Measuring Inequality with Ordinal data

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## Outline

### Motivation
- Introduction and Previous work
- Basics
- Examples

### Approach
- Model
- Characterisation

### Inequality Measures
- Main properties
- Sensitivity

### Empirical aspects
- Implementation
- Performance
- Application

### Summary
Introduction

• Ordinal data issue widespread in inequality analysis
• Many applications proceed just as though cardinal:
  - health status: Van Doorslaer and Jones (2003)

• Small literature that takes ordinal problem seriously
  - early approaches using 1st order dominance, the median
  - but these have limitations

• Present approach based on Cowell and Flachaire (2014)
Income Inequality

- 3 ingredients:
  - “income”: family income, earnings, wealth $x \in X \subseteq \mathbb{R}$.
  - “income-receiving unit”: $n$ persons
  - method of aggregation: function $X^n \rightarrow \mathbb{R}$

- Usually work with $X^\mu_n \subseteq \mathbb{R}$

- $X^\mu_n$: Distributions obtainable from a given total income $n\mu$ using lump-sum transfers

- Obviously can’t do that here: $\mu$ is undefined
Utility
Cardinalisation and inequality

- 3 ingredients:
  - “income”: \( u = U(x) \)
  - “income-receiving unit”: \( n \) persons (as before)
  - method of aggregation: function \( \mathbb{U}^n \rightarrow \mathbb{R} \)

- Problem of cardinalisation

- But just assuming cardinal utility is no use
  - Already pointed out in Atkinson (1970)
  - Dalton (1920) suggested inequality of (cardinal) utility
  - But if, for all \( i \), you multiply \( u_i \) by \( \lambda \in (0, 1) \) and add \( \delta = \mu [1 - \lambda] \ldots \)
    - ...this will automatically reduce measured inequality.

- Is this just a technicality?
- Can we proceed just as with regular income?
## Categorical variable

**Example: Access to Services**

<table>
<thead>
<tr>
<th>Case</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both Gas and Electricity</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>Electricity only</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>Gas only</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>Neither</td>
<td>25</td>
<td>0</td>
</tr>
</tbody>
</table>

- Suppose we have no information about needs / usage
- It seems clear that Case 1 is more unequal than Case 2
Example self-reported health

- **World Health Survey (WHS)**
  - a general population survey
  - developed by WHO

- **Question: Health State Descriptions**
  - overall health
  - including both physical and mental health

- **In general, how would you rate your health today?**
  - Very good
  - Good
  - Moderate
  - Bad
  - Very Bad

- **Compare distributions across countries**
SRH Results: four countries

<table>
<thead>
<tr>
<th></th>
<th>Austria</th>
<th>UK</th>
<th>Mexico</th>
<th>Bangladesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of responses</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very good</td>
<td>423</td>
<td>318</td>
<td>7193</td>
<td>494</td>
</tr>
<tr>
<td>Good</td>
<td>390</td>
<td>498</td>
<td>18112</td>
<td>1949</td>
</tr>
<tr>
<td>Moderate</td>
<td>200</td>
<td>278</td>
<td>11221</td>
<td>2132</td>
</tr>
<tr>
<td>Bad</td>
<td>36</td>
<td>82</td>
<td>2002</td>
<td>741</td>
</tr>
<tr>
<td>Very bad</td>
<td>4</td>
<td>17</td>
<td>218</td>
<td>228</td>
</tr>
</tbody>
</table>

- For all countries: rank categories in order
- For each country: compute freq distributions across categories
- How to evaluate inequality?
SRH Inequality: Gini

<table>
<thead>
<tr>
<th></th>
<th>At</th>
<th>UK</th>
<th>Mx</th>
<th>BD</th>
<th>Country Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2,3,4,5)</td>
<td>0.111</td>
<td>0.130</td>
<td>0.116</td>
<td>0.154</td>
<td>BD, UK, Mx, At</td>
</tr>
<tr>
<td>(1,2,3,4,1000)</td>
<td>0.593</td>
<td>0.725</td>
<td>0.800</td>
<td>0.884</td>
<td>BD, Mx, UK, At</td>
</tr>
<tr>
<td>(-1000,2,3,4,5)</td>
<td>0.608</td>
<td>0.821</td>
<td>0.856</td>
<td>2.377</td>
<td>BD, Mx, UK, At</td>
</tr>
</tbody>
</table>
SRH Inequality: Coeff of Variation

