s}(\varsigma, i), \mathbf{s}'(\varsigma, i) \in S^n$ satisfy$(\mathbf{s}(\varsigma, i), e) \sim (\mathbf{s}'(\varsigma, i), e)$ for some ς then $(\mathbf{s}(\varsigma, i), e) \sim (\mathbf{s}'(\varsigma, i), e)$ for all ς

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- **[Anonymity]** For all $\mathbf{s} \in S^n$ and permutation matrix Π: $(\Pi \mathbf{s}, e) \sim (\mathbf{s}, e)$

Standard result

Theorem

Continuity, Monotonicity, Independence, Anonymity jointly imply $\succeq$ *is representable by the continuous function* $I : S_e^n \to \mathbb{R}$ *where* $I(\mathbf{s}; e) = \Phi\left(\sum_{i=1}^{n} d(s_i, e), e\right)$, *where* $d : S \to \mathbb{R}$ *is a continuous function that is strictly increasing (decreasing) in its first argument if* $s_i > e$ $(s_i < e)$.

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Corollary

Inequality is total "distance" from equality. Distance d is continuous. $d(s, e)$ *is increasing in status if you move away from the reference point.*

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Corollary

Inequality represented as $I_\alpha(\mathbf{s}; e) := \frac{1}{\alpha[\alpha-1]}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^{\alpha} - e^{\alpha}\right]$

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A usable inequality index?

- A *class* of functions available as inequality measures:
  - $\Phi(I_\alpha(\mathbf{s};e),e)$
  - $e=\eta(\mathbf{s})$, the reference point
  - $I_\alpha(\mathbf{s};e) := \frac{1}{\alpha[\alpha-1]}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - e^\alpha\right]$

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  - $I_\alpha(\mathbf{s};e) := \frac{1}{\alpha[\alpha-1]}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - e^\alpha\right]$
- Do functions $\Phi(I_\alpha(\mathbf{s};e), e)$ "look like" inequality measures?
  - transfer principle?
  - reference point?
  - sensitivity to parameters
- What is the appropriate form for Φ?
  - may depend on the reference status e
  - may depend on interpretation

Outline

Four distributional scenarios (1)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

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$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- n_k is # persons in category $k \in \{B, E, G, N\}$

Four distributional scenarios (1)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	3/4	50	1	25	3/4
G	25	1/2	25	1/2	50	1/2	50	1/2
N	25	1/4	25	1/4	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- n_k is # persons in category $k \in \{B, E, G, N\}$
- $s_i = \frac{1}{n} \sum_{\ell=1}^{k(i)} n_\ell$ – *downward*-looking status

Four distributional scenarios (1′)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s'_i	n_k	s'_i	n_k	s'_i	n_k	s'_i
B	0		25	$1/4$	0		25	$1/4$
E	50	$1/2$	25	$1/2$	50	$1/2$	25	$1/2$
G	25	$3/4$	25	$3/4$	50	1	50	1
N	25	1	25	1	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

Four distributional scenarios (1′)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s'_i	n_k	s'_i	n_k	s'_i	n_k	s'_i
B	0		25	1/4	0		25	1/4
E	50	1/2	25	1/2	50	1/2	25	1/2
G	25	3/4	25	3/4	50	1	50	1
N	25	1	25	1	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

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	Case 0		Case 1		Case 2		Case 3	
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E	50	$1/2$	25	$1/2$	50	$1/2$	25	$1/2$
G	25	$3/4$	25	$3/4$	50	1	50	1
N	25	1	25	1	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- n_k is # persons in category $k \in \{\mathrm{B}, \mathrm{E}, \mathrm{G}, \mathrm{N}\}$
- $s'_i = \frac{1}{n} \sum_{\ell=k(i)}^{K} n_\ell$ – *upward*-looking status

Four distributional scenarios (2)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- Case 0 to Case 1:

Four distributional scenarios (2)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- Case 0 to Case 1:
  - 25 people promoted from E to B
  - if e equals to any of values taken by $\mu(\mathbf{s})$
  - then inequality increases

Four distributional scenarios (3)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- Case 0 to Case 2:

Four distributional scenarios (3)

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- Case 0 to Case 2:
  - 25 people promoted from N to G
  - if e equals to any of values taken by $\mu(\mathbf{s})$
  - then inequality decreases

“Transfer Principle”?

