Working Paper Series
Department of Economics
University of Verona

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WP Number: 8 August 2023

ISSN: 2036-2919 (paper), 2036-4679 (online)

A Procedure for Solving a Linear Complementarity System

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Abstract

We present a procedure for finding a solution of a linear complementarity system. The procedure is based on the theory of the simplex method and generates iterative cuts of the relaxed system, until one of the solutions is obtained. This solution can be used in the first step of the method we have described in [4].

Keywords Complementarity systems, Linear Programming, Simplex Algorithm

Mathematics Subject Classification 90C05, 90C33

JEL Classification C61

1 Introduction

Let us consider the following Linear Complementarity System

$$\text{LCS} \qquad \begin{cases} Ax + By \geq b, \\ x \geq 0, y \geq 0, \\ \langle x, y \rangle = 0, \end{cases}$$

where $A, B \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

The system LCS is the feasible region of the Linear Problem with Complementarity Constraints (LPCC for short) which we have considered in [4]:

$$\text{P} \qquad \begin{cases} \min(\langle c, x \rangle + \langle d, y \rangle) \\ (x, y) \in K_0 := \{(x, y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0, \langle x, y \rangle = 0\}, \end{cases}$$

where $c, d \in \mathbb{R}^n$.

A crucial aspect of the iterative method proposed in [4] to solve P consists in finding a first feasible solution, namely a solution of LCS; nevertheless, the problem of solving a linear complementarity system is an interesting problem in itself (to this purpose, see [1, 2]).

With the aim of finding a solution of LCS, let us consider an objective function which is bounded from below on the set

$$K := \{(x, y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0\},$$

as for example $\ell(x, y) = \sum_{i=1}^{n} (x_i + y_i)$.

Now, let us introduce the following linear problem in the standard form, whose feasible set is obtained by dropping the complementarity constraint from P:

$$\text{RCP} \qquad \begin{cases} \min \ \ell(x, y) \\ Ax + By \pm t = b, \\ x \geq 0, y \geq 0, t \geq 0 \end{cases}$$

where $t \in \mathbb{R}^m$ and $b \geq 0$. In order to have $b \geq 0$, as in the standard form, $\pm t$ is necessary.

Let us observe that, if the function $\langle c, x \rangle + \langle d, y \rangle$ is bounded from below on K, we may consider in RCP $\ell(x, y) = \langle c, x \rangle + \langle d, y \rangle$.

We suppose that all the vertices of the feasible region of RLP correspond to a non-degenerate basic solution. Under these assumptions, in Section 2 we will describe a procedure for finding a feasible solution of LCS. To better illustrate such a procedure, some numerical examples will be proposed in Section 3.

2 Description of the Procedure

Solve RCP and let $(\bar{x}, \bar{y}, \bar{t})$ be one of its optimal basic solutions. If $\langle \bar{x}, \bar{y} \rangle = 0$, $(\bar{x}, \bar{y})$ is a solution of LCS and the procedure ends. If $\langle \bar{x}, \bar{y} \rangle > 0$, let $(\bar{z}_D, \bar{z}_N)$ be the partition of

$\bar{z} := (\bar{x}, \bar{y}, \bar{t})$ into basic and nonbasic components and (D, N) the corresponding partition into basic and nonbasic matrices of $(A, B, \tilde{I}_m)$, where $|\tilde{I}_m| = I_m$ (suppose, without any loss of generality, that the basic components are the first m). We have:

$$\bar{z} = (\bar{z}_D = D^{-1}b - D^{-1}N\bar{z}_N, \bar{z}_N = 0).$$

Denote by $(\beta_{i0}),\ i = 1, \ldots, m$, the $m-$vector $D^{-1}b$, and by $(\beta_{ij}),\ i = 1, \ldots, m,\ j = 1, \ldots, 2n$, the $(m \times 2n)$ matrix $D^{-1}N$. For any $k \in I_N := \{m+1, \ldots, m+2n\}$ let z_k^{sup} defined as in the simplex algorithm [3]:

$$z_k^{sup} := \begin{cases} +\infty & \text{if} \quad \beta_{ik} \leq 0,\ i = 1, \ldots, m \\ \\ \min\limits_{\beta_{ik}>0} \left\{ \dfrac{\beta_{i0}}{\beta_{ik}} \right\}, & \text{otherwise.} \end{cases} \tag{1}$$

