Statistical tools for dissimilarity analysis

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Motivation

- **Dissimilarity comparisons** of **sets** of distributions.
- **Question**: which of the two **distribution matrices** displays more dissimilarity between its rows?

$$\mathbf{A} = \begin{matrix} & cl1 & cl2 & cl3 \\ gr1 & 0.6 & 0 & 0.4 \\ gr2 & 0 & 0.25 & 0.75 \\ gr3 & \frac{6}{16} & \frac{2}{16} & \frac{8}{16} \end{matrix} \quad \text{and} \quad \mathbf{B} = \begin{matrix} & cl1 & cl2 & cl3 \\ gr1 & 0.6 & 0.2 & 0.2 \\ gr2 & 0.375 & 0.25 & 0.375 \\ gr3 & \frac{8}{16} & \frac{4}{16} & \frac{4}{16} \end{matrix}$$

- **Focus** on matrices representing relative frequencies distributions of **groups** across **classes** in $\mathcal{M}_d$, like:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dn} \end{pmatrix} \quad \text{with} \quad a_{ij} \in [0,1]\ \forall i,j \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} = 1 \quad \forall i.$$

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- **Proposal :** underpin **statistical tools** that allows to rank **B** as better than **A** when **B** is closer to a **similarity matrix** than **B** is.

$$\textbf{Similarity matrix:} \quad = \begin{pmatrix} a_1, & a_2, & \cdots, & a_k \\ a_1, & a_2, & \cdots, & a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_1, & a_2, & \cdots, & a_k \end{pmatrix}$$

- **Gini 1914** defines two or more (say d) (relative) **frequency distributions** of the same variate (taking on k values) to be **similar** if:

> *"for any modality [...] the absolute frequencies are proportional. If two distributions are similar they can have different sizes but their syntheses which are based on relative frequencies are equal"*

Motivation

- Many potential applications:
  - **Segregation**
    (Duncan Duncan ASR55, Massey Denton SF88, Hutchens MSS91, Frankel Volij JET2011)
  - **Discrimination**
    (Le Breton et al. JET2012, Gastwirth AS75, Jenkins JEmetrics94, Butler McDonald JBES87)
  - **(Intergenerational) mobility analysis**
    (Shorrocks ECMA78, Tchen 1981, Markandya EER82, Dardanoni JET93)
  - **Inequality**, uni- and multi-dimensional
    (Marshal Olking Arnolds 2011, Koshevoy Mosler JASA96, Ebert Moyes ECMA2005)
  - **Distance analysis**
    (Ebert JET1984)
  - **Statistics/Linear algebra/Informativeness**
    (Ali Sivlerey RRSA61, Blackwell AMS53, Koshevoy Mosler JASA96, Torgersen 1992, Dahl LAA99)

Partial and complete orders

- These phenomena are all related to dissimilarity comparisons of two or more distributions
- One simple principle of evaluation consists in compressing the distributional information into an evaluation function (which is an index number)
  - *This is conclusive*: given two situations, they can always be ranked.
  - *This is not robust*: if you challenge the evaluation function, you may obtain different rankings.
- Geometric tests have been proposed to reflect agreement in a class of evaluation functions.
  - *Geometric* means that they can be empirically assessed via linear programming.
- In the case of **two groups**, research has focused on comparisons ("lies always above or below" tyeps of arguments) of **curves**, i.e. transformations of the data.
  - Ex: Segregation curves, Discrimination curves, Concentration curves, Lorenz curves.
  - *These tests produce robust evaluations* that reflect agreement in interesting classes of evaluation functions.
  - *These tests might be inconclusive*, since when two curves cross, nothing can be said on the extent of agreement.
- When there are **many groups (more than two)**, robust partial orders implemented by geometric tests become tricky. This is where **indicators kick in**.
  - The extensions of the geometric tests, and their characterization, for the multigroup case are in Andreoli Zoli (2014).
- In dissimilarity analysis, we consider two situations where different empirical criteria apply: the cases where classes are **permutable** and where they are exogenously **ordered**.

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Empirical cases

Phenomenon:	*Classes:* Non-order	*Classes:* Order	*Classes:* Cardinal	*Groups:* Non-order	*Groups:* Order
School segregation	✓			✓	
Inequality	✓			✓	
Earnings discrimination		✓	✓	✓	
Mobility		✓	✓		✓

- **Objective** : We will study partial orders of distribution matrices ranking $\mathbf{B} \preccurlyeq \mathbf{A}$ iff $\mathbf{A}$ displays "at leas as much dissimilarity/segregation/discrimination/mobility as" $\mathbf{B}$, that are based on geometric comparisons of curves. These curves represent the degree of dissimilarity among the distributions involved in the comparisons.

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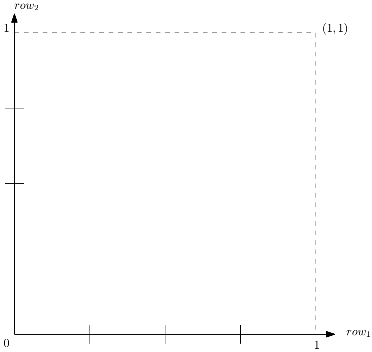
Geometric criteria

- Consider a distribution matrix **A** and the cumulation of its classes:

$$\mathbf{A} = \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.25 & 0 & 0.75 \\ X & X & X \end{pmatrix}$$

Zonotope $Z(\mathbf{A})$:



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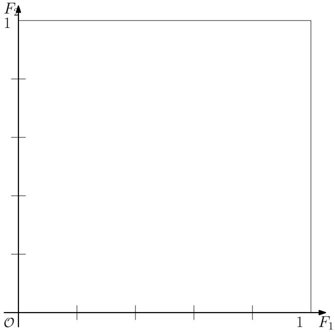

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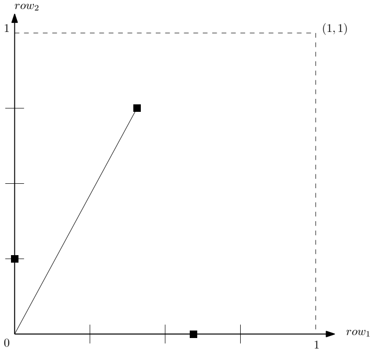

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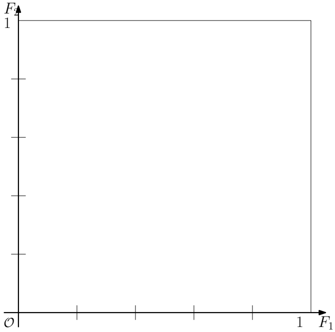

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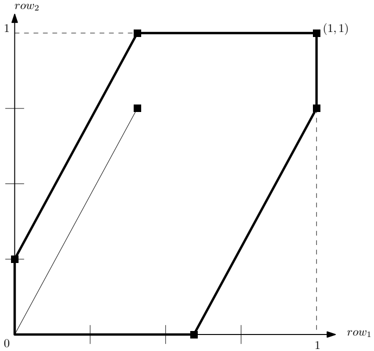

