Measuring Inequality with Ordinal data

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Università di Verona: Alba di Canazei
Winter School

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Outline

Motivation
  Introduction and Previous work
  Basics
  Examples

Approach
  Model
  Characterisation

Inequality Measures
  Main properties
  Example
  Reference point and sensitivity

Empirical aspects
  Implementation
  Performance
  Application

Summary

References
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Summary
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- Ordinal data issue widespread in inequality analysis
- Many applications proceed just as though cardinal:
  - health status: Van Doorslaer and Jones (2003)
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- Small literature that takes ordinal problem seriously
  - early approaches using 1st order dominance, the median
  - but these have limitations
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• Many applications proceed just as though cardinal:
  • health status: Van Doorslaer and Jones (2003)

• Small literature that takes ordinal problem seriously
  • early approaches using 1st order dominance, the median
  • but these have limitations

• Present approach based on Cowell and Flachaire (2014)
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Summary
Income Inequality

• 3 ingredients:

  • “income”: family income, earnings, wealth $x \in X \subseteq \mathbb{R}$.
  • “income-receiving unit”: $n$ persons
  • method of aggregation: function $X^n \rightarrow \mathbb{R}$
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- Obviously can’t do that here: $\mu$ is undefined
Utility
Cardinalisation and inequality

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- But just assuming cardinal utility is no use
  - Already pointed out in Atkinson (1970)
  - Dalton (1920) suggested inequality of (cardinal) utility
  - But if, for all \( i \), you multiply \( u_i \) by \( \lambda \in (0, 1) \) and add \( \delta = \mu [1 - \lambda] \)... 
  - ...this will automatically reduce measured inequality.
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• Is this just a technicality?
• Can we proceed just as with regular income?
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Categorical variable
Example: Access to Services

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- Compare distributions across countries
## SRH Results: four countries

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• For all countries: rank categories in order
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• How to evaluate inequality?
SRH Inequality: Gini

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<th>Mx</th>
<th>BD</th>
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| (1,2,3,4,5)    | 0.111| 0.130| 0.116| 0.154| (BD, UK, Mx, At)
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(BD,UK,Mx,At)
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<td>(-1000,2,3,4,5)</td>
<td>0.608</td>
<td>0.821</td>
<td>0.856</td>
<td>2.377</td>
<td>BD,Mx,UK,At</td>
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SRH Inequality: Coeff of Variation

- At: 0.209
- UK: 0.244
- Mx: 0.219
- BD: 0.287

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<td>Bad</td>
<td>1.210</td>
<td>1.638</td>
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Status and Information

• Step 1 is to define status
  • depends on the purpose of inequality analysis
  • depends on structure of information
  • conventional inequality approach only works in narrowly defined information structure

• In some cases a person's status is self-defining
  • income
  • wealth

• In some cases defined given additional distribution-free information
  • example: if it is known that utility is log (x)

• In some cases requires information on distribution
  • GRE, TOEFL
  • “opportunity” (de Barros et al. 2008)
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• *i*'s status uniquely defined for a given distribution of *u*
• *i*’s status uniquely defined for a given distribution of *u*

![Graph Showing Status and Distribution](image)

• disposes of the problem of cardinalisation
  • *U* and \( V = \varphi(u) \) two cardinalisations of the utility of *x*
  • for each *i*: \( u_i \) and \( v_i \) map into \( s_i \)
### Status and distribution (2)

This approach works for categorical data. We just have an ordered arrangement of categories $1, 2, \ldots, k, \ldots, K$ and the numbers in each category $n_1, n_2, \ldots, n_k, \ldots, n_K$.

**Merger principle**
- Merge two adjacent categories that are irrelevant for $i$.
- Then this should leave $i$'s status unaltered.

**Principle implies that status should be additive in the numbers** $n_k$.
- Downward-looking status: $\sum_{k=1}^{\ell} (i) = n_\ell$.
- Upward-looking status: $\sum_{k=\ell+1}^{K} (i) = n_\ell$.