Recall that z_k^{sup} is the maximum value that can be assumed by the k-th nonbasic component of z_N and preserving the feasibility of the solution. Under the assumption that the feasible set of RCP is bounded, we have $z_k^{sup} < +\infty,\ \forall k \in I_N$; moreover, if $\bar{z}$ is a nondegenerate basic solution, then $z_k^{sup} > 0,\ \forall k \in I_N$.

Define the following vector $\hat{z} \in \mathbb{R}^{m+2n}$:

$$\begin{cases} \hat{z}_j = \bar{z}_j,\ \forall j = 1, \ldots, m \\ (\hat{z}_{m+1}, \hat{z}_{m+2}, \ldots, \hat{z}_{m+2n}) = (\lambda_1 z_{m+1}^{sup}, \ldots, \lambda_{2n} z_{m+2n}^{sup}) = \displaystyle\sum_{s=1}^{2n} \lambda_s z^{m+s} \end{cases} \tag{2}$$

with $\lambda_s \geq 0$ and $\sum_{s=1}^{2n} \lambda_s < 1$, and

$$z^{m+1} = (z_{m+1}^{sup}, 0, \ldots, 0),\ z^{m+2} = (0, z_{m+2}^{sup}, \ldots, 0), \ldots,\ z^{m+2n} = (0, 0, \ldots, z_{m+2n}^{sup}),$$

Proposition 2.1. *The vector $\hat{z} = (\hat{x}, \hat{y}, \hat{t})$ defined in (2) is such that $\langle \hat{x}, \hat{y} \rangle > 0$.*

Proof. The thesis is immediate if we observe that the null components in the solution $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$ are now positive in the vector $\hat{z} = (\hat{x}, \hat{y}, \hat{t})$, while all the other components are equal to those of $\bar{z}$. □

Let H_0 be the (unique) hyperplane passing though the $2n$ points z^{m+s}, $s = 1, \ldots, 2n$. We have

$$H_0 = \left\{ (z_{m+1}, z_{m+2}, \ldots, z_{m+2n}) \in \mathbb{R}^{2n} : \frac{z_{m+1}}{z_{m+1}^{sup}} + \frac{z_{m+2}}{z_{m+2}^{sup}} + \ldots + \frac{z_{m+2n}}{z_{m+2n}^{sup}} = 1 \right\}.$$

Proposition 2.2. *If $(z_{m+1}, z_{m+2}, \ldots, z_{m+2n})$ belongs to the set*

$$H_{\geq 0}^{-} = \left\{ (z_{m+1}, z_{m+2}, \ldots, z_{m+2n}) \in \mathbb{R}_+^{2n} : \frac{z_{m+1}}{z_{m+1}^{sup}} + \frac{z_{m+2}}{z_{m+2}^{sup}} + \ldots + \frac{z_{m+2n}}{z_{m+2n}^{sup}} < 1 \right\}$$

and (x, y, t) is the corresponding feasible solution to RCP, then $\langle x, y \rangle > 0$.

Proof. If $(z_{m+1}, z_{m+2}, \ldots, z_{m+2n}) \in H^-_{\geq 0}$, then by setting $\frac{z_{m+s}}{z^{sup}_{m+s}} = \lambda_s,\ s = 1, \ldots, 2n$, we have

$$(z_{m+1}, z_{m+2}, \ldots, z_{m+2n}) = (\lambda_1 z^{sup}_{m+1}, \ldots, \lambda_{2n} z^{sup}_{m+2n}) = \sum_{s=1}^{2n} \lambda_s z^{m+s}$$

and

$$\lambda_s \geq 0 \quad s = 1, \ldots, 2n, \text{ and } \sum_{s=1}^{2n} \lambda_s = \sum_{s=1}^{2n} \frac{z^{m+s}}{z^{sup}_{m+s}} < 1.$$