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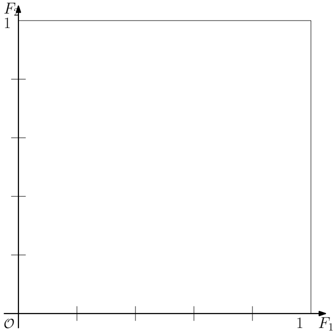

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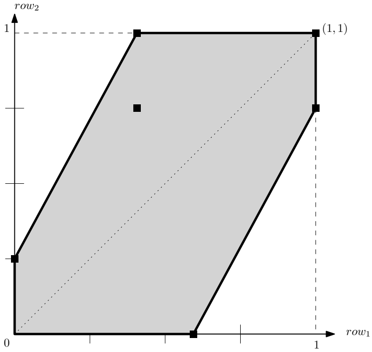

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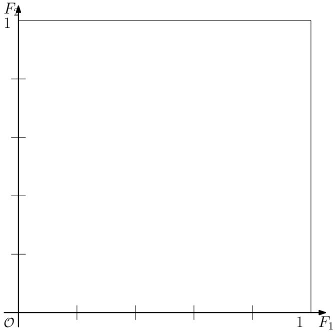

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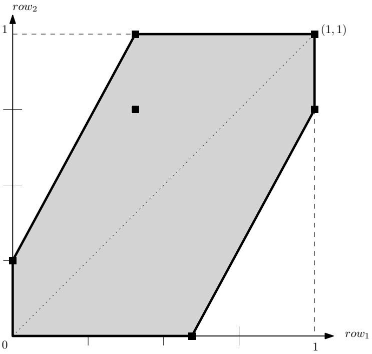

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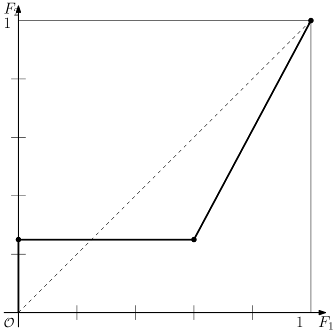

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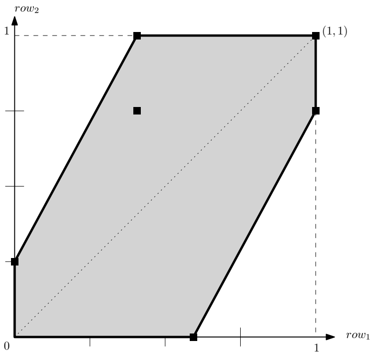

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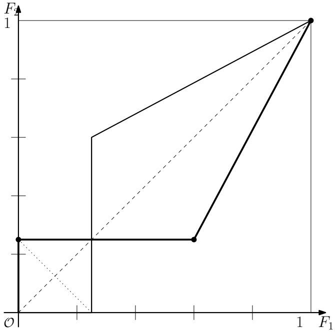

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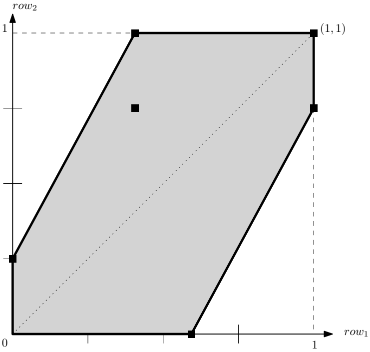

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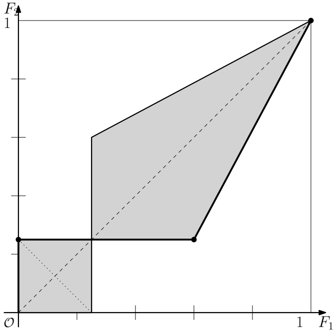

The Zonotopes inclusion test ($d = 2$)

- A suitable test for segregation comparisons, where matrices **A** and **B** may represent two cities/school districts/labor markets, and classes are neighborhoods/schools/jobs.
  - **Segregation curves** (Duncan Duncan ASR1955, Hutchens MSS1991) are the lower bound of the Zonotope. Thei ordering is related related to segregation-reducing movements of population: *when some members of the group overrepresented in a class move to a class where their group is underrepresented, segregation is reduced*.
  - **Local segregation curves** (Alonso-Villar, del Rio MSS2010) are segregation curves contrasting the distriution of each group overall population.

- **Bivariate case**: $d = 2$ groups and $n = 3$ classes

$$\mathbf{A}' := \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0 & 0.25 & 0.75 \\ X & X & X \end{pmatrix}$$

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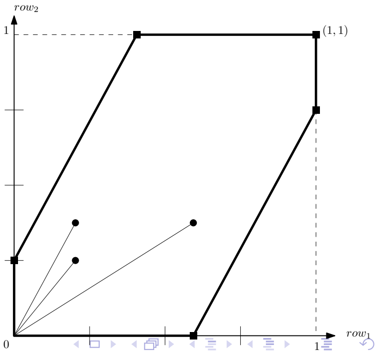

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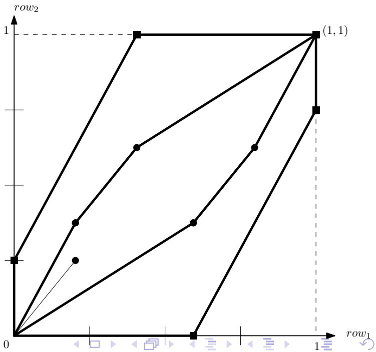

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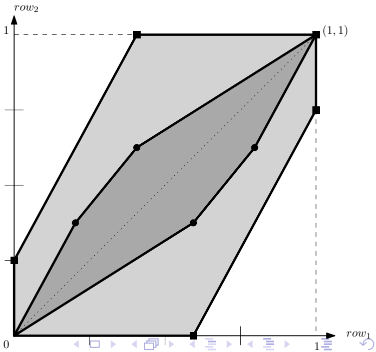

The Zonotopes inclusion test ($d > 2$)

- Extend to the multi-group case the segregation curve analysis (characterized by Andreoli and Zoli 2014)
- Its bivariate projections induce orderings coherent with segregation curves, although Zonotopes inclusion reflects the perspective of all projections.

- **Multi-group case**: $d = 3$ groups and $n = 3$ classes

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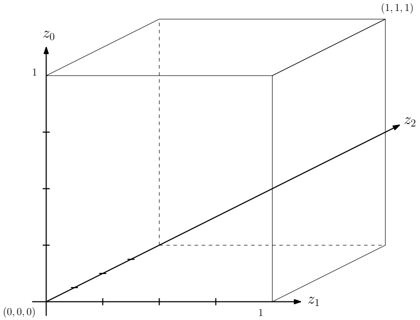

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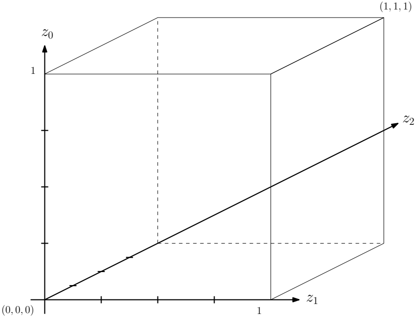