"Transfer Principle"?

	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

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	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
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N	25	$1/4$	25	$1/4$	0		0	
$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

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	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
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$\mu(\mathbf{s})$		$11/16$		$5/8$		$3/4$		$11/16$

- Case 0 to Case 1: inequality increases
- Case 0 to Case 2: inequality decreases

"Transfer Principle"?

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	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$^3/_4$	50	1	25	$^3/_4$
G	25	$^1/_2$	25	$^1/_2$	50	$^1/_2$	50	$^1/_2$
N	25	$^1/_4$	25	$^1/_4$	0		0	
$\mu(\mathbf{s})$		$^{11}/_{16}$		$^5/_8$		$^3/_4$		$^{11}/_{16}$

- Case 0 to Case 1: inequality increases
- Case 0 to Case 2: inequality decreases
- Case 0 to Case 3: combination results in ambiguous change

Outline

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  - for categorical data, there is no counterpart to fixed-mean assumption in income-inequality analysis

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- **Min status**: $e = 0$
  - counterpart for peer-exclusive case

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- If $\alpha = 0$ then $I_0(\mathbf{s};e) = -\frac{1}{n}\sum_{i=1}^{n}\log s_i + \log e$,
- If $e = \mu(\mathbf{s})$ and $\alpha = 1$ then $\frac{1}{n}\sum_{i=1}^{n} s_i \log s_i - e\log e$

Behaviour of $I_0(\mathbf{s}; e)$

	Case 0	Case 1	Case 2	Case 3
$\mu(\mathbf{s})$	$11/16$	$5/8$	$3/4$	$11/16$
$\text{med}_1(\mathbf{s})$	$3/4$	$5/8$	$3/4$	$5/8$
$\text{med}_2(\mathbf{s})$	$1/2$	$1/2$	$1/2$	$1/2$
$I_0(\mathbf{s};\ \mu(\mathbf{s}))$	0.1451	0.1217	0.0588	0.0438
$I_0(\mathbf{s};\ \text{med}_1(\mathbf{s}))$	0.2321	0.1217	0.0588	-0.0515
$I_0(\mathbf{s};\ \text{med}_2(\mathbf{s}))$	-0.1732	-0.1013	-0.3465	-0.2746
$I_0(\mathbf{s};\ 1)$	0.5198	0.5917	0.3465	0.4184

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- $I_0(\mathbf{s};\ \mu(\mathbf{s}))$, $I_0(\mathbf{s};\ \text{med}_1(\mathbf{s}))$: inequality *decreases* for
  - Case 0 to 1, or Case 2 to 3
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$\mathrm{med}_1(\mathbf{s})$	$^{3}/_{4}$	$^{5}/_{8}$	$^{3}/_{4}$	$^{5}/_{8}$
$\mathrm{med}_2(\mathbf{s})$	$^{1}/_{2}$	$^{1}/_{2}$	$^{1}/_{2}$	$^{1}/_{2}$
$I_0(\mathbf{s};\ \mu(\mathbf{s}))$	0.1451	0.1217	0.0588	0.0438
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- $I_0(\mathbf{s};\ \mathrm{med}_2(\mathbf{s})) < 0$ for *all* cases in example!
- But $I_0(\mathbf{s}; 1)$ seems sensible

Inequality measure

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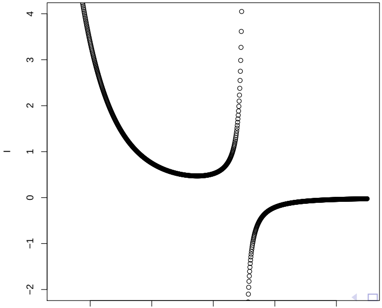
- $I_\alpha(\mathbf{s},1) = \begin{cases} \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - 1\right], & \text{if } \alpha \neq 0,\ \alpha<1 \\ -\frac{1}{n}\sum_{i=1}^{n}\log s_i. & \text{if } \alpha=0 \end{cases}$



Outline

Implementation

- Description of sample

$$x_i = \begin{cases} 1 & \text{with sample proportion } p_1 \\ 2 & \text{with sample proportion } p_2 \\ \dots & \\ K & \text{with sample proportion } p_K \end{cases},$$