See also Yitzhaki (1979).
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  - status determined from utility?
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  - could also depend on status vector $e = \eta(s)$
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- Inequality: aggregate distance from $e$
  - don’t need an explicit distance function
  - implicitly define through inequality ordering $\succeq$
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Basic Axioms

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- **[Monotonicity]** If $s, s' \in S^n$ differ only in their $i$th component then (a) if $s'_i \geq e : s_i > s'_i \iff (s, e) \succ (s', e)$; (b) if $s'_i \leq e$: $\iff (s, e) \succ (s', e)$
Basic Axioms

- **[Continuity]** \( \succeq \) is continuous on \( S^{n+1} \)

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- **[Independence]** If \( s(\zeta, i), s'(\zeta, i) \in S^n \) satisfy \( s(\zeta, i), e) \sim (s'(\zeta, i), e) \) for some \( \zeta \) then \( s(\zeta, i), e) \sim (s'(\zeta, i), e) \) for all \( \zeta \)
Basic Axioms

- **[Continuity]** \( \succeq \) is continuous on \( S^{n+1} \)

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- **[Anonymity]** For all \( s \in S^n \) and permutation matrix \( \Pi \):
  \( (\Pi s, e) \sim (s, e) \)
Standard result

Theorem

Continuity, Monotonicity, Independence, Anonymity jointly imply $\succeq$ is representable by the continuous function $I : S^n_e \to \mathbb{R}$ where $I(s; e) = \Phi(\sum_{i=1}^{n} d(s_i, e), e)$, where $d : S \to \mathbb{R}$ is a continuous function that is strictly increasing (decreasing) in its first argument if $s_i > e$ ($s_i < e$).
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Continuity, Monotonicity, Independence, Anonymity jointly imply \( \succeq \) is representable by the continuous function \( I : S^n_e \rightarrow \mathbb{R} \) where

\[
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\]

where \( d : S \rightarrow \mathbb{R} \) is a continuous function that is strictly increasing (decreasing) in its first argument if \( s_i > e \) \( (s_i < e) \).

Corollary

Inequality is total “distance” from equality. Distance \( d \) is continuous. \( d(s, e) \) is increasing in status if you move away from the reference point.
Structure Theorem

- We need more structure on the problem
We need more structure on the problem

[Scale invariance 1] For all $\lambda \in \mathbb{R}_+$: if $s, s', \lambda s, \lambda s' \in S^n$ and $e, e' \in S$ then $(s, e) \sim (s', e') \Rightarrow (\lambda s, e) \sim (\lambda s', e')$.

[Scale invariance 2] For all $\lambda \in \mathbb{R}_+$: if $s, s', \lambda s, \lambda s' \in S^n$ and $e, e', \lambda e, \lambda e' \in S$ then $(s, e) \sim (s', e') \Rightarrow (\lambda s, \lambda e) \sim (\lambda s', \lambda e')$.
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**Theorem**

*Impose also Scale irrelevance 1. Then* \( d(s, e) = A(e)s^{\alpha(e)} \)
Structure Theorem

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**Theorem**

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**Theorem**

*Impose instead Scale Invariance 2. Then* $d(s, e) = e^\beta \phi \left( \frac{s}{e} \right)$ *where* $\beta$ *is a constant and* $\phi$ *is arbitrary*
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**Theorem**

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**Corollary**

*Inequality represented as* $I^\alpha(s; e) := \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - e^\alpha \right]$
Outline

Motivation
  Introduction and Previous work
  Basics
  Examples

Approach
  Model
  Characterisation

Inequality Measures
  Main properties
  Example
  Reference point and sensitivity

Empirical aspects
  Implementation
  Performance
  Application

Summary
A usable inequality index?

- A class of functions available as inequality measures:
  - $\Phi(I_\alpha(s; e), e)$
  - $e = \eta(s)$, the reference point
  - $I_\alpha(s; e) := \frac{1}{\alpha(\alpha-1)} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - e^\alpha \right]$
A usable inequality index?

- A *class* of functions available as inequality measures:
  - \( \Phi(I_\alpha(s; e), e) \)
  - \( e = \eta(s) \), the reference point
  - \( I_\alpha(s; e) := \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - e^\alpha \right] \)

- Do functions \( \Phi(I_\alpha(s; e), e) \) “look like” inequality measures?
  - transfer principle?
  - reference point?
  - sensitivity to parameters

- What is the appropriate form for \( \Phi \)?
  - may depend on the reference status \( e \)
  - may depend on interpretation
Outline

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Summary
### Four distributional scenarios (1)

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<tr>
<th>Case</th>
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<th>$s_i$</th>
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| $\mu(s)$  | 11/16 | 5/8   | 3/4   | 11/16 |

- $n_k$ is the number of persons in category $k$.
- $s_i$ is the $i$-th downward-looking status.