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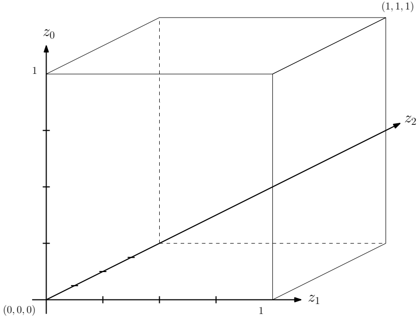

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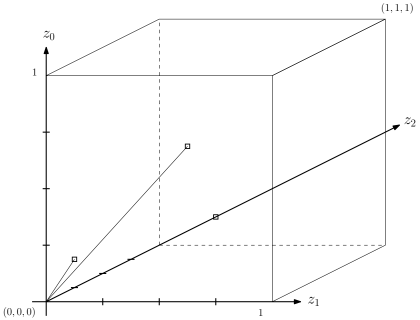

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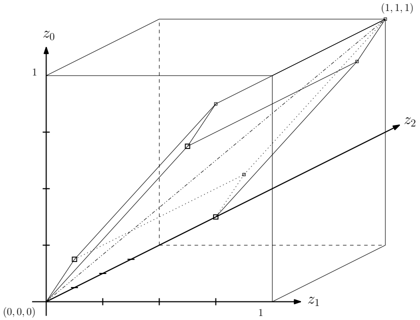

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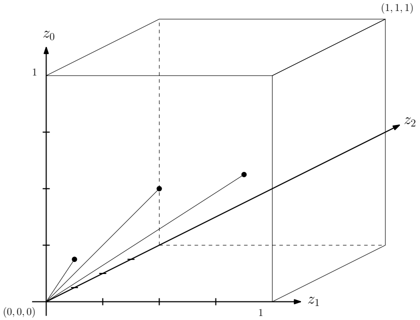

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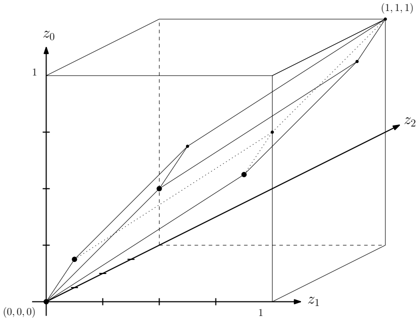

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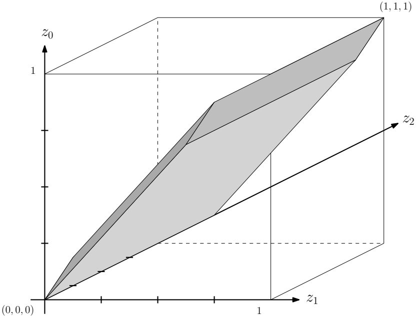

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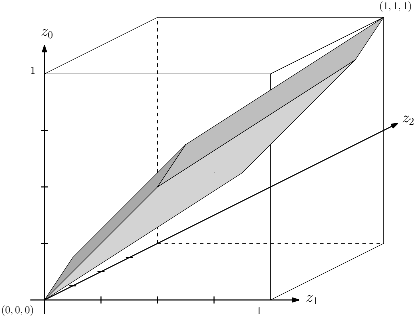

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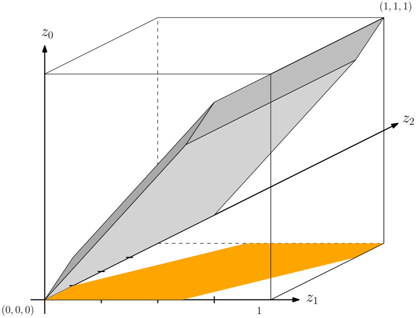

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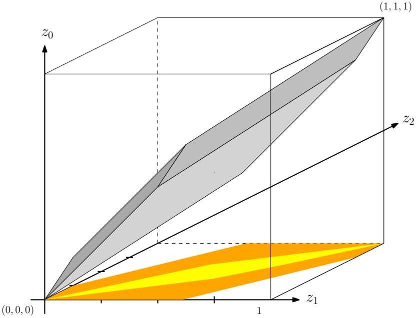

Inequality as dissimilarity

- **Every** inequality comparison **is** a dissimilarity comparison, not the other way-round.
- **Income distributions**: $\mathbf{a} = (1,\ 1,\ 4)$ and $\mathbf{b} = (1,\ 2,\ 3)$
  - **b** is obtained from **a** through a set of **rich to poor** transfers.
  - vector **b** is obtained from **a** by a **T-transform**:

$$(1, 1, 4) \cdot (\mathbf{T} - \mathbf{transform}) = (1, 2, 3)$$

where:

$$(\mathbf{T} - \mathbf{transform}) := \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{1} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

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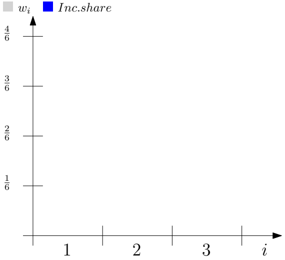
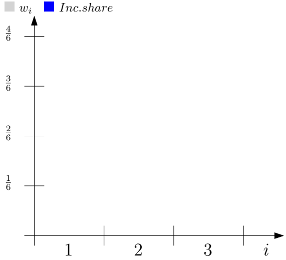
- T-transform are related to operations that reduce the overall dissimilarity between income shares distributions and the population weight distribution.

- Equivalently for **dissimilarity matrices**:

$$\mathbf{A} = \begin{array}{c} \\ w_i \\ Inc.sh. \end{array} \begin{array}{c} \begin{array}{ccc} i_1 & i_2 & i_3 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \end{pmatrix} \end{array} \quad \text{and} \quad \mathbf{B} = \begin{array}{c} \\ w_i \\ Inc.sh. \end{array} \begin{array}{c} \begin{array}{ccc} i_1 & i_2 & i_3 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{pmatrix} \end{array}$$





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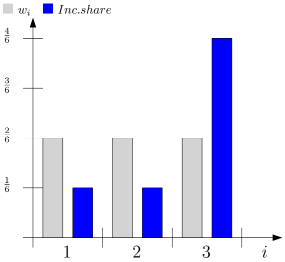

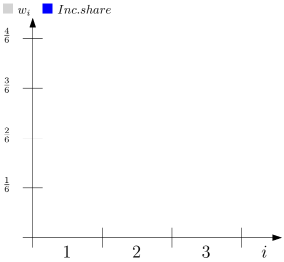

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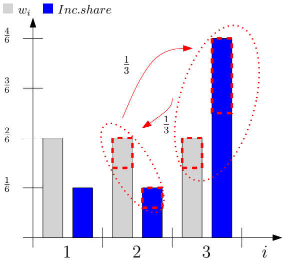

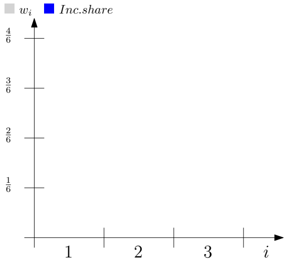

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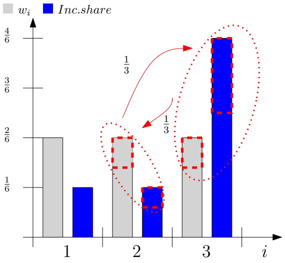