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- Point estimate of the index:
  - $I_\alpha = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[\sum_{i=1}^{K} p_i \left[\sum_{j=1}^{i} p_j \right]^\alpha - 1 \right] & \text{if } \alpha \neq 0,1 \\ -\sum_{i=1}^{K} p_i \log \left[\sum_{j=1}^{i} p_j \right] & \text{if } \alpha = 0 \end{cases}$
  - function of K parameter estimates $(p_1, p_2, \dots, p_K)$ following a multinomial

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$$\Sigma = \frac{1}{n}\begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \ldots & -p_1p_K \\ -p_2p_1 & p_2(1-p_2) & \ldots & -p_2p_K \\ \vdots & \vdots & \vdots & \vdots \\ -p_Kp_1 & -p_Kp_2 & \ldots & p_K(1-p_K) \end{bmatrix}$$

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- $\widehat{\text{Var}}(I_\alpha) = D\Sigma D^\top \qquad \text{with} \qquad D = \left[\frac{\partial I_\alpha}{\partial p_1};\ \frac{\partial I_\alpha}{\partial p_2};\dots;\ \frac{\partial I_\alpha}{\partial p_K}\right]$
- $\frac{\partial I_\alpha}{\partial p_l} = \frac{1}{\alpha(\alpha-1)}\left(\left[\sum_{i=1}^{l} p_i\right]^\alpha + \alpha\sum_{i=l}^{K-1} p_i\left[\sum_{j=1}^{i} p_j\right]^{\alpha-1}\right), \alpha \neq 0$
- $\frac{\partial I_0}{\partial p_l} = -\log\left[\sum_{j=1}^{l} p_j\right] - \sum_{i=l}^{K-1} p_i\left[\sum_{j=1}^{i} p_j\right]^{-1}$

Outline

Confidence Intervals

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- 3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized Bootstrap
- $CI_{asym} = [I_\alpha - c_{0.975}\widehat{\text{Var}}(I_\alpha)^{1/2}; I_\alpha + c_{0.975}\widehat{\text{Var}}(I_\alpha)^{1/2}]$
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- Bootstraps: generate resamples, $b = 1, \ldots, B$
  - for each resample b compute the inequality index
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  - also B bootstrap t-statistics $t_\alpha^b = (I_\alpha^b - I_\alpha)/\widehat{\text{Var}}(I_\alpha^b)^{1/2}$

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- $CI_{stud} = [I_\alpha - c^*_{0.975}\widehat{\mathrm{Var}}(I_\alpha)^{1/2}\,;\ I_\alpha - c^*_{0.025}\widehat{\mathrm{Var}}(I_\alpha)^{1/2}]$
  - $c^*_{0.025}$ and $c^*_{0.975}$ are from EDF of the bootstrap t-statistics

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  - coverage error rate should be close to nominal rate, 0.05
- Check coverage error rate of CIs as sample size increases
  - $\alpha = -1, 0, 0.5, 0.99$
  - 199 bootstraps
  - 10 000 replications to compute error rates
  - $n = 20, 50, 100, 200, 500, 1000$

Estimation Methods Compared

	α	-1	0	0.5	0.99
Asymptotic B	$n = 20$	0.0606	0.0417	0.0598	0.0491
	$n = 500$	0.0523	0.0492	0.0521	0.0523
	$n = 1000$	0.0485	0.0540	0.0552	0.0549
Percentile B	$n = 20$	0.0384	0.0981	0.0912	0.1023
	$n = 500$	0.0509	0.0513	0.0552	0.0554
	$n = 1000$	0.0482	0.0556	0.0547	0.0551
Studentized B	$n = 20$	0.1275	0.0843	0.1041	0.1377
	$n = 500$	0.0518	0.0478	0.0429	0.0465
	$n = 1000$	0.0473	0.0522	0.0493	0.0503

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	$n = 1000$	0.0485	0.0540	0.0552	0.0549
Percentile B	$n = 20$	0.0384	0.0981	0.0912	0.1023
	$n = 500$	0.0509	0.0513	0.0552	0.0554
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Studentized B	$n = 20$	0.1275	0.0843	0.1041	0.1377
	$n = 500$	0.0518	0.0478	0.0429	0.0465
	$n = 1000$	0.0473	0.0522	0.0493	0.0503

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- Asymptotic CIs perform OK in finite sample
- Percentile bootstrap performs well for $n > 50$
- Studentized bootstrap does not do well for small samples
- Reliable results for $\alpha = 0.99$ (index is undefined for $\alpha = 1$)