$\mu(s)$ represents the average downward-looking status.
### Four distributional scenarios (1)

<table>
<thead>
<tr>
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</table>

$\mu(s) = \frac{11}{16}$  \quad $= \frac{5}{8}$  \quad $= \frac{3}{4}$  \quad $= \frac{11}{16}$

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$
## Four distributional scenarios (1)

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\[ \mu(s) = \frac{11}{16} \quad \frac{5}{8} \quad \frac{3}{4} \quad \frac{11}{16} \]

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$
- $s_i = \frac{1}{n} \sum_{\ell=1}^{k(i)} n_\ell$ – *downward*-looking status
Four distributional scenarios (1')

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<td>$\mu(s)$</td>
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$n_k$ is the number of persons in category $k$, $s'_i$ is the upward-looking status.
### Four distributional scenarios (1')

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\[
\mu(s) = \begin{cases} 
11/16 & \text{Case 0} \\
5/8 & \text{Case 1} \\
3/4 & \text{Case 2} \\
11/16 & \text{Case 3}
\end{cases}
\]

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$
Four distributional scenarios (1')

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$\mu(s) = \frac{11}{16}$, $5/8$, $3/4$, $\frac{11}{16}$

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$

- $s_i' = \frac{1}{n} \sum_{\ell=k(i)}^{K} n_\ell$ – upward-looking status
## Four distributional scenarios (2)

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<td>$\mu(s)$</td>
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</table>

- Case 0 to Case 1:
  
  • 25 people promoted from E to B
  
  • if $e$ equals to any of values taken by $\mu(s)$
  
  then inequality increases
Four distributional scenarios (2)

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$\mu(s)$ 11/16 5/8 3/4 11/16

- Case 0 to Case 1:
  - 25 people promoted from E to B
  - if $e$ equals to any of values taken by $\mu(s)$
  - then inequality increases
## Four distributional scenarios (3)

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\[ \mu(s) = \frac{11}{16}, \frac{5}{8}, \frac{3}{4}, \frac{11}{16} \]

- Case 0 to Case 2:
### Four distributional scenarios (3)

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| $\mu(s)$ | 11/16 | 5/8 | 3/4 | 11/16 |

- Case 0 to Case 2:
  - 25 people promoted from N to G
  - if $e$ equals to any of values taken by $\mu(s)$
  - then inequality decreases
“Transfer Principle”?
“Transfer Principle”?

<table>
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<td>25</td>
<td>1/4</td>
<td>25</td>
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</tr>
<tr>
<td>$\mu(s)$</td>
<td></td>
<td>11/16</td>
<td></td>
<td>5/8</td>
</tr>
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- Case 0 to Case 1: inequality increases
- Case 0 to Case 2: inequality decreases
- Case 0 to Case 3: combination results in ambiguous change
“Transfer Principle”? 

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</tr>
<tr>
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<td>0</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>50</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td><strong>G</strong></td>
<td>25 1/2</td>
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</tr>
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\[ \mu(s) = \frac{11}{16}, \quad \frac{5}{8}, \quad \frac{3}{4}, \quad \frac{11}{16} \]

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\[
\mu(s) = \begin{cases} 
11/16 & \text{Case 0 to Case 1: inequality increases} \\
5/8 & \text{Case 0 to Case 2: inequality decreases} \\
3/4 \\
11/16
\end{cases}
\]
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$\mu(s)$

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   Basics
   Examples

Approach
   Model
   Characterisation

Inequality Measures
   Main properties
   Example
   Reference point and sensitivity

Empirical aspects
   Implementation
   Performance
   Application

Summary

References
Reference point
Reference point

- **Mean status:** $e = \eta(s) = \mu(s)$
  - for continuous distributions will equal 0.5
  - for categorical data, there is no counterpart to fixed-mean assumption in income-inequality analysis
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- **Median status:** \( e = \eta (s) = \text{med}(s) \)
  - not well-defined: any value in interval \( M(s) \)
  - \( M(s) = [\frac{1}{2}, 1) \) in cases 0 and 2
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- **Max status:** \( e = 1 \)
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- **Min status:** \( e = 0 \)
  - counterpart for peer-exclusive case
Sensitivity