<table>
<thead>
<tr>
<th>Condition</th>
<th>At</th>
<th>UK</th>
<th>Mx</th>
<th>BD</th>
<th>Country Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2,3,4,5)</td>
<td>0.209</td>
<td>0.244</td>
<td>0.219</td>
<td>0.287</td>
<td>(BD,UK,Mx,At)</td>
</tr>
<tr>
<td>(1,2,3,4,1000)</td>
<td>1.210</td>
<td>1.638</td>
<td>2.056</td>
<td>3.088</td>
<td>(BD,Mx,UK,At)</td>
</tr>
<tr>
<td>(-1000,2,3,4,5)</td>
<td>187.5</td>
<td>11.43</td>
<td>40.45</td>
<td>5.264</td>
<td>(At,Mx,UK,BD)</td>
</tr>
</tbody>
</table>
Status and Information

- Step 1 is to define status
  - depends on the purpose of inequality analysis
  - depends on structure of information
  - conventional inequality approach only works in narrowly defined information structure

- In some cases a person’s status is self-defining
  - income
  - wealth

- In some cases defined given additional distribution-free information
  - example: if it is known that utility is \( \log(x) \)

- In some cases requires information on distribution
  - GRE, TOEFL
  - “opportunity” (de Barros et al. 2008)
Status and Distribution (1)

- $i$’s status uniquely defined for a given distribution of $u$

- disposes of the problem of cardinalisation
  - $U$ and $V = \varphi(U)$ two cardinalisations of the utility of $x$
  - for each $i: u_i$ and $v_i$ map into $s_i$
• This approach works for categorical data
  • we just have an ordered arrangement of categories $1, 2, \ldots, k, \ldots, K$
  • and the numbers in each category $n_1, n_2, \ldots, n_k, \ldots, n_K$

• Merger principle
  • merge two adjacent categories that are irrelevant for $i$
  • then this should leave $i$’s status unaltered

• Merger principle implies that $s$ should be additive in the $n_k$
  • upward-looking status: $\sum_{\ell=1}^{k(i)} n_\ell$
  • downward-looking status: $\sum_{\ell=k(i)}^{K} n_\ell$
  • see also Yitzhaki (1979)
Elements of the Model

- Individual’s status is given by $s \in S \subseteq \mathbb{R}$
  - status determined from utility?

- Vector of status in a population of size $n$ : $s \in S^n$

- $e \in S$ : an equality-reference point
  - could be specified exogenously
  - could also depend on status vector $e = \eta (s)$
  - $\eta$ need not be increasing in each component of $s$

- Inequality: aggregate distance from $e$
  - don’t need an explicit distance function
  - implicitly define through inequality ordering $\succeq$
Basic Axioms

- **[Continuity]** $\succeq$ is continuous on $S^n$
- **[Monotonicity]** If $s, s' \in S^n_e$ differ only in their $i$th component then (a) if $s'_i \geq e : s_i > s'_i \iff s \succ s'$; (b) if $s'_i \leq e : s'_i > s_i \iff s \succ s'$
- **[Independence]** For $s, s' \in S^n_e$, if $s \sim s'$ and $s_i = s'_i$ for some $i$ then $s(\varsigma, i) \sim s'(\varsigma, i)$ for all $\varsigma \in [s_{i-1}, s_{i+1}] \cap [s'_{i-1}, s'_{i+1}]$
- **[Anonymity]** For all $s \in S^n$ and permutation matrix $P$, $Ps \sim s$.
Theorem
Continuity, Monotonicity, Independence, Anonymity jointly imply \( \succeq \) is representable by the continuous function \( I : S^n_e \to \mathbb{R} \) where
\[
I(s; e) = \Phi \left( \sum_{i=1}^{n} d(s_i, e), e \right),
\]
where \( d : S \to \mathbb{R} \) is a continuous function that is strictly increasing (decreasing) in its first argument if \( s_i > e \) (\( s_i < e \)).