From Proposition 2.1 and (2) the thesis follows. $\square$

Therefore, from Proposition 2.2, if we add the inequality

$$\frac{z_{m+1}}{z^{sup}_{m+1}} + \frac{z_{m+2}}{z^{sup}_{m+2}} + \ldots + \frac{z_{m+2n}}{z^{sup}_{m+2n}} \geq 1 \tag{3}$$

as a constraint in the feasible region of RCP, such inequality is a cut of the feasible set K_0 which does not exclude any solution of P fulfilling the complementarity condition $\langle x, y\rangle = 0$. We add the inequality (3) to the feasible region of RCP and let RCP_1 the problem so obtained from RCP. Then we reapply the described procedure to problem RCP_1.

Remark 2.1. We have obtained the above results under the assumption that the feasible set of RCP is bounded. If this is not the case, in (1) at least one of the value z^{sup}_k could be equal to $+\infty$. Suppose that this happens for the first p's, with $m + 1 \leq p < m + 2n$. Then the cut (3) is replaced by

$$\frac{z_{m+p+1}}{z^{sup}_{m+p+1}} + \ldots + \frac{z_{m+2n}}{z^{sup}_{m+2n}} \geq 1. \tag{4}$$

If $p = m + 2n$, i.e., all the values z^{sup}_k are equal to $+\infty$, then RCP is a problem such that

$$\begin{cases} \inf \ell(x, y) = -\infty \\ Ax + By - t = b, \\ x \geq 0, y \geq 0, t \geq 0 \end{cases}$$

and this contradicts the fact that $\ell(x, y)$ is bounded from below on K.

The other assumption made on RCP is that all its basic solutions are nondegenerate. If $(\bar{x}, \bar{y}, \bar{t})$ is a degenerate basic solution of RCP, we have to apply one of the classic method for handling with degeneration; for instance, the lexicographic rule or the $\varepsilon-$perturbation. In such a way, in (1) we obtain $z^{sup}_k > 0\ \forall k \in I_N$.

Remark 2.2. At each step of the procedure, the cut (3) or (4) excludes a subset of K that contains no solutions (x, y) such that $\langle x, y\rangle = 0$. The procedure ends when one of the optimizations of RCP gives as optimal solution a vector $(\bar{x}, \bar{y}, \bar{t})$ such that $\langle \bar{x}, \bar{y}\rangle = 0$. Hence, if this optimization is performed with the original objective function, i.e. $\langle c, x\rangle + \langle d, y\rangle$, then $(\bar{x}, \bar{y})$ is also an optimal solution of P; in fact, K_0 is evidently a proper subset of the feasible set K reduced by the cut. Otherwise, if the objective function of RCP is $\ell(x, y) = \sum_{i=1}^{n}(x_i + y_i)$, the procedure stops having found a feasible solution of P.

3 Examples

In this section, we propose some examples to clarify the procedure described in Section 2. The examples of problem P are with $n = 1$ because in such a way a geometric representation in $\mathbb{R}^2$ is possible.

Example 1

Let us consider the following example of problem P with $n = 1$, $m = 4$:

$$\begin{cases} \min(-2x - y) \\ \begin{array}{rcl} x \;\; - 2y & \le & 4 \\ x \qquad\; & \le & 8 \\ x \;\; + 2y & \le & 18 \\ -x + 2y & \le & 10 \end{array} \\ x \ge 0, y \ge 0 \\ \langle x, y\rangle = 0 \end{cases}$$

If we introduce the relaxed problem, obtained by dropping the complementarity condition, its feasible region is represented in Figure 1, together with two level sets (in red) of the objective function corresponding to its maximum and minimum values.