- $\alpha$ captures the sensitivity of measured inequality
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• If $\alpha$ is high $I_\alpha (s; e) = \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^{\alpha} - e^{\alpha} \right]$, sensitive to high status-inequality
Sensitivity

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**Sensitivity**

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- If $\alpha = 0$ then $I_0(s; e) = -\frac{1}{n} \sum_{i=1}^{n} \log s_i + \log e$,

- If $e = \mu(s)$ and $\alpha = 1$ then $\frac{1}{n} \sum_{i=1}^{n} s_i \log s_i - e \log e$
## Behaviour of $I_0(s; e)$

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu(s)$</th>
<th>$\text{med}_1(s)$</th>
<th>$\text{med}_2(s)$</th>
<th>$I_0(s; \mu(s))$</th>
<th>$I_0(s; \text{med}_1(s))$</th>
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<th>$I_0(s; 1)$</th>
</tr>
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<tr>
<td><strong>Case 0</strong></td>
<td>$11/16$</td>
<td>$3/4$</td>
<td>$1/2$</td>
<td>0.1451</td>
<td>0.2321</td>
<td>-0.1732</td>
<td>0.5198</td>
</tr>
<tr>
<td><strong>Case 1</strong></td>
<td>$5/8$</td>
<td>$5/8$</td>
<td>$1/2$</td>
<td>0.1217</td>
<td>0.1217</td>
<td>-0.1013</td>
<td>0.5917</td>
</tr>
<tr>
<td><strong>Case 2</strong></td>
<td>$3/4$</td>
<td>$5/8$</td>
<td>$1/2$</td>
<td>0.0588</td>
<td>0.0588</td>
<td>-0.3465</td>
<td>0.3465</td>
</tr>
<tr>
<td><strong>Case 3</strong></td>
<td>$11/16$</td>
<td>$5/8$</td>
<td>$1/2$</td>
<td>0.1217</td>
<td>0.0438</td>
<td>-0.2746</td>
<td>0.4184</td>
</tr>
</tbody>
</table>

- $I_0(s; \mu(s))$, $I_0(s; \text{med}_1(s))$: inequality decreases for Case 0 to 1, or Case 2 to 3.
- Movement changes both the $\mu(s)$ and med$_1(s)$ ref points.
- $I_0(s; \text{med}_2(s)) < 0$ for all cases in example!
- But $I_0(s; 1)$ seems sensible.
## Behaviour of $I_0(s; e)$

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</tr>
<tr>
<td>$med_2(s)$</td>
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Inequality measure

- For ordinal data, peer-inclusive status
Inequality measure

- For ordinal data, peer-inclusive status

\[ I_\alpha(s, 1) = \begin{cases} 
\frac{1}{\alpha(\alpha-1)} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - 1 \right], & \text{if } \alpha \neq 0, \alpha < 1 \\
- \frac{1}{n} \sum_{i=1}^{n} \log s_i. & \text{if } \alpha = 0 
\end{cases} \]
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Summary
Implementation

- Description of sample

\[
x_i = \begin{cases} 
1 & \text{with sample proportion } p_1 \\
2 & \text{with sample proportion } p_2 \\
\vdots \\
K & \text{with sample proportion } p_K 
\end{cases}
\]
Implementation

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1 & \text{with sample proportion } p_1 \\
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\vdots \\
K & \text{with sample proportion } p_K 
\end{cases}
\]

- Point estimate of the index:

\[
I_\alpha = \begin{cases} 
\frac{1}{\alpha(\alpha-1)} \left[ \sum_{i=1}^{K} p_i \left[ \sum_{j=1}^{i} p_j \right]^\alpha \right] - 1 & \text{if } \alpha \neq 0,1 \\
- \sum_{i=1}^{K} p_i \log \left[ \sum_{j=1}^{i} p_j \right] & \text{if } \alpha = 0
\end{cases}
\]

- function of \( K \) parameter estimates \((p_1, p_2, \ldots, p_K)\) following a multinomial
Asymptotics

• From the CLT $I_\alpha$ is asymptotically Normally distributed

• Estimator of cov matrix of $(p_1, p_2, ..., p_k)$ is $\Sigma = \frac{1}{n} \begin{bmatrix} p_1(1-p_1) & \dots & -p_1p_k \\ \vdots & \ddots & \vdots \\ -p_1p_k & \dots & \frac{1}{n} \end{bmatrix}$

• $\hat{\text{Var}}(I_\alpha) = D \Sigma D^\top$ with $D = [\frac{\partial I_\alpha}{\partial p_1}; \frac{\partial I_\alpha}{\partial p_2}; \dots; \frac{\partial I_\alpha}{\partial p_k}]$.