Outline

World Values Survey

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- Life satisfaction question:

All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are "completely dissatisfied" and 10 means you are "completely satisfied" where would you put your satisfaction with your life as a whole? (code one number):

Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied

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Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied

- Health question:

All in all, how would you describe your state of health these days? Would you say it is (read out):

1 Very good, 2 Good, 3 Fair, 4 Poor.

GDP and Life satisfaction

- Cross-country comparison of life satisfaction and GDP/head
  - happiness-income paradox (Easterlin 1974, Clark and Senik 2011)
  - weak relation happiness-income internationally? (Easterlin 1995, Easterlin et al. 2010)
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  - simple linearity of Likert scale? or exponential scale?
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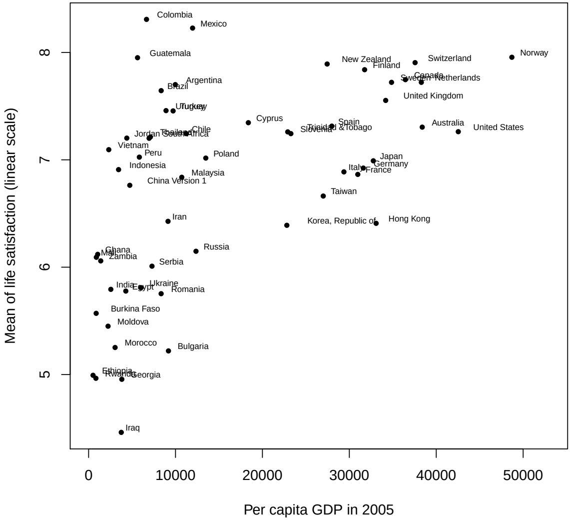
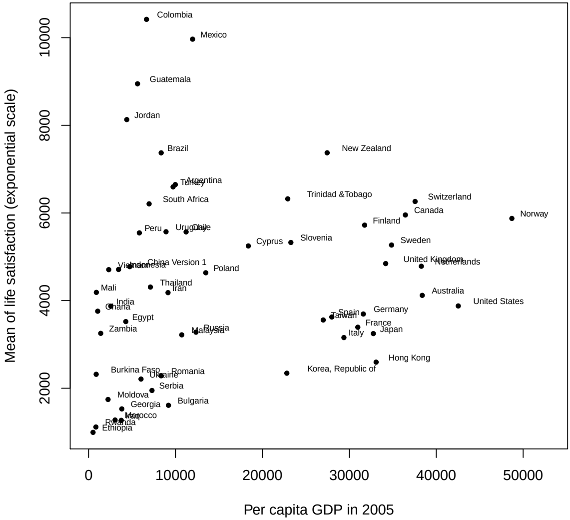
  - Use I_0 and other members of the same family

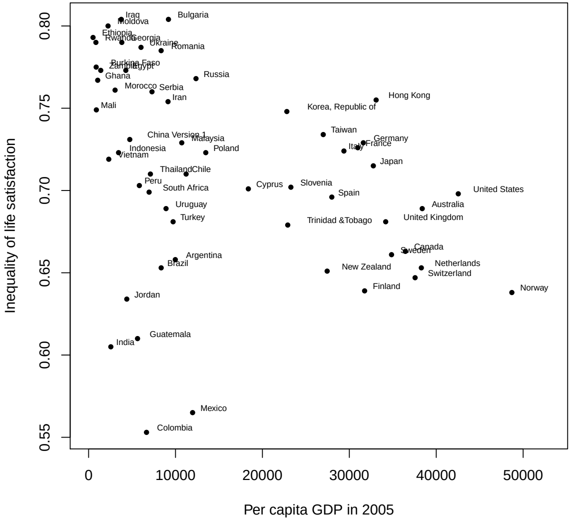
GDP and Life satisfaction (Linear)