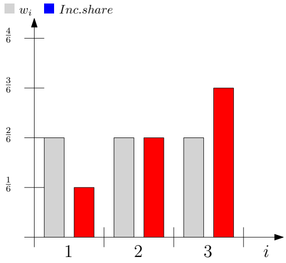

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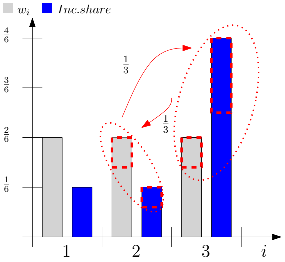

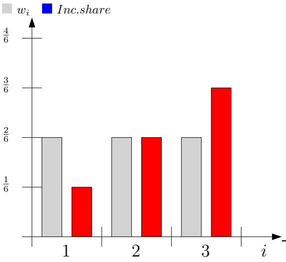

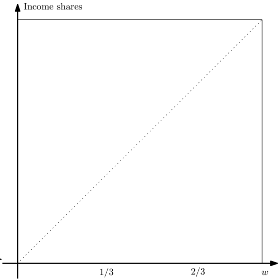

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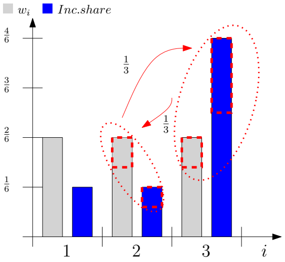

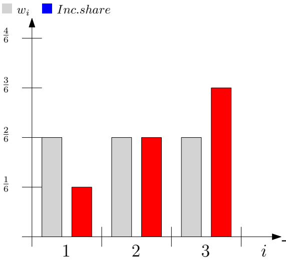

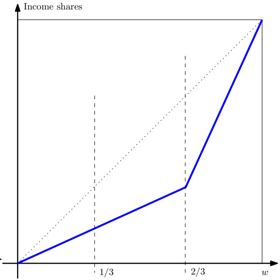

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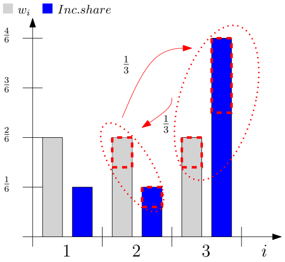

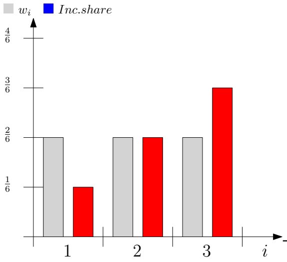

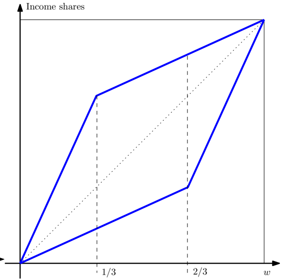

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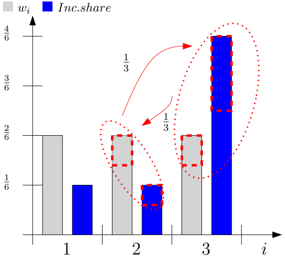

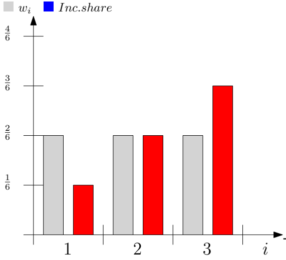

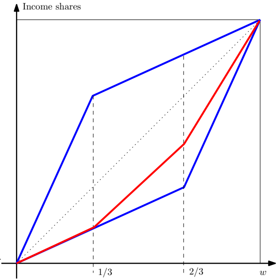

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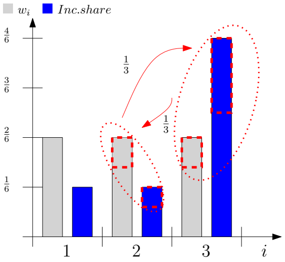
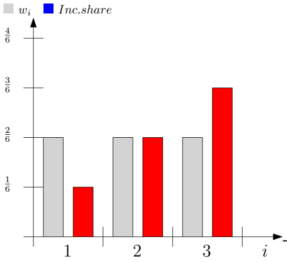
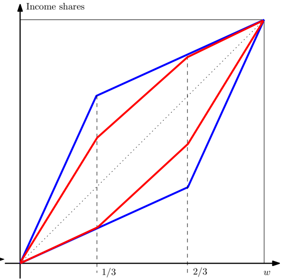
$$\mathbf{A} = \begin{matrix} & i_1 & i_2 & i_3 \\ w_i & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ Inc.sh. & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \end{matrix} \quad \text{and} \quad \mathbf{B} = \begin{matrix} & i_1 & i_2 & i_3 \\ w_i & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ Inc.sh. & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{matrix}$$



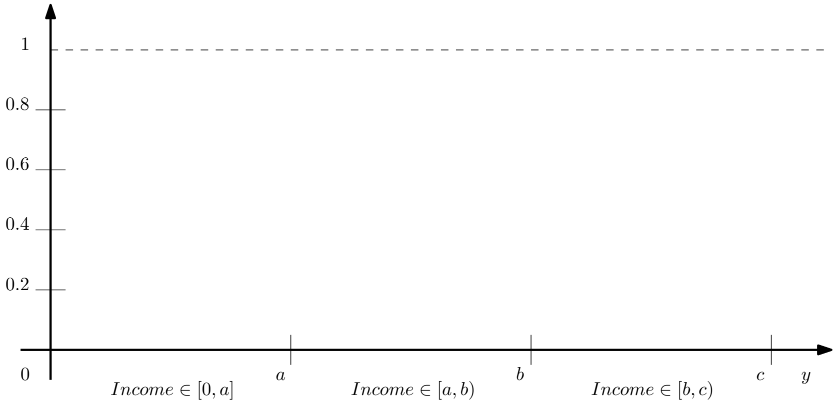




Incoherence of the merge in the ordered case

- **Ordered case**: *cdfs* are defined over exogenously ordered classes:

$$\mathbf{A}' = \begin{pmatrix} cdf \;\; of \;\; group \; 1 \\ cdf \;\; of \;\; group \; 2 \\ cdf \;\; of \;\; group \; 3 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0.6} & \mathbf{0.4} \\ \mathbf{0.25} & \mathbf{0} & \mathbf{0.75} \\ X & X & X \end{pmatrix}$$



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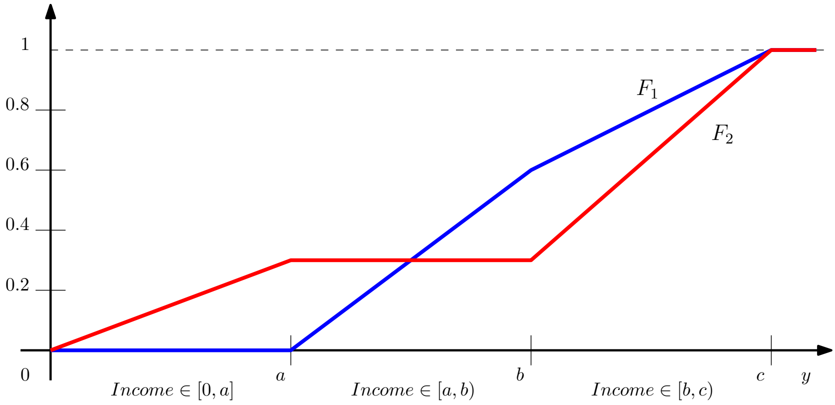