• $\frac{\partial I_\alpha}{\partial p_l} = \frac{1}{\alpha} \left( \frac{\alpha-1}{\sum_{i=1}^l p_i} + \alpha \sum_{i=l+1}^K p_i \left( \sum_{j=1}^K p_j \right)^{-1} \right)$, $\alpha \neq 0$

• $\frac{\partial I_0}{\partial p_l} = -\log \left( \sum_{j=1}^n p_j - \sum_{i=l+1}^K p_i \left( \sum_{j=1}^K p_j \right)^{-1} \right)$
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 -p_2p_1 & p_2(1-p_2) & \cdots & -p_2p_k \\
 \vdots & \vdots & \ddots & \vdots \\
 -p_{Kp} & -p_{Kp_2} & \cdots & p_K(1-p_K)
\end{bmatrix}
$$
Asymptotics

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- $\frac{\partial I_\alpha}{\partial p_l} = \frac{1}{\alpha(\alpha-1)} \left( \left[ \sum_{i=1}^l p_i \right]^{\alpha} + \alpha \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{\alpha-1} \right), \alpha \neq 0

- $\frac{\partial I_0}{\partial p_l} = -\log \left[ \sum_{j=1}^l p_j \right] - \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{-1}$
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Confidence Intervals

- **Asymptotic CI**
  
  \[ I_{\alpha} - c_{0.975} \hat{\text{Var}}(I_{\alpha})^{1/2}; I_{\alpha} + c_{0.975} \hat{\text{Var}}(I_{\alpha})^{1/2} \]

  - \( c_{0.975} \) from the Student distribution \( T(n-1) \)

- **Bootstrap CIs**
  
  - Generate resamples, \( b = 1, \ldots, B \)
  
  - For each resample, compute the inequality index
  
  - Obtain \( B \) bootstrap statistics, \( I_b \alpha \)
  
  - Also \( B \) bootstrap \( t \)-statistics

\[ t_b \alpha = \frac{I_b \alpha - I_{\alpha}}{\hat{\text{Var}}(I_b \alpha)^{1/2}} \]

- **Percentile Bootstrap CI**
  
  \[ I_{\alpha} - c_{b_{0.025}} \hat{\text{Var}}(I_{\alpha})^{1/2}; I_{\alpha} + c_{b_{0.975}} \hat{\text{Var}}(I_{\alpha})^{1/2} \]

  - \( c_{b_{0.025}} \) and \( c_{b_{0.975}} \) are from EDF of bootstrap statistics

- **Studentized Bootstrap CI**
  
  \[ I_{\alpha} - c_{\star_{0.025}} \hat{\text{Var}}(I_{\alpha})^{1/2}; I_{\alpha} - c_{\star_{0.975}} \hat{\text{Var}}(I_{\alpha})^{1/2} \]

  - \( c_{\star_{0.025}} \) and \( c_{\star_{0.975}} \) are from EDF of the bootstrap \( t \)-statistics
Confidence Intervals

- 3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized Bootstrap
- \( CI_{asym} = [I_\alpha - c_{0.975} \hat{\text{Var}}(I_\alpha)^{1/2} ; I_\alpha + c_{0.975} \hat{\text{Var}}(I_\alpha)^{1/2}] \)
  - \( c_{0.975} \) from the Student distribution \( T(n-1) \)
  - do not always perform well in finite samples
Confidence Intervals

- 3 variants of CIs: **Asymptotic**, **Percentile Bootstrap**, **Studentized Bootstrap**

- \( CI_{\text{asym}} = [I_\alpha - c_{0.975} \hat{\text{Var}}(I_\alpha)^{1/2} ; I_\alpha + c_{0.975} \hat{\text{Var}}(I_\alpha)^{1/2}] \)
  - \( c_{0.975} \) from the Student distribution \( T(n - 1) \)
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- Bootstraps: generate resamples, \( b = 1, \ldots, B \)
  - for each resample \( b \) compute the inequality index
  - obtain \( B \) bootstrap statistics, \( I^b_\alpha \)
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Confidence Intervals