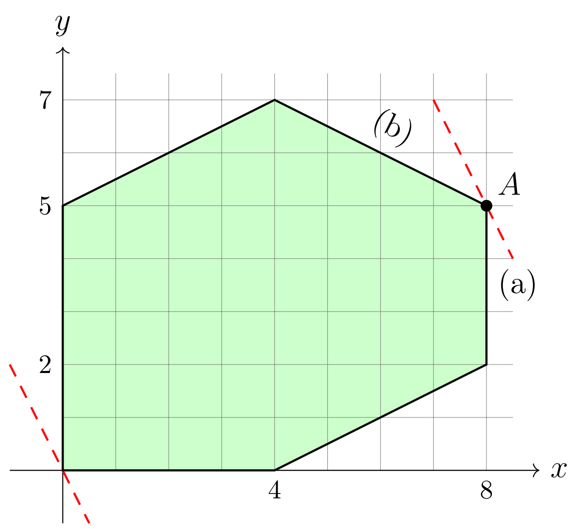


Figure 1

$$\begin{cases} \min(-2x - y) \\ \begin{array}{rcl} x \;\; - 2y & \le & 4 \\ x \qquad\; & \le & 8 \\ x \;\; + 2y & \le & 18 \\ -x + 2y & \le & 10 \end{array} \\ x \ge 0, y \ge 0 \end{cases}$$

The standard form is

$$\begin{cases} \min(-2x - y) \\ x \ - 2y + \ t_1 \qquad\qquad\qquad = \ 4 \\ x \qquad\qquad\quad + t_2 \qquad\qquad = \ 8 \\ x \ + 2y \qquad\qquad + t_3 \qquad = 18 \\ -x + 2y \qquad\qquad\qquad + t_4 = 10 \\ x \geq 0, y \geq 0 \end{cases}$$

The first and last tableau of the simplex iterations are [3]

$$\begin{array}{rrrrrr|r} -2 & -1 & 0 & 0 & 0 & 0 & \\ \hline 1 & -2 & 1 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & 0 & 8 \\ 1 & 2 & 0 & 0 & 1 & 0 & 18 \\ -1 & 2 & 0 & 0 & 0 & 1 & 10 \\ \hline x & y & t_1 & t_2 & t_3 & t_4 & \end{array} \qquad \begin{array}{cccccc|c} 0 & 0 & 0 & 3/2 & 1/2 & 0 & \\ \hline 1 & 0 & 0 & 2/2 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 & 5 \\ 0 & 0 & 1 & -4/2 & 2/2 & 0 & 6 \\ 0 & 0 & 0 & 4/2 & -2/2 & 1 & 8 \\ \hline x & y & t_1 & t_2 & t_3 & t_4 & \end{array}$$

From the latter tableau we have that the optimal solution is

$$(\overline{x}, \overline{y}, \overline{t}_1, \overline{t}_2, \overline{t}_3, \overline{t}_4) = (8, 5, 6, 0, 0, 8)$$

where the partition in basic and nonbasic components is

$$\overline{z}_D = (\overline{x}, \overline{y}, \overline{t}_1, \overline{t}_4) \quad \text{and} \quad \overline{z}_N = (\overline{t}_2, \overline{t}_3).$$

The solution corresponds to the vertex $A = (8, 5)$ in Figure 1. Clearly, the complementarity condition is not fulfilled.

From the latter tableau we can get the information which takes us to the first cut.

The two variables that can enter into the basis are t_2 and t_3. Their maximum values, allowing to stay in the feasible region, by applying (1), are

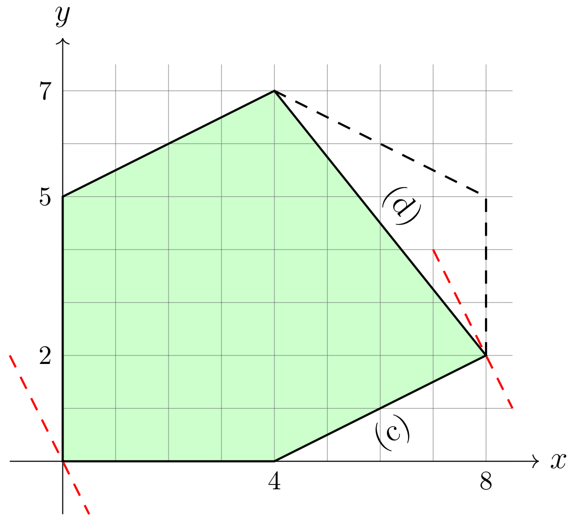
$$z_{t_2}^{sup} = \min(8, 4) = 4 \quad \text{and} \quad z_{t_3}^{sup} = \min(10, 6) = 6.$$