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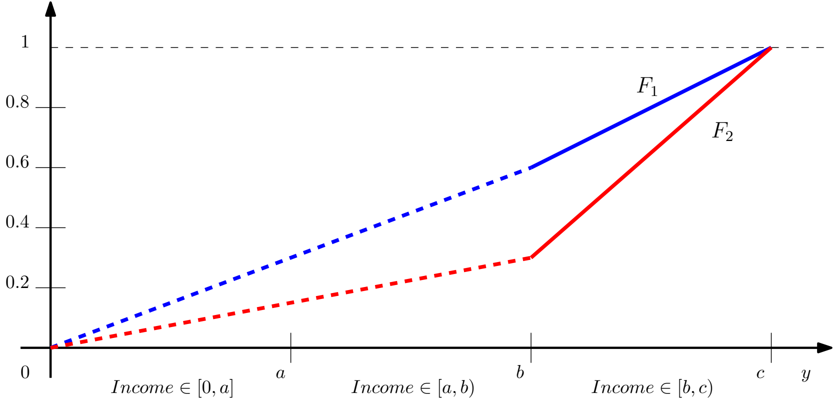

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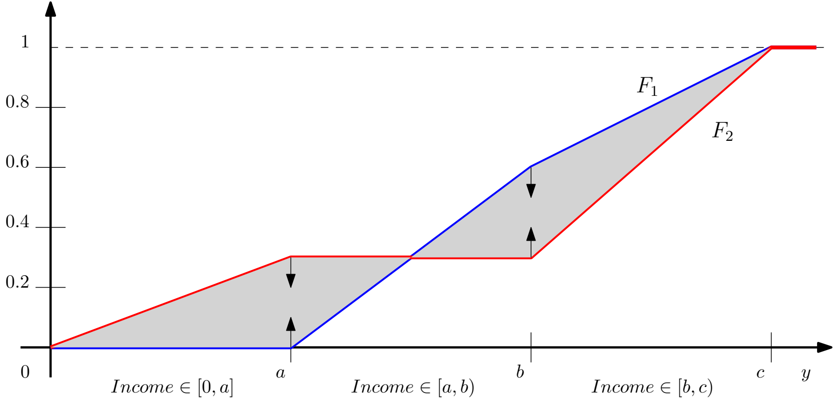

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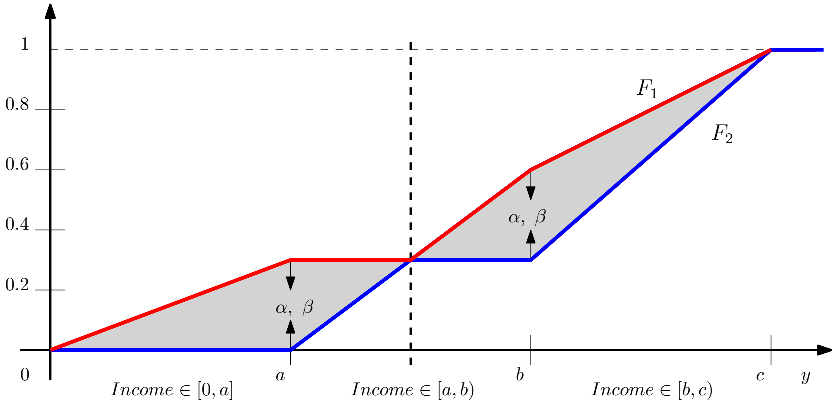

Dissimilarity reducing transfers for ordered classes

Axiom (The exchange operation (Tchen AP1980, Van de gaer et al Ecmica2001):)

Every transfer of population masses (ϵ) across adjacent classes that is progressive for the **dominating group** *and regressive for the* **dominated group** *reduces dissimilarity.*

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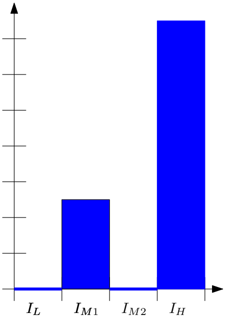

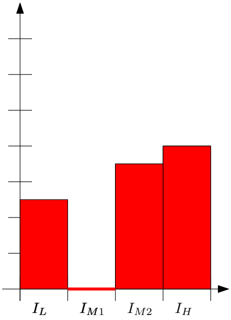

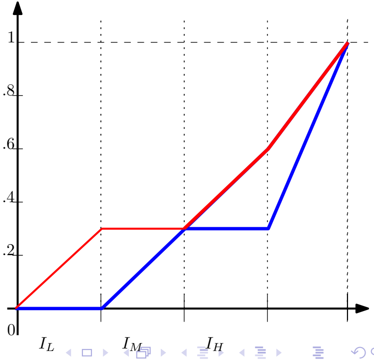

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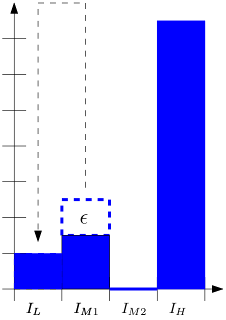

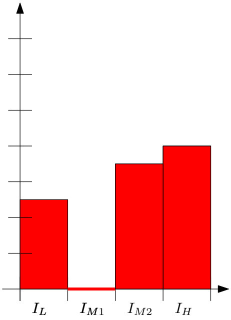

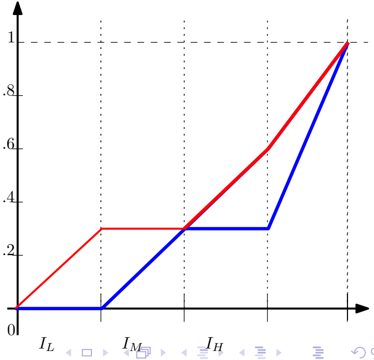

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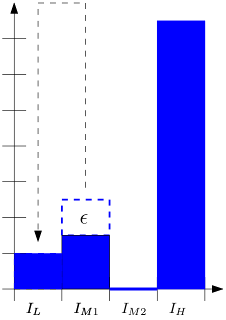