- 3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized Bootstrap
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- \( CI_{perc} = [c_{0.025}^b ; c_{0.975}^b] \)
  - \( c_{0.025}^b \) and \( c_{0.975}^b \) are from EDF of bootstrap statistics
Confidence Intervals

3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized

• \( C_{\text{asym}} = [I_{\alpha - 0.975 \hat{\text{Var}}(I_{\alpha})^{1/2}}; I_{\alpha + 0.975 \hat{\text{Var}}(I_{\alpha})^{1/2}}] \)

• \( C_{\text{perc}} = [c_{b, 0.025}^b; c_{b, 0.975}^b] \)

• \( C_{\text{stud}} = [I_{\alpha - c_b^* 0.975 \hat{\text{Var}}(I_{\alpha})^{1/2}}; I_{\alpha + c_b^* 0.975 \hat{\text{Var}}(I_{\alpha})^{1/2}}] \)

Confidence Intervals

- Asymptotic
- Percentile Bootstrap
- Studentized

3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized

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• also B bootstrap t-statistics, \( t_b^b = (I_b^b - I_{\alpha}) / \hat{\text{Var}}(I_{\alpha})^{1/2} \)

• do not always perform well in finite samples

• \( c_{0.025}^* \) and \( c_{0.975}^* \) are from EDF of bootstrap t-statistics

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Performance Test
Performance Test

• Take an example with 3 ordered categories \((K = 3)\)
Performance Test

- Take an example with 3 ordered categories ($K = 3$)

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  - if we are using 95% CIs of $I_\alpha$
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  • if we are using 95% CIs of \(I_\alpha\)
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• Check coverage error rate of CIs as sample size increases
  • \(\alpha = -1, 0, 0.5, 0.99\)
  • 199 bootstraps
  • 10 000 replications to compute error rates
  • \(n = 20, 50, 100, 200, 500, 1000\)
## Estimation Methods Compared

<table>
<thead>
<tr>
<th>Method</th>
<th>α</th>
<th>-1</th>
<th>0</th>
<th>0.5</th>
<th>0.99</th>
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<tr>
<td><strong>Asymptotic B</strong></td>
<td>n = 20</td>
<td>0.0606</td>
<td>0.0417</td>
<td>0.0598</td>
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<td></td>
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<td>0.0540</td>
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</tr>
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<td>n = 20</td>
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- Asymptotic CIs perform OK in finite sample
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- Reliable results for $\alpha = 0.99$ (index is undefined for $\alpha = 1$)
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  Introduction and Previous work
  Basics
  Examples
Approach
  Model
  Characterisation
Inequality Measures
  Main properties
  Example
  Reference point and sensitivity
Empirical aspects
  Implementation
  Performance
  Application
Summary
World Values Survey

Life satisfaction question:
All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are "completely dissatisfied" and 10 means you are "completely satisfied" where would you put your satisfaction with your life as a whole? (code one number):
Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied

Health question:
All in all, how would you describe your state of health these days? Would you say it is (read out):
1 Very good, 2 Good, 3 Fair, 4 Poor.
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- Cross-country comparison of life satisfaction and GDP/head
  - happiness-income paradox (Easterlin 1974, Clark and Senik 2011)
  - weak relation happiness-income internationally? (Easterlin 1995, Easterlin et al. 2010)
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- Is inequality of life satisfaction related to GDP/head?
  - Use $I_0$ and other members of the same family
GDP and Life satisfaction (Linear)
GDP and Life satisfaction (Exponential)
GDP and Inequality of Life satisfaction
Income inequality and Inequality of Life satisfaction

Inequality of income (Gini) vs. Inequality of life satisfaction for various countries.