Corollary
Inequality is total “distance” from equality. Distance \( d \) is continuous. \( d(s, e) \) is increasing in status if you move away from the reference point.
Structure Theorem

• We need more structure on the problem

• [Scale invariance 1] For all $\lambda \in \mathbb{R}^+$: if $s, s', \lambda s, \lambda s' \in S^n$ and $e, e' \in S$ then $(s, e) \sim (s', e') \Rightarrow (\lambda s, e) \sim (\lambda s', e')$.

• [Scale invariance 2] For all $\lambda \in \mathbb{R}^+$: if $s, s', \lambda s, \lambda s' \in S^n$ and $e, e', \lambda e, \lambda e' \in S$ then $(s, e) \sim (s', e') \Rightarrow (\lambda s, \lambda e) \sim (\lambda s', \lambda e')$

Theorem

*Impose also Scale irrelevance 1. Then $d(s, e) = A(e)s^{\alpha (e)}$*

Theorem

*Impose instead Scale Invariance 2. Then $d(s, e) = e^{\beta} \phi \left( \frac{s}{e} \right)$.* where $\beta$ is a constant and $\phi$ is arbitrary

Corollary

*Inequality represented as $I_{\alpha}(s; e) := \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^{\alpha} - e^{\alpha} \right]$*
A usable inequality index?

- A class of functions available as inequality measures:
  - $\Phi(I_\alpha(s; e), e)$
  - $e = \eta(s)$, the reference point
  - $I_\alpha(s; e) := \frac{1}{\alpha [\alpha - 1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - e^\alpha \right]$

- Do functions $\Phi(I_\alpha(s; e), e)$ “look like” inequality measures?
  - transfer principle?
  - reference point?
  - sensitivity to parameters

- What is the appropriate form for $\Phi$?
  - may depend on the reference status $e$
  - may depend on interpretation
## Four distributional scenarios (1)

<table>
<thead>
<tr>
<th></th>
<th>Case 0</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n_k$</td>
<td>$s_i$</td>
<td>$n_k$</td>
<td>$s_i$</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>50</td>
<td>1</td>
<td>25</td>
<td>3/4</td>
</tr>
<tr>
<td><strong>G</strong></td>
<td>25</td>
<td>1/2</td>
<td>25</td>
<td>1/2</td>
</tr>
<tr>
<td><strong>N</strong></td>
<td>25</td>
<td>1/4</td>
<td>25</td>
<td>1/4</td>
</tr>
</tbody>
</table>

$$\mu(s) = \frac{11}{16} \quad \frac{5}{8} \quad \frac{3}{4} \quad \frac{11}{16}$$

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$
- $s_i = \frac{1}{n} \sum_{\ell=1}^{k(i)} n_\ell$ – *downward*-looking status
### Four distributional scenarios

<table>
<thead>
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<th>Case 0</th>
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<tbody>
<tr>
<td></td>
<td>$n_k$</td>
<td>$s'_i$</td>
<td>$n_k$</td>
<td>$s'_i$</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>25/4</td>
<td>0</td>
<td>25/4</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>1/2</td>
<td>25</td>
<td>1/2</td>
</tr>
<tr>
<td>G</td>
<td>25</td>
<td>3/4</td>
<td>25</td>
<td>3/4</td>
</tr>
<tr>
<td>N</td>
<td>25</td>
<td>1</td>
<td>25</td>
<td>1</td>
</tr>
</tbody>
</table>

$\mu(s)$

- $n_k$ is # persons in category $k \in \{B, E, G, N\}$
- $s'_i = \frac{1}{n} \sum_{\ell=k(i)}^{K} n_\ell$ - *upward*-looking status
Four distributional scenarios (2)

<table>
<thead>
<tr>
<th></th>
<th>Case 0</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n_k$</td>
<td>$s_i$</td>
<td>$n_k$</td>
<td>$s_i$</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>1</td>
<td>25</td>
<td>3/4</td>
</tr>
<tr>
<td>G</td>
<td>25</td>
<td>1/2</td>
<td>25</td>
<td>1/2</td>
</tr>
<tr>
<td>N</td>
<td>25</td>
<td>1/4</td>
<td>25</td>
<td>1/4</td>
</tr>
</tbody>
</table>