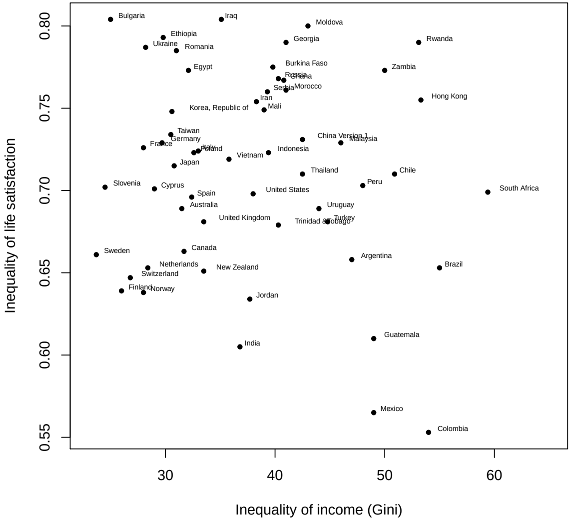
GDP and Life satisfaction (Exponential)



GDP and Inequality of Life satisfaction



Income inequality and Inequality of Life satisfaction



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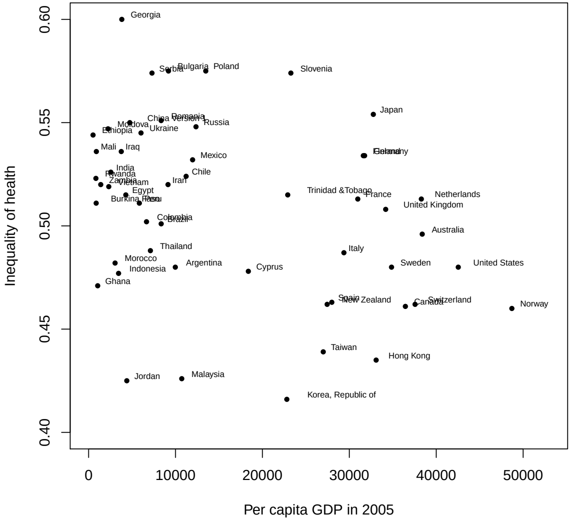
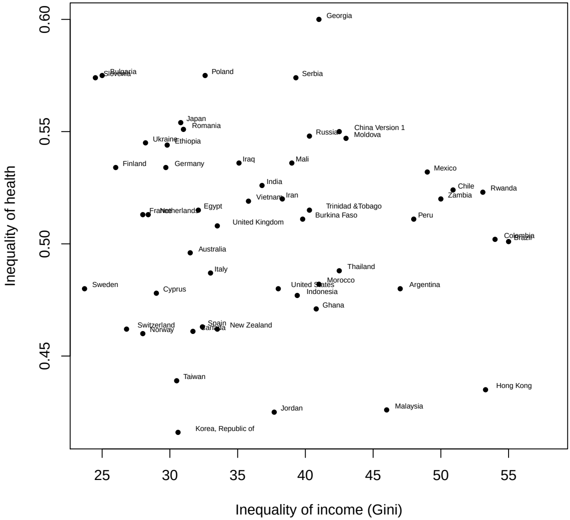
- Cross-country comparison of inequality of health and Inequality of life satisfaction
  - use same inequality index as for life satisfaction

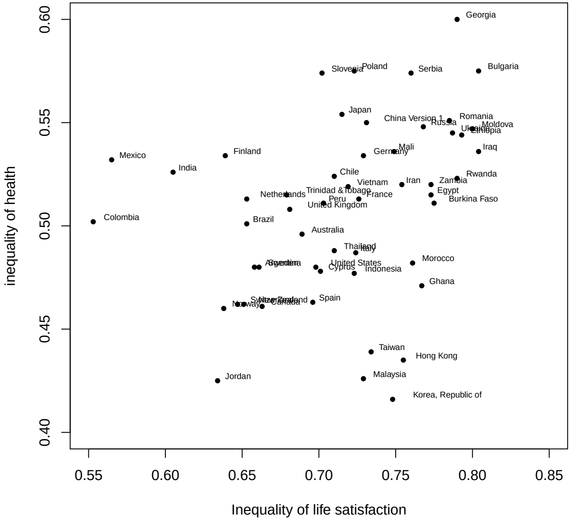
GDP and Inequality of health



Income inequality and health inequality



Inequality of life satisfaction and health inequality



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- No clear relationship between I_0(health) on GDP per capita
- OLS estimate of I_0(health) on I_0(life satisfaction) produces a slope coefficient not significantly different from zero
  - health-life satisfaction relationship is not significant

Summary

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Summary

- Inequality with ordinal data is a widespread phenomenon
- Conventional *I*-measures may make no sense
- Cowell and Flachaire (2014) approach:
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  - allows you to choose “reference status”
  - gives a family of measures
- Nice properties empirically