Countries included:
- Argentina
- Australia
- Burkina Faso
- Bulgaria
- Brazil
- Canada
- Switzerland
- Chile
- China
- Colombia
- Cyprus
- Egypt
- Spain
- Ethiopia
- Ethiopia
- Georgia
- Germany
- Ghana
- Greece
- Guatemala
- Hong Kong
- Indonesia
- Iran
- Iraq
- Italy
- Jordan
- Japan
- Korea, Republic of
- Kuwait
- Latvia
- Lebanon
- Lithuania
- Luxembourg
- Malaysia
- Malta
- Mauritius
- Mexico
- Moldova
- Morocco
- Netherland
- Norway
- New Zealand
- Nigeria
- Pakistan
- Poland
- Romania
- Russia
- Rwanda
- Serbia
- Slovenia
- Slovakia
- South Africa
- Spain
- Sri Lanka
- Sweden
- Switzerland
- Trinidad & Tobago
- Tunisia
- Turkey
- Ukraine
- Uruguay
- United States
- United Kingdom
- Vietnam
- Yemen
- Zambia
- Zimbabwe
Health status

- Health is HRS
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- Cross-country comparison of health and GDP
  - a significant positive relationship? (Deaton 2008)
Health status

- Health is HRS

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  - a significant positive relationship? (Deaton 2008)

- Cross-country comparison of inequality of health and Inequality of life satisfaction
  - use same inequality index as for life satisfaction
GDP and Inequality of health

![Graph showing the relationship between per capita GDP in 2005 and inequality of health across various countries.](image-url)
Income inequality and health inequality

Inequality of income (Gini)

Inequality of health
Inequality of life satisfaction and health inequality
Application: overview

• Satisfaction / GDP results sensitive to the cardinal interpretation
  - linear: positive relation below $15,000, flat after that (Layard 2003)
  - exponential: no relation
• OLS estimate of $I_0$ (life satisfaction) on the GDP per capita: small and negative
• Happiness-income relationship is weak in cross-country comparisons
• No clear relationship between $I_0$ (health) on GDP per capita
• OLS estimate of $I_0$ (health) on $I_0$ (life satisfaction): produces a slope coefficient not significantly different from zero
• Health-life satisfaction relationship is not significant
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  • allows you to choose “reference status”
  • gives a family of measures
Summary

- Inequality with ordinal data is a widespread phenomenon
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- Cowell and Flachaire (2014) approach:
  - separates out the issue of status from that of inequality-aggregation
  - allows you to choose “reference status”
  - gives a family of measures
- Nice properties empirically
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Proofs

Empirical argument
Median: definition and in our cases

- \( \text{med}(s) \) defined as \( e \in S \) such that \( \#(s_i \leq e) \geq \frac{n}{2}, \#(s_i \geq e) \geq \frac{n}{2} \)
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<td>0</td>
<td>25</td>
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<td>3/4</td>
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- \( \text{med}(s) \) could be any value in interval \( M(s) \)
Median example 1

- Three ordered categories
- Same proportion of individuals in each category
Median example 1

- Three ordered categories
- Same proportion of individuals in each category
- The status vector is \( s = \left( \frac{1}{3}, \frac{2}{3}, 1 \right) \)
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- The status vector is \( s = (\frac{1}{3}, \frac{2}{3}, 1) \)
- conventional definition is \( \text{med}(s) = m := \frac{2}{3} \):
  - \( \frac{2}{3} \) of the population has a status less or equal to \( m \)
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- Median as “half-way” point is misleading
Median example 2

- Two ordered categories (B better than A)
- Three distributions
  1. $n_A = 500$, $n_B = 500$
  2. $n_A = 499$, $n_B = 501$
  3. $n_A = 999$, $n_B = 1$
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SRH Inequality: Gini (median norm’d)

(1,2,3,4,5) At 0.107  UK 0.135  Mx 0.123  BD 0.140  (BD,UK,Mx,At)*
SRH Inequality: Gini (median norm’d)

<table>
<thead>
<tr>
<th></th>
<th>At</th>
<th>UK</th>
<th>Mx</th>
<th>BD</th>
<th>Reference</th>
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<td>(1,2,3,4,5)</td>
<td>0.107</td>
<td>0.135</td>
<td>0.123</td>
<td>0.140</td>
<td>(BD,UK,Mx,At)</td>
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<tr>
<td>(1,2,3,4,1000)</td>
<td>0.006</td>
<td>0.011</td>
<td>0.017</td>
<td>0.029</td>
<td>(BD,Mx,UK,At)</td>
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<td>(BD,Mx,UK,At)*</td>
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<tr>
<td>(-1000,2,3,4,5)</td>
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<td>-0.315</td>
<td>-1.844</td>
<td>-0.188</td>
<td>(At,Mx,UK,BD)</td>
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</table>
SRH Inequality: C of V (median norm’d)