$\mu(s) = \frac{11}{16}, \frac{5}{8}, \frac{3}{4}, \frac{11}{16}$

Case 0 to Case 1:

- 25 people promoted from E to B
- if $e$ equals to any of values taken by $\mu(s)$
- then inequality increases
### Four distributional scenarios (3)

<table>
<thead>
<tr>
<th></th>
<th>Case 0</th>
<th></th>
<th>Case 1</th>
<th></th>
<th>Case 2</th>
<th></th>
<th>Case 3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n_k$</td>
<td>$s_i$</td>
<td>$n_k$</td>
<td>$s_i$</td>
<td>$n_k$</td>
<td>$s_i$</td>
<td>$n_k$</td>
<td>$s_i$</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>25 1</td>
<td>25</td>
<td>3/4</td>
<td>0</td>
<td>1</td>
<td>25</td>
<td>3/4</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>1</td>
<td>25</td>
<td>1/2</td>
<td>50</td>
<td>1/2</td>
<td>50</td>
<td>1/2</td>
</tr>
<tr>
<td>G</td>
<td>25</td>
<td>1/4</td>
<td>25</td>
<td>1/4</td>
<td>50</td>
<td>1</td>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>N</td>
<td>25</td>
<td>1/4</td>
<td>25</td>
<td>1/4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\mu(s)$</td>
<td></td>
<td>11/16</td>
<td>5/8</td>
<td></td>
<td>3/4</td>
<td></td>
<td>11/16</td>
<td></td>
</tr>
</tbody>
</table>

- **Case 0 to Case 2:**
  - 25 people promoted from N to G
  - if $e$ equals to any of values taken by $\mu(s)$
  - then inequality decreases
## Transfer Principle again

<table>
<thead>
<tr>
<th>Case 0</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_k )</td>
<td>( s_i )</td>
<td>( n_k )</td>
<td>( s_i )</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>G</td>
<td>25</td>
<td>( \frac{1}{2} )</td>
<td>25</td>
</tr>
<tr>
<td>N</td>
<td>25</td>
<td>( \frac{1}{4} )</td>
<td>25</td>
</tr>
</tbody>
</table>

\[ \mu(s) = \frac{11}{16}, \quad \frac{5}{8}, \quad \frac{3}{4}, \quad \frac{11}{16} \]

- Case 0 to Case 1: inequality increases
- Case 0 to Case 2: inequality decreases
- Case 0 to Case 3: combination results in ambiguous change
Reference point

- **Mean status:** $e = \eta(s) = \mu(s)$
  - for continuous distributions will equal 0.5
  - for categorical data, there is no counterpart to fixed-mean assumption in income-inequality analysis

- **Median status:** $e = \eta(s) = \text{med}(s)$
  - not well-defined: any value in interval $M(s)$
  - $M(s) = [1/2, 1)$ in cases 0 and 2
  - $M(s) = [1/2, 3/4)$ in cases 1 and 3

- **Max status:** $e = 1$
  - for constant $e$ this is only value that makes sense

- **Min status:** $e = 0$
  - counterpart for peer-exclusive case
• $\alpha$ captures the sensitivity of measured inequality

• If $\alpha$ is high $I_\alpha (s; e) = \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - e^\alpha \right]$, sensitive to high status-inequality

• If $\alpha = 0$ then $I_0 (s; e) = -\frac{1}{n} \sum_{i=1}^{n} \log s_i + \log e$,

• If $e = \mu (s)$ and $\alpha = 1$ then $\frac{1}{n} \sum_{i=1}^{n} s_i \log s_i - e \log e$
Behaviour of $I_0(s; e)$

<table>
<thead>
<tr>
<th></th>
<th>Case 0</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu(s)$</td>
<td>$\frac{11}{16}$</td>
<td>$\frac{5}{8}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{11}{16}$</td>
</tr>
<tr>
<td>med$_1(s)$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{5}{8}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{5}{8}$</td>
</tr>
<tr>
<td>med$_2(s)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$I_0(s; \mu(s))$</td>
<td>0.1451</td>
<td>0.1217</td>
<td>0.0588</td>
<td>0.0438</td>
</tr>
<tr>
<td>$I_0(s; \text{med}_1(s))$</td>
<td>0.2321</td>
<td>0.1217</td>
<td>0.0588</td>
<td>-0.0515</td>
</tr>
<tr>
<td>$I_0(s; \text{med}_2(s))$</td>
<td>-0.1732</td>
<td>-0.1013</td>
<td>-0.3465</td>
<td>-0.2746</td>
</tr>
<tr>
<td>$I_0(s; 1)$</td>
<td>0.5198</td>
<td>0.5917</td>
<td>0.3465</td>
<td>0.4184</td>
</tr>
</tbody>
</table>