Summary_

Outline

Median: definition and in our cases

- $\text{med}(\mathbf{s})$ defined as $e \in S$ such that $\#(s_i \leq e) \geq \frac{n}{2}, \#(s_i \geq e) \geq \frac{n}{2}$

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	Case 0		Case 1		Case 2		Case 3	
	n_k	s_i	n_k	s_i	n_k	s_i	n_k	s_i
B	0		25	1	0		25	1
E	50	1	25	$3/4$	50	1	25	$3/4$
G	25	$1/2$	25	$1/2$	50	$1/2$	50	$1/2$
N	25	$1/4$	25	$1/4$	0		0	
$M(\mathbf{s})$	$[1/2, 1)$		$[1/2, 3/4)$		$[1/2, 1)$		$[1/2, 3/4)$	

- med(**s**) could be any value in interval $M(\mathbf{s})$

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- The status vector is $\mathbf{s} = ({}^{1}\!/_{3}, {}^{2}\!/_{3}, 1)$
- conventional definition is $\text{med}(\mathbf{s}) = m := {}^{2}\!/_{3}$:
  - ${}^{2}\!/_{3}$ of the population has a status less or equal to m
  - ${}^{2}\!/_{3}$ of the population has a status greater than or equal to m
- Median as “half-way” point is misleading

Median example 2

- Two ordered categories (B better than A)
- Three distributions

1. $n_{\mathrm{A}} = 500,\ n_{\mathrm{B}} = 500$
2. $n_{\mathrm{A}} = 499,\ n_{\mathrm{B}} = 501$
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  1. $n_A = 500,\ n_B = 500$
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- Status and median in each case:
  1. $\mathbf{s} = (0.5, 1)$, $\text{med}(\mathbf{s}) = 0.5$
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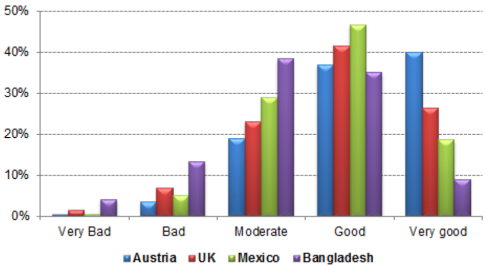
SRH Inequality: Gini (median norm'd)



	At	UK	Mx	BD	
(1,2,3,4,5)	0.107	0.135	0.123	0.140	(BD,UK,Mx,At)*

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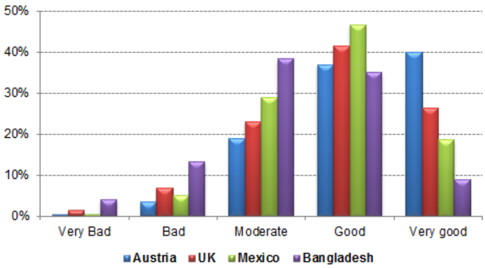
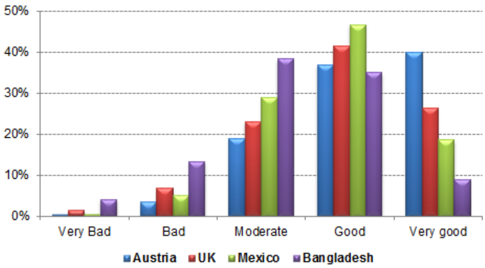


	At	UK	Mx	BD	
(1,2,3,4,5)	0.107	0.135	0.123	0.140	(BD,UK,Mx,At)*
(1,2,3,4,1000)	0.006	0.011	0.017	0.029	(BD,Mx,UK,At)*

SRH Inequality: Gini (median norm’d)



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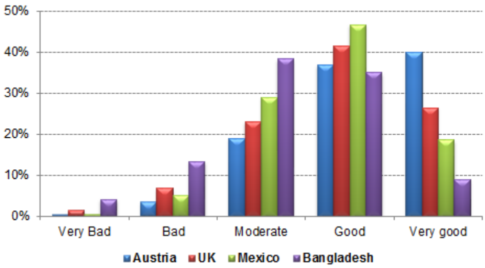
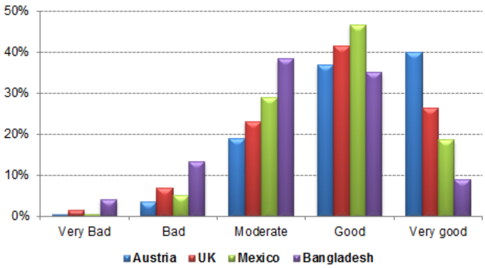
(1,2,3,4,1000)	0.006	0.011	0.017	0.029	(BD,Mx,UK,At)*
(-1000,2,3,4,5)	7.39	-0.315	-1.844	-0.188	(At,Mx,UK,BD)