These two values define the inequality (see (3))

$$\frac{t_2}{4} + \frac{t_3}{6} \geq 1 \qquad \text{or equivalently} \qquad 3t_1 + 2t_2 \geq 12.$$

We can easily get the equivalent form in terms of the original variables by using the equations in the standard form $t_2 = 8 - x$ and $t_3 = 18 - x - 2y$ and we get $5x + 4y \leq 48$.

If we add the inequality $5x + 4y \leq 48$ to the feasible set of P, constraints (a) and (b) in Figure 1, namely $x \leq 8$ and $x + 2y \leq 18$ became redundant; therefore we have the new problem:



$$\begin{cases} \min(-2x - y) \\ x \ - 2y \leq \ 4 \\ -x + 2y \leq 10 \\ 5x \ + 4y \leq 48 \\ x \geq 0, y \geq 0 \end{cases}$$

Figure 2

From Figure 2 we may say that the optimal solution is now $(8, 2)$. The standard form is the following, together with the corresponding simplex tableau

$$\begin{cases} \min(-2x - y) \\ x \ - 2y + \ t_1 \qquad\qquad = \ 4 \\ -x + 2y \qquad + t_4 \qquad = 10 \\ 5x \ + 4y \qquad\qquad + t_5 = 48 \\ x \geq 0, y \geq 0 \end{cases}
\qquad
\begin{array}{ccccc|c} -2 & -1 & 0 & 0 & 0 & \\ \hline 1 & -2 & 1 & 0 & 0 & 4 \\ -1 & 2 & 0 & 1 & 0 & 10 \\ 5 & 4 & 0 & 0 & 1 & 48 \\ \hline x & y & t_1 & t_4 & t_5 & \end{array}$$

The tableau related to the optimal solution can be directly obtained by considering that correspondingly to solution $(8, 2)$ we have the basic variables x, y, t_4. With the suitable matrix for the basis change we get the tableau

$$\begin{array}{ccccc|c} 0 & 0 & {}^{3}/_{14} & 0 & {}^{5}/_{14} & \\ \hline 1 & 0 & {}^{4}/_{14} & 0 & {}^{2}/_{14} & 8 \\ 0 & 0 & {}^{14}/_{14} & 1 & 0 & 14 \\ 0 & 1 & -{}^{5}/_{14} & 0 & {}^{1}/_{14} & 2 \\ \hline x & y & t_1 & t_4 & t_5 & \end{array}$$

The optimal solution is then

$$(\overline{x}, \overline{y}, \overline{t}_1, \overline{t}_4, \overline{t}_5) = (8, 2, 0, 14, 0)$$

where the partition in basic and nonbasic components is

$$\overline{z}_D = (\overline{x}, \overline{y}, \overline{t}_4) \quad \text{and} \quad \overline{z}_N = (\overline{t}_1, \overline{t}_5).$$

From the tableau we can get the information for the second cut.

The two variables that can enter into the basis are t_1 and t_5. Their maximum values, allowing to stay in the feasible region, by applying (1), are

$$z_{t_1}^{sup} = \min(28, 14) = 14 \quad \text{and} \quad z_{t_5}^{sup} = \min(56, 28) = 28.$$

These two values define the inequality (see (3))

$$\frac{t_1}{14} + \frac{t_5}{28} \geq 1 \qquad \text{or equivalently} \qquad 2t_1 + t_5 \geq 28.$$

We can easily get the equivalent form in terms of the original variables by using the equations in the standard form $t_1 = 4 - x + 2y$ and $t_5 = 48 - 5x - 4y$ and we get $x \leq 4$.