- $I_0(s; \mu(s)), I_0(s; \text{med}_1(s))$: inequality decreases for
  - Case 0 to 1, or Case 2 to 3
  - movement changes both the $\mu(s)$ and $\text{med}_1(s)$ ref points
- $I_0(s; \text{med}_2(s)) < 0$ for all cases in example!
- But $I_0(s; 1)$ seems sensible
Inequality measure

- For ordinal data, peer-inclusive status

\[
I_\alpha(s, 1) = \begin{cases} 
\frac{1}{\alpha(\alpha-1)} \left[ \frac{1}{n} \sum_{i=1}^{n} s_i^\alpha - 1 \right], & \text{if } \alpha \neq 0, \alpha < 1 \\
- \frac{1}{n} \sum_{i=1}^{n} \log s_i, & \text{if } \alpha = 0
\end{cases}
\]
Implementation

- Description of sample
  
  \[ x_i = \begin{cases} 
  1 & \text{with sample proportion } p_1 \\
  2 & \text{with sample proportion } p_2 \\
  \vdots \\
  K & \text{with sample proportion } p_K 
  \end{cases} \]

- Point estimate of the index:
  
  \[ I_\alpha = \begin{cases} 
  \frac{1}{\alpha(\alpha-1)} \left[ \sum_{i=1}^{K} p_i \left[ \sum_{j=1}^{i} p_j \right]^\alpha \right] - 1 & \text{if } \alpha \neq 0,1 \\
  -\sum_{i=1}^{K} p_i \log \left[ \sum_{j=1}^{i} p_j \right] & \text{if } \alpha = 0 
  \end{cases} \]

- Function of \( K \) parameter estimates \((p_1, p_2, \ldots, p_K)\) following a multinomial
Asymptotics

- From the CLT $I_{\alpha}$ is asymptotically Normally distributed

- Estimator of cov matrix of $(p_1, p_2, \ldots, p_k)$ is

$$
\sum = \frac{1}{n} \begin{bmatrix}
p_1(1-p_1) & -p_1p_2 & \ldots & -p_1p_k \\
-p_2p_1 & p_2(1-p_2) & \ldots & -p_2p_k \\
\vdots & \vdots & \ddots & \vdots \\
-p_kp_1 & -p_kp_2 & \ldots & p_k(1-p_k)
\end{bmatrix}
$$

- $\hat{\text{Var}}(I_{\alpha}) = D\Sigma D^\top$ with $D = \left[ \frac{\partial I_{\alpha}}{\partial p_1}; \frac{\partial I_{\alpha}}{\partial p_2}; \ldots; \frac{\partial I_{\alpha}}{\partial p_k} \right]$  

- $\frac{\partial I_{\alpha}}{\partial p_l} = \frac{1}{\alpha(\alpha-1)} \left( \left[ \sum_{i=1}^{l} p_i \right]^\alpha + \alpha \sum_{i=l+1}^{K-1} p_i \left[ \sum_{j=1}^{i} p_j \right]^{\alpha-1} \right), \alpha \neq 0$

- $\frac{\partial I_0}{\partial p_l} = -\log \left[ \sum_{j=1}^{l} p_j \right] - \sum_{i=l+1}^{K-1} p_i \left[ \sum_{j=1}^{i} p_j \right]^{-1}$
Confidence Intervals