SRH Inequality: C of V (median norm'd)



		At	UK	Mx	BD	
(1,2,3,4,5)		0.202	0.253	0.232	0.260	(BD,UK,Mx,At) *

SRH Inequality: C of V (median norm'd)



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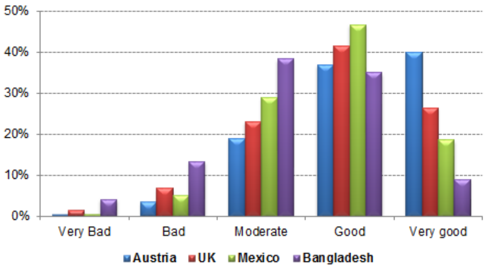

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(1,2,3,4,1000)	0.012	0.024	0.044	0.101	(BD,Mx,UK,At)*
(-1000,2,3,4,5)	2276	-4.39	-87.2	-0.42	(At,BD,UK,Mx)

Outline

Proof of Theorem 1

- Two cases to consider
  - data are categorical: S is set of non-negative rational numbers, $\mathbb{Q}_+$.
  - data have cardinal significance: S can be taken as an interval in $\mathbb{R}$.
- In either case $(S, +, \succ)$ forms a strictly ordered group (Krantz 1964, Luce and Tukey 1964, Wakker 1988)
- From Theorem 5.3 of Fishburn (1970) Axioms jointly imply that, for a given e, $\succeq$ is representable by a continuous function $S^{n+1} \to \mathbb{R}$:$\sum_{i=1}^{n} d_i(s_i, e), \forall (\mathbf{s}, e) \in S^{n+1}$where, for each i, $d_i : S \to \mathbb{R}$ is a continuous function.
- By monotonicity this is increasing in s_i if $s_i > e$ and *vice versa*.
- By anonymity the functions d_i must all be identical
- ordering $\succeq$ is also representable any monotonic transform

"Maximum inequality"

- Take the case where status is downward-looking and peer-inclusive
- Suppose that the status of each member of category k is s
- If a person is promoted from category k to category $k+1$
  - status increases to $s + n_{k+1}/n$
  - status of each of the remaining $n_k - 1$ members of category k falls to $s - 1/n$.
- The resulting change in inequality iproportional to $\left[d\left(s+\frac{n_{k+1}}{n},e\right)-d\left(s,e\right)\right]+\left[n_k-1\right]\left[d\left(s-\frac{1}{n},e\right)-d\left(s,e\right)\right]$
- If d is differentiable then this expression is approximately $d'\left(s,e\right)\frac{n_{k+1}}{n}-\frac{n_k-1}{n}d'\left(s,e\right)$
  - which equals $\frac{1}{n}d'\left(s,e\right)\left[n_{k+1}-n_k+1\right]$.
- If $s < e$ then monotonicity implies $d'\left(s,e\right) < 0$
  - the change in inequality is negative if $n_{k+1} \geq n_k$.

Outline

Dispersion

- Model: $\text{LifeSatisf}_i = \alpha + \beta\,\text{GDP}_i + \varepsilon_i, \qquad \varepsilon_i \sim N(0, \sigma_i^2)$
  - β is a significant coefficient and R^2 is large
  - strong (linear) relationship between LifeSatisf and GDP
- If LifeSatisf equation is homoskedastic:
  - no relationship between GDP and the dispersion of LifeSatisf
  - whatever is GDP, the dispersion of LifeSatisf is the same
- If LifeSatisf equation heteroskedastic dispersion of LifeSatisf may or may not be related to GDP
  - the form of the heteroskedasticity cannot be deduced from the relationship between the dependent variable and the covariate.
- If every i has different GDP, σ_i^2 measures the dispersion of LifeSatisf for i
- taking the measure I_0 as a measure of dispersion, the same reasoning applies

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