The constraints (c) and (d) in Figure 2 become redundant and the new problem is

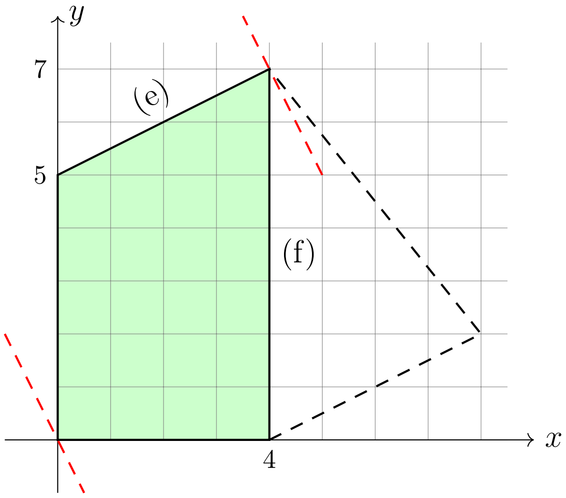


Figure 3

$$\begin{cases} \min(-2x - y) \\ -x + 2y \leq 10 \\ x \qquad\quad \leq \ 4 \\ x \geq 0, y \geq 0 \end{cases}$$

From Figure 3 we may say that the optimal solution is now $(4, 7)$. The standard form is the following, together with the corresponding simplex tableau

$$\begin{cases} \min(-2x - y) \\ -x + 2y + t_4 \qquad = 10 \\ x \qquad\qquad + t_6 = \ 4 \\ x \geq 0, y \geq 0 \end{cases} \qquad \begin{array}{cccc|c} -2 & -1 & 0 & 0 & \\ \hline -1 & 2 & 1 & 0 & 10 \\ 1 & 0 & 0 & 1 & 4 \\ \hline x & y & t_4 & t_6 & \end{array}$$

The tableau corresponding to the optimal solution can be directly obtained by considering that correspondingly to solution $(4, 7)$ we have the basic variables x, y. With the suitable matrix for the basis change we get the tableau

$$\begin{array}{cccc|c} 0 & 0 & 1/2 & 5/2 & 4 \\ \hline 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1/2 & 1/2 & 7 \\ \hline x & y & t_4 & t_6 & \end{array}$$

The optimal solution is then

$$(\overline{x}, \overline{y}, \overline{t}_4, \overline{t}_6) = (4, 7, 0, 0)$$

where the partition in basic and nonbasic components is

$$\overline{z}_D = (\overline{x}, \overline{y}) \quad \text{and} \quad \overline{z}_N = (\overline{t}_4, \overline{t}_6).$$

From the tableau we can get the information for the third cut.

The two variables that can enter into the basis are t_4 and t_6. Their maximum values, allowing to stay in the feasible region, by applying (1), are

$$z_{t_4}^{sup} = 14 \quad \text{and} \quad z_{t_6}^{sup} = \min(4, 14) = 4.$$

These two values define the inequality (see (3))

$$\frac{t_4}{14} + \frac{t_6}{4} \geq 1 \qquad \text{or equivalently} \qquad 2t_4 + 7t_6 \geq 28.$$

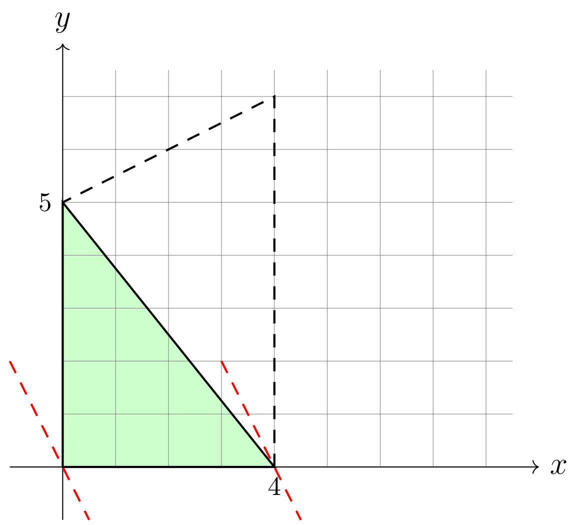
We can easily get the equivalent form in terms of the original variables by using the equations in the standard form

$$t_4 = 10 + x - 2y \qquad \text{and} \qquad t_6 = 4 - x$$

and we get

$$5x + 4y \leq 20.$$

The constraints (e) and (f) in Figure 3 become redundant and the new problem is



$$\begin{cases} \min(-2x - y) \\ 5x + 4y \leq 20 \\ x \geq 0, y \geq 0 \end{cases}$$

The new optimal solution is now $(4, 0)$, that satisfies the complementarity condition and hence is a feasible solution of P. Observe that $(4, 0)$ is also the optimal solution of the given problem (see Remark 2.2).