- 3 variants of CIs: **Asymptotic**, **Percentile Bootstrap**, **Studentized Bootstrap**
- \( CI_{sym} = \left[ I_\alpha - c_{0.975} \sqrt{\text{Var}(I_\alpha)}^{1/2} ; I_\alpha + c_{0.975} \sqrt{\text{Var}(I_\alpha)}^{1/2} \right] \)
  - \( c_{0.975} \) from the Student distribution \( T(n-1) \)
  - do not always perform well in finite samples
- Bootstraps: generate resamples, \( b = 1, \ldots, B \)
  - for each resample \( b \) compute the inequality index
  - obtain \( B \) bootstrap statistics, \( I_\alpha^b \)
  - also \( B \) bootstrap \( t \)-statistics \( t_\alpha^b = (I_\alpha^b - I_\alpha)/\sqrt{\text{Var}(I_\alpha^b)}^{1/2} \)
- \( CI_{perc} = \left[ c_{0.025}^b ; c_{0.975}^b \right] \)
  - \( c_{0.025}^b \) and \( c_{0.975}^b \) are from EDF of bootstrap statistics
- \( CI_{stud} = \left[ I_\alpha - c_{0.975}^* \sqrt{\text{Var}(I_\alpha)}^{1/2} ; I_\alpha - c_{0.025}^* \sqrt{\text{Var}(I_\alpha)}^{1/2} \right] \)
  - \( c_{0.025}^* \) and \( c_{0.975}^* \) are from EDF of the bootstrap \( t \)-statistics
Performance Test

- Take an example with 3 ordered categories ($K = 3$)

- Samples are drawn from a multinomial distribution with probabilities $\pi = (0.3, 0.5, 0.2)$

- Is asymptotic or bootstrap distribution a good approximation of the exact distribution of the statistic?
  - if we are using 95% CIs of $I_\alpha$
  - coverage error rate should be close to nominal rate, 0.05

- Check coverage error rate of CIs as sample size increases
  - $\alpha = -1, 0, 0.5, 0.99$
  - 199 bootstraps
  - 10 000 replications to compute error rates
  - $n = 20, 50, 100, 200, 500, 1000$
## Estimation Methods Compared

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>-1</th>
<th>0</th>
<th>0.5</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Asymptotic B</strong></td>
<td>$n = 20$</td>
<td>0.0606</td>
<td>0.0417</td>
<td>0.0598</td>
<td>0.0491</td>
</tr>
<tr>
<td></td>
<td>$n = 500$</td>
<td>0.0523</td>
<td>0.0492</td>
<td>0.0521</td>
<td>0.0523</td>
</tr>
<tr>
<td></td>
<td>$n = 1000$</td>
<td>0.0485</td>
<td>0.0540</td>
<td>0.0552</td>
<td>0.0549</td>
</tr>
<tr>
<td><strong>Percentile B</strong></td>
<td>$n = 20$</td>
<td>0.0384</td>
<td>0.0981</td>
<td>0.0912</td>
<td>0.1023</td>
</tr>
<tr>
<td></td>
<td>$n = 500$</td>
<td>0.0509</td>
<td>0.0513</td>
<td>0.0552</td>
<td>0.0554</td>
</tr>
<tr>
<td></td>
<td>$n = 1000$</td>
<td>0.0482</td>
<td>0.0556</td>
<td>0.0547</td>
<td>0.0551</td>
</tr>
<tr>
<td><strong>Studentized B</strong></td>
<td>$n = 20$</td>
<td>0.1275</td>
<td>0.0843</td>
<td>0.1041</td>
<td>0.1377</td>
</tr>
<tr>
<td></td>
<td>$n = 500$</td>
<td>0.0518</td>
<td>0.0478</td>
<td>0.0429</td>
<td>0.0465</td>
</tr>
<tr>
<td></td>
<td>$n = 1000$</td>
<td>0.0473</td>
<td>0.0522</td>
<td>0.0493</td>
<td>0.0503</td>
</tr>
</tbody>
</table>

- Asymptotic CIs perform OK in finite sample
- Percentile bootstrap performs well for $n > 50$
- Studentized bootstrap does not do well for small samples
- Reliable results for $\alpha = 0.99$ (index is undefined for $\alpha = 1$)
World values survey

- Life satisfaction question:

  All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are “completely dissatisfied” and 10 means you are “completely satisfied” where would you put your satisfaction with your life as a whole? (code one number):

  Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied

- Health question:

  All in all, how would you describe your state of health these days? Would you say it is (read out):