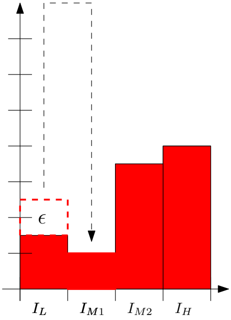

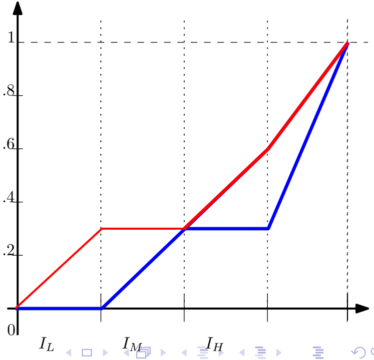

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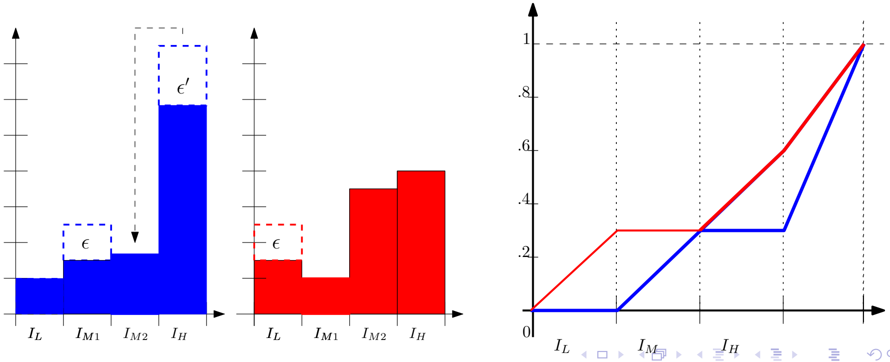

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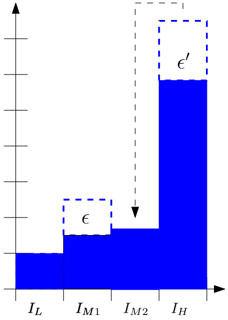

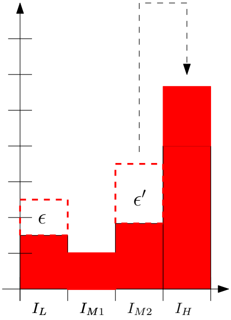

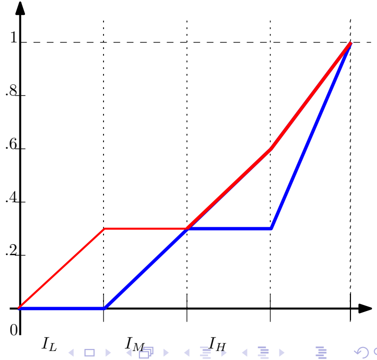

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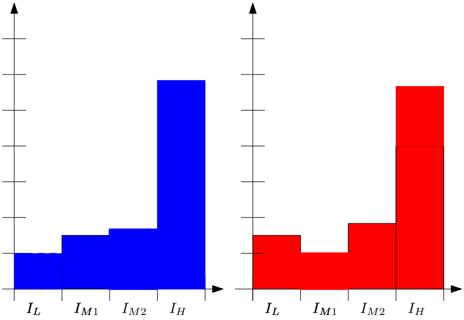

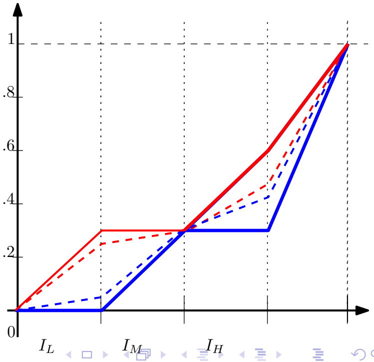

Exchange operations and the Path Polytope

- The Path Pilytope inclusion order detects the existence of exchanges, as below:

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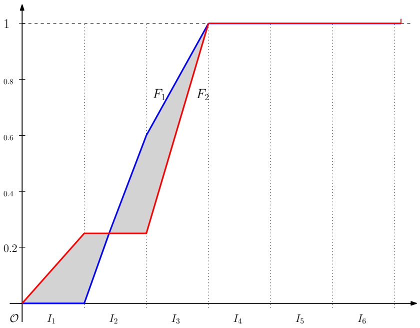

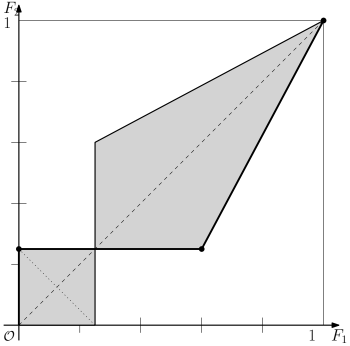

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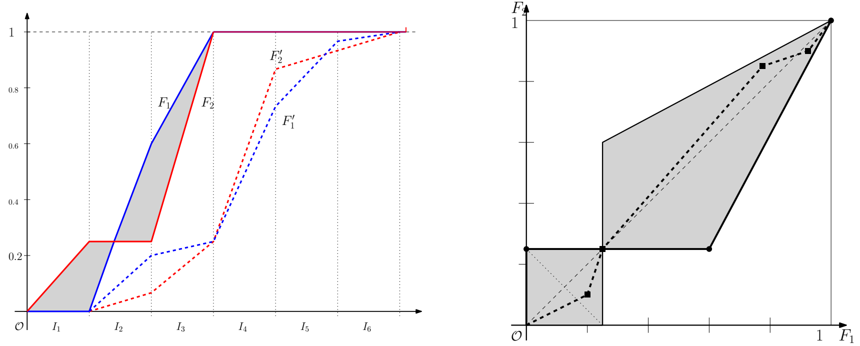

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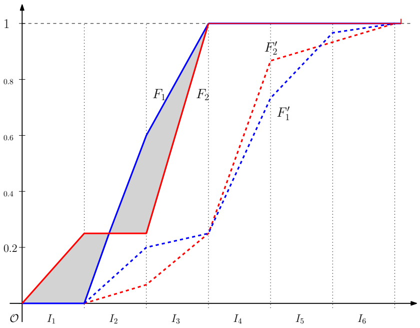

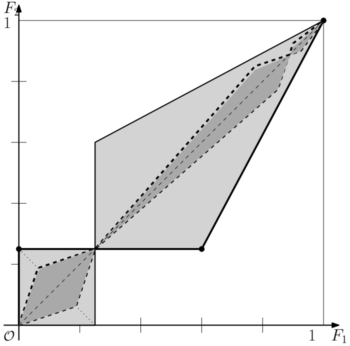

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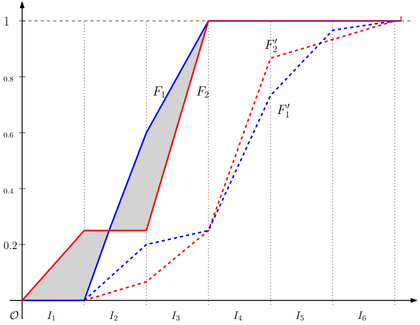

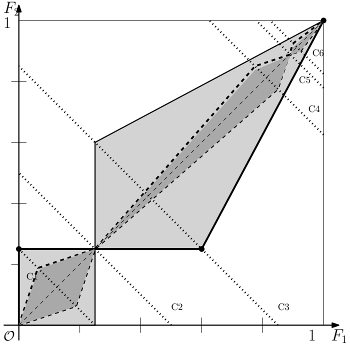

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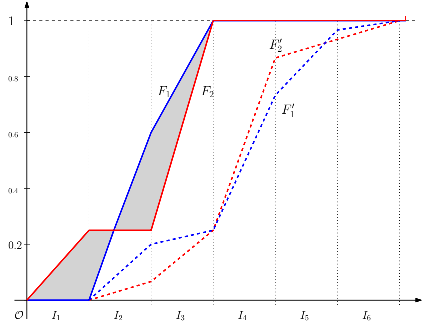