Example 2

Let us consider the following example of problem P with $n = 1$, $m = 3$:

$$\begin{cases} \min(-x - y) \\ \ x \ + 4y \geq \ 4 \\ 2x + \ y \ \geq \ 2 \\ 2x - \ y \ \geq -4 \\ x \geq 0, y \geq 0 \\ \langle x, y \rangle = 0. \end{cases}$$

If we introduce the relaxed problem, obtained by dropping the complementarity condition, its feasible region is represented in Figure 4.

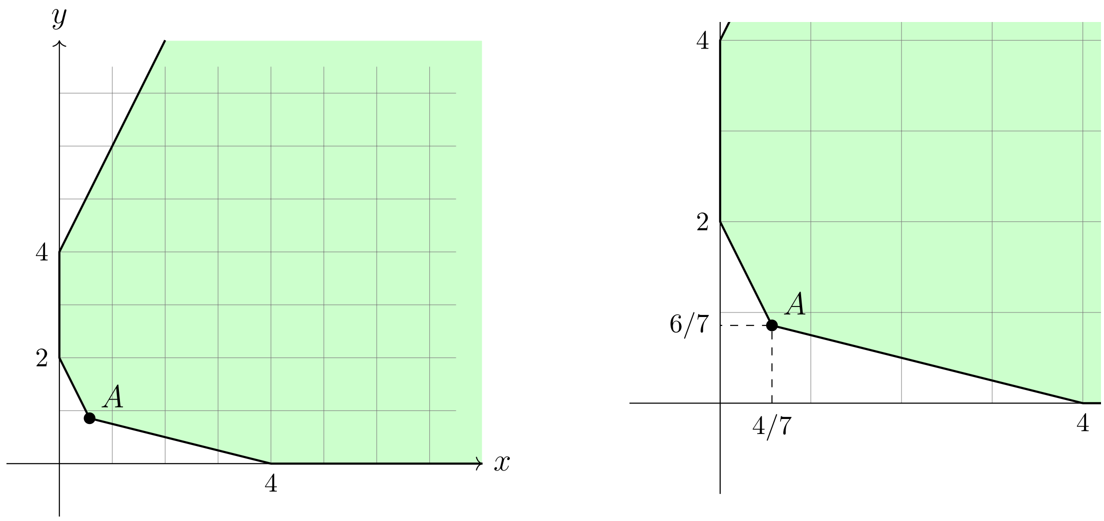


Figure 4

Obvioulsy, the objective function $(-x - y)$ is not bounded from below on the feasible region; therefore, we substitute it with $(x+y)$ and we obtain the following linear problem in the standard form

$$\begin{cases} \min\ (x + y) \\ \ x \ \ + 4y - t_1 \qquad\qquad = 4 \\ \ 2x \ + \ y \qquad - t_2 \qquad = 2 \\ -2x + \ y \qquad\qquad + t_3 = 4 \\ x, y \geq 0, t_i \geq 0, i = 1, 2, 3. \end{cases}$$

The application of the simplex algorithm determines the optimal solution

$$(\overline{x}, \overline{y}, \overline{t}_1, \overline{t}_2, \overline{t}_3) = \left(\frac{4}{7}, \frac{6}{7}, 0, 0, \frac{30}{7}\right)$$

where the partition in basic and nonbasic components is

$$\overline{z}_D = (\overline{x}, \overline{y}, \overline{t}_3) \quad \text{and} \quad \overline{z}_N = (\overline{t}_1, \overline{t}_2).$$

The solution corresponds to the vertex $A = (\frac{4}{7}, \frac{6}{7})$ in Figure 4.