  1 Very good, 2 Good, 3 Fair, 4 Poor.
GDP and Life satisfaction

- Cross-country comparison of life satisfaction and GDP/head
  - happiness-income paradox (Easterlin 1974, Clark and Senik 2011)
  - weak relation happiness-income internationally? (Easterlin 1995, Easterlin et al. 2010)
  - or a strong relationship? (Hagerty and Veenhoven 2003, Deaton 2008, Stevenson and Wolfers 2008a, Inglehart et al. 2008)

- How should we quantify life satisfaction?
  - simple linearity of Likert scale? or exponential scale?
  - Ng (1997), Ferrer-i-Carbonell and Frijters (2004), Kristoffersen (2011)

- Is inequality of life satisfaction related to GDP/head?
  - Use $I_0$ and other members of the same family
GDP and Life satisfaction (Linear)
GDP and Life satisfaction (Exponential)
GDP and Inequality of Life satisfaction

Inequality of life satisfaction vs. Per capita GDP in 2005

Countries plotted:
- Argentina
- Australia
- Burkina Faso
- Bulgaria
- Brazil
- Canada
- Switzerland
- Chile
- China
- Colombia
- Cyprus
- Egypt
- Spain
- Ethiopia
- Finland
- France
- United Kingdom
- Georgia
- Germany
- Ghana
- Greece
- Guatemala
- Hong Kong
- Indonesia
- India
- Iran
- Iraq
- Italy
- Jordan
- Japan
- Korea, Republic of
- Morocco
- Moldova
- Mali
- Mauritania
- Mexico
- Moldova
- Nepal
- Nigeria
- Norway
- New Zealand
- Peru
- Poland
- Portugal
- Russia
- Rwanda
- Senegal
- Colombia
- Singapore
- Slovakia
- Slovenia
- South Africa
- Spain
- Sweden
- Thailand
- Trinidad & Tobago
- Turkey
- Ukraine
- Uruguay
- United States
- Vietnam
- South Africa
- Zambia
- Zimbabwe

Legend:
- Argentina
- Australia
- Burkina Faso
- Bulgaria
- Brazil
- Canada
- Switzerland
- Chile
- China
- Colombia
- Cyprus
- Egypt
- Spain
- Ethiopia
- Finland
- France
- United Kingdom
- Georgia
- Germany
- Ghana
- Greece
- Guatemala
- Hong Kong
- Indonesia
- India
- Iran
- Iraq
- Italy
- Jordan
- Japan
- Korea, Republic of
- Morocco
- Moldova
- Mali
- Mauritania
- Mexico
- Moldova
- Nepal
- Nigeria
- Norway
- New Zealand
- Peru
- Poland
- Portugal
- Russia
- Rwanda
- Senegal
- Colombia
- Singapore
- Slovakia
- Slovenia
- South Africa
- Spain
- Sweden
- Thailand
- Trinidad & Tobago
- Turkey
- Ukraine
- Uruguay
- United States
- Vietnam
- South Africa
- Zambia
- Zimbabwe
Health status

- Health is HRS

- Cross-country comparison of health and GDP
  - a significant positive relationship? (Deaton 2008)

- Cross-country comparison of inequality of health and Inequality of life satisfaction
  - use same inequality index as for life satisfaction
Inequality of health and GDP
Inequality of health

Inequality of health measures in various countries.
Application: overview

- Satisfaction / GDP results sensitive to the cardinal interpretation of the answers
  - linear: positive relation below $15,000, flat after that (Layard 2003)
  - exponential: no relation

- OLS estimate of $I_0$(life satisfaction) on the GDP per capita small and negative
  - happiness-income relationship is weak in cross-country comparisons

- No clear relationship between $I_0$(health) on GDP per capita

- OLS estimate of $I_0$(health) on $I_0$(life satisfaction) produces a slope coefficient not significantly different from zero
  - health-life satisfaction relationship is not significant
Summary

- Inequality with ordinal data is a widespread phenomenon
- Conventional $I$-measures may make no sense
- Cowell and Flachaire (2014) approach:
  - separates out the issue of status from that of inequality-aggregation
  - allows you to choose “reference status”
  - gives a family of measures
- Nice properties empirically