(1,2,3,4,5) 0.202 0.253 0.232 0.260 (BD,UK,Mx,At) *

(1,2,3,4,1000) 0.012 0.024 0.044 0.101 (BD,Mx,UK,At) *

(-1000,2,3,4,5) 2276 -4.39 -87.2 -0.42 (At,BD,UK,Mx)
SRH Inequality: C of V (median norm’d)

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### SRH Inequality: C of V (median norm’d)

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<th>Mx</th>
<th>BD</th>
<th>Country Combination</th>
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<td>(1,2,3,4,5)</td>
<td>0.202</td>
<td>0.253</td>
<td>0.232</td>
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<td>(BD,UK,Mx,At) *</td>
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<td>(-1000,2,3,4,5)</td>
<td>2276</td>
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Outline

Motivation
  Introduction and Previous work
  Basics
  Examples

Approach
  Model
  Characterisation

Inequality Measures
  Main properties
  Example
  Reference point and sensitivity

Empirical aspects
  Implementation
  Performance
  Application

Summary

Proofs

Empirical argument
Proof of Theorem 1

• Two cases to consider
  • data are categorical: $S$ is set of non-negative rational numbers, $\mathbb{Q}_+$.  
  • data have cardinal significance: $S$ can be taken as an interval in $\mathbb{R}$.

• In either case $(S, +, \succ)$ forms a strictly ordered group (Krantz 1964, Luce and Tukey 1964, Wakker 1988)

• From Theorem 5.3 of Fishburn (1970) Axioms jointly imply that, for a given $e$, $\succeq$ is representable by a continuous function $S^{n+1} \rightarrow \mathbb{R} : \sum_{i=1}^{n} d_i(s_i, e), \forall (s, e) \in S^{n+1}$ where, for each $i$, $d_i : S \rightarrow \mathbb{R}$ is a continuous function.

• By monotonicity this is increasing in $s_i$ if $s_i > e$ and vice versa.

• By anonymity the functions $d_i$ must all be identical

• ordering $\succeq$ is also representable any monotonic transform
Take the case where status is downward-looking and peer-inclusive

Suppose that the status of each member of category $k$ is $s$

If a person is promoted from category $k$ to category $k + 1$
  - status increases to $s + \frac{n_{k+1}}{n}$
  - status of each of the remaining $n_k - 1$ members of category $k$ falls to $s - \frac{1}{n}$.

The resulting change in inequality is proportional to
\[
\left[ d \left( s + \frac{n_{k+1}}{n}, e \right) - d \left( s, e \right) \right] + \left[ n_k - 1 \right] \left[ d \left( s - \frac{1}{n}, e \right) - d \left( s, e \right) \right]
\]

If $d$ is differentiable then this expression is approximately
\[
d' \left( s, e \right) \left( \frac{n_{k+1}}{n} - \frac{n_{k-1}}{n} \right) + d' \left( s, e \right)
\]
  which equals $\frac{1}{n} d' \left( s, e \right) \left[ n_{k+1} - n_k + 1 \right]$.

If $s < e$ then monotonicity implies $d' \left( s, e \right) < 0$
  - the change in inequality is negative if $n_{k+1} \geq n_k$. 

“Maximum inequality”
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Dispersion

- Model: \( \text{LifeSatisf}_i = \alpha + \beta \text{GDP}_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma_i^2) \)
  - \( \beta \) is a significant coefficient and \( R^2 \) is large
  - strong (linear) relationship between LifeSatisf and GDP

- If LifeSatisf equation is homoskedastic:
  - no relationship between GDP and the dispersion of LifeSatisf
  - whatever is GDP, the dispersion of LifeSatisf is the same

- If LifeSatisf equation heteroskedastic dispersion of LifeSatisf may or may not be related to GDP
  - the form of the heteroskedasticity cannot be deduced from the relationship between the dependent variable and the covariate.

- If every \( i \) has different GDP, \( \sigma_i^2 \) measures the dispersion of LifeSatisf for \( i \)

- taking the measure \( I_0 \) as a measure of dispersion, the same reasoning applies