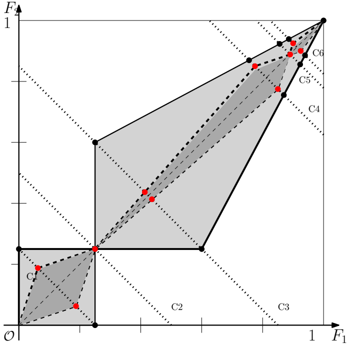

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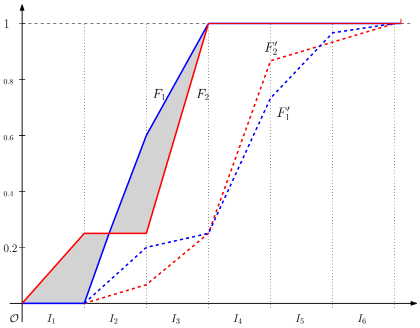

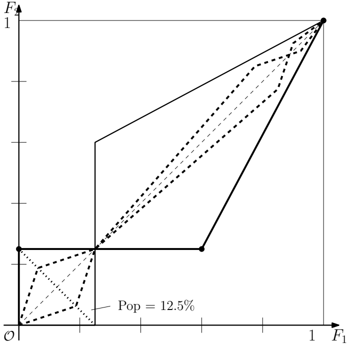

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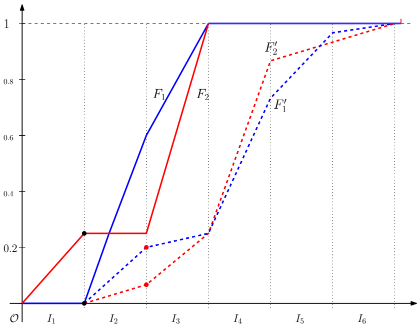

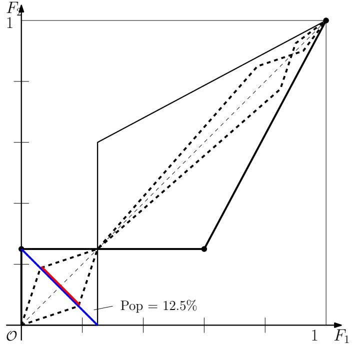

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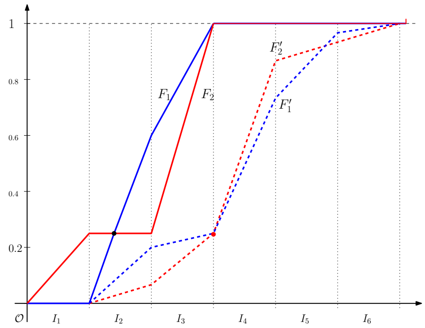

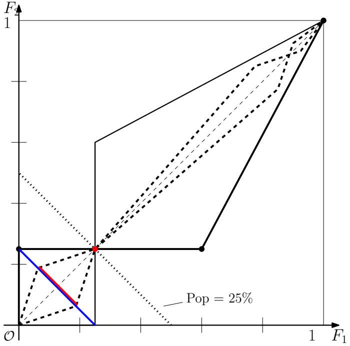

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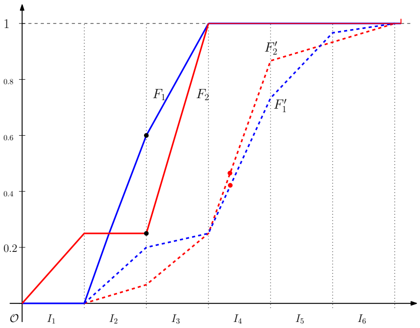
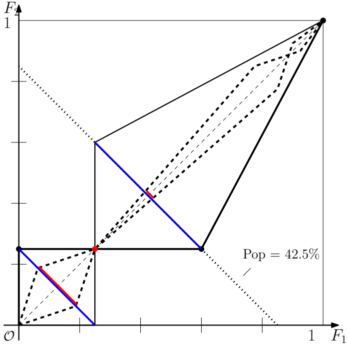
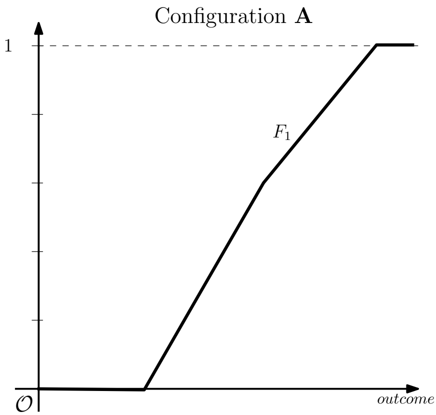
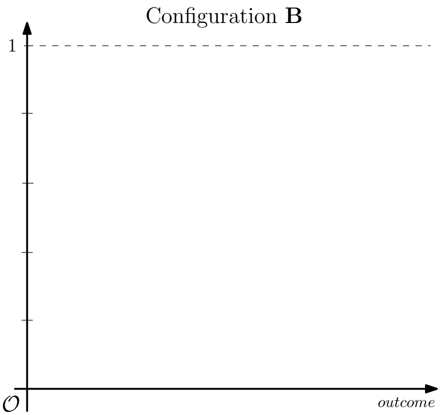




Discrimination comparisons with many distributions.

- Have do do with reductions in dispersion across distributions at every population proportion.





- **A** defined on classes c_1, c_2, c_3, . . .
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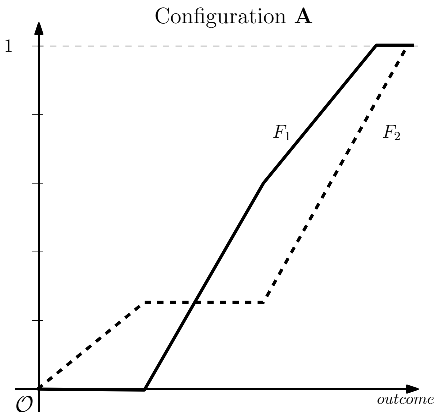

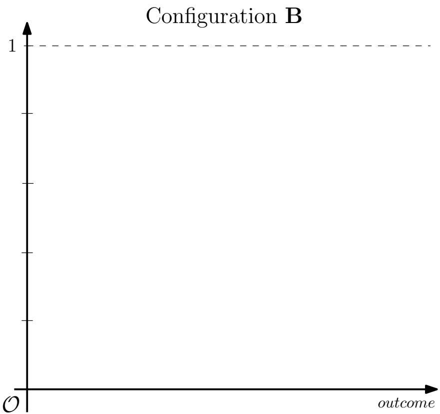

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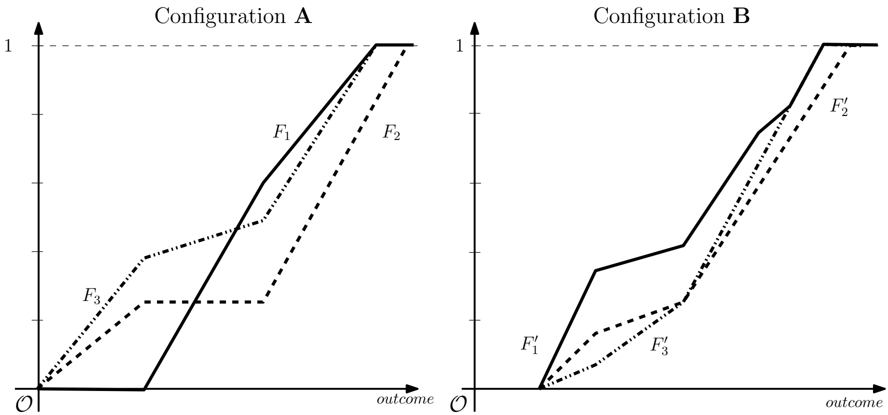