Clearly, the complementarity condition is not fulfilled. The corresponding simplex tableau is:

$$
\begin{array}{ccccc|c}
0 & 0 & 1/7 & 3/7 & 0 & \\
\hline
1 & 0 & 1/7 & -4/7 & 0 & 4/7 \\
0 & 1 & -2/7 & 1/7 & 0 & 6/7 \\
0 & 0 & 4/7 & -9/7 & 1 & 30/7 \\
\hline
x & y & t_1 & t_2 & t_3 &
\end{array}
$$

and the application of (1) determines

$$z_{t_1}^{sup} = \min(4, 15/2) = 4 \quad \text{and} \quad z_{t_2}^{sup} = 6.$$

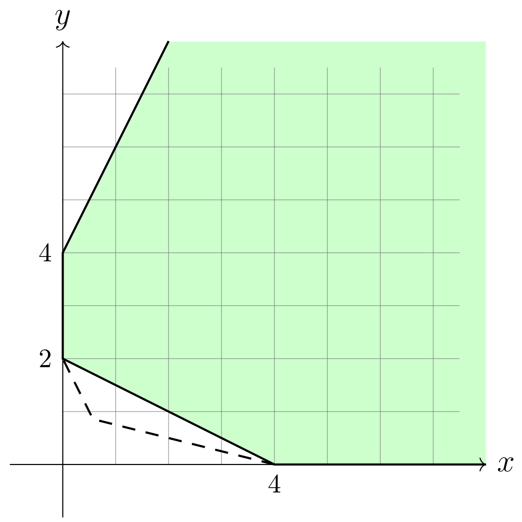
These two values define the inequality (see (3))

$$\frac{t_1}{4} + \frac{t_2}{6} \geq 1 \qquad \text{or equivalently} \qquad 3t_1 + 2t_2 \geq 12.$$

Taking into account that

$$t_1 = x + 4y - 4 \quad \text{and} \quad t_2 = 2x + y - 2,$$

we obtain the cut $x + 2y \geq 4$ of the feasible set:



If we add the inequality $x + 2y \geq 4$ to the feasible set of P, $x + 4y \geq 4$ and $2x + y \geq 2$ became redundant; therefore we have the new problem:

$$\begin{cases} \min(x+y) \\ 2x - \ y \geq -4 \\ \ x \ + 2y \geq \ 4 \\ x \geq 0, y \geq 0. \end{cases}$$

A further application of the simplex algorithm finds the optimal solution $(\overline{x}, \overline{y}) = (2, 0)$ which satisfies the complementarity condition and hence is a feasible solution of P.

Example 3

In this example we want to illustrate the case where at least one of the values z_k^{sup} defined in (1) is equal to $+\infty$. So, let us consider the following example of problem P with $n = 1$, $m = 3$:

$$\begin{cases} \min(x+2y) \\ -x + 2y \geq -1 \\ \;\; x \;+\; y \;\geq\; 2 \\ 2x \;-\; y \;\geq -4 \\ x \geq 0, y \geq 0 \\ \langle x, y \rangle = 0. \end{cases}$$

The standard form is:

$$\begin{cases} \min\;(x+2y) \\ \;\; x \quad - 2y + t_1 \qquad\qquad\quad = 1 \\ \;\; x \quad + \; y \qquad\; - t_2 \qquad\; = 2 \\ -2x + \; y \qquad\qquad\qquad + t_3 = 4 \\ x, y \geq 0, t_i \geq 0, i = 1, 2, 3. \end{cases}$$

The optimal solution of the problem is

$$(\overline{x}, \overline{y}, \overline{t}_1, \overline{t}_2, \overline{t}_3) = \left(\frac{5}{3}, \frac{1}{3}, 0, 0, 7\right)$$

with corresponding partition in basic and nonbasic components

$$\overline{z}_D = (\overline{x}, \overline{y}, \overline{t}_3) \qquad \text{and} \qquad