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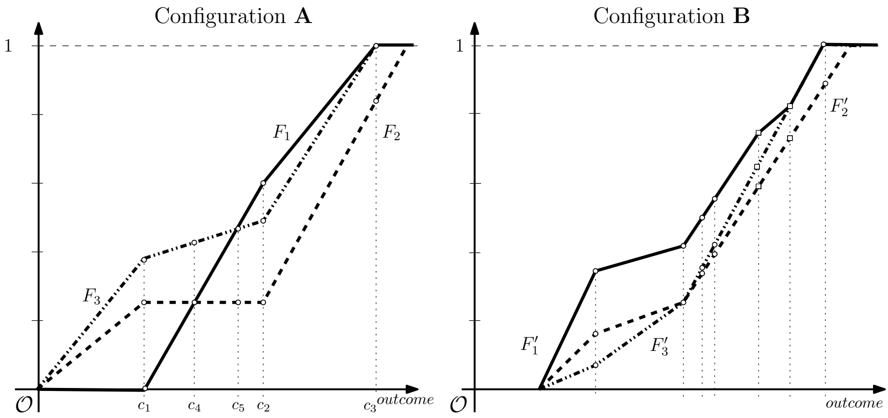

- **A** defined on classes c_1, c_2, c_3, ...
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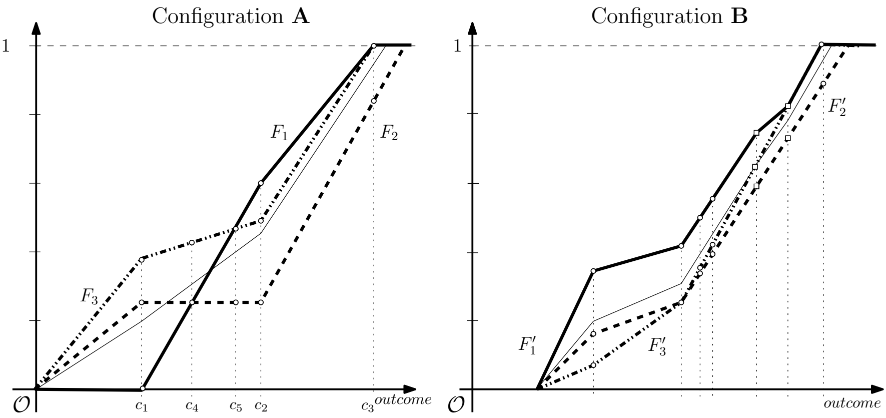

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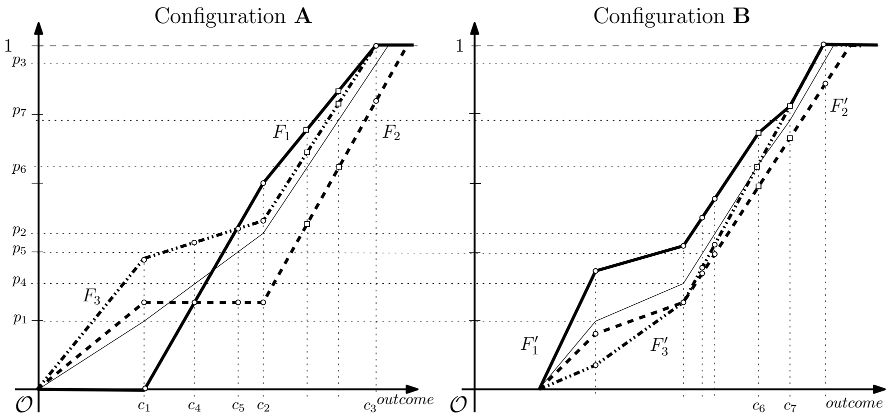

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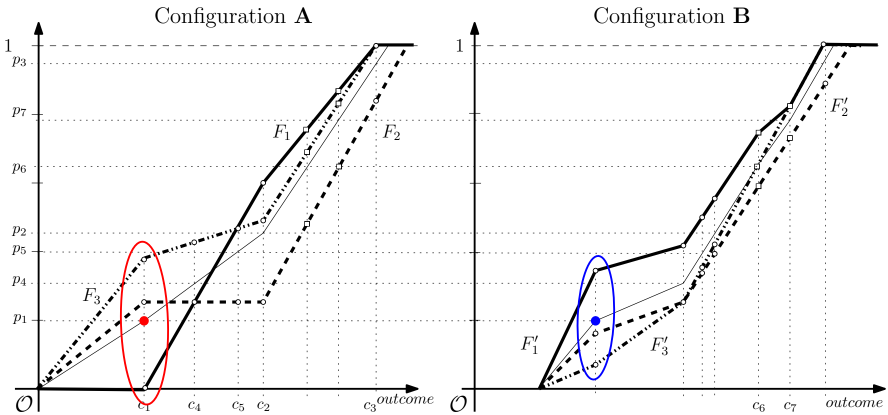

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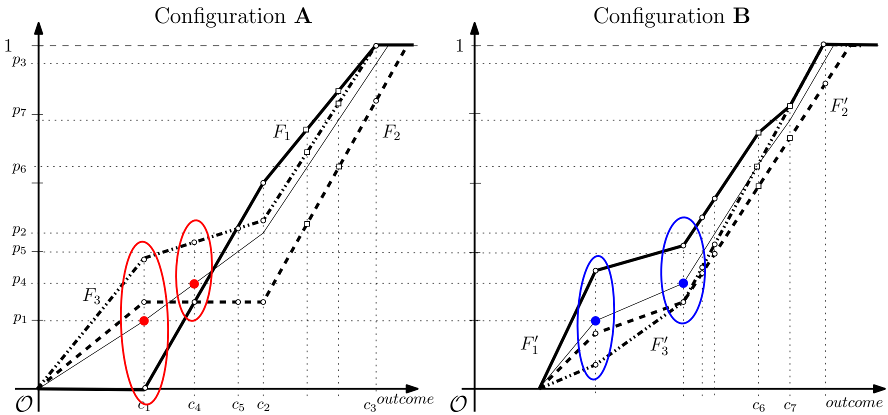

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Path Polytopes in the literature

- The **concentration curves** for $d = 2$ (Butler McDonald JBES1987) correspond to the arrangement of the ordered segments (corresponding to exogenously ordered classes)

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with h convex. It is a model for segregation indices.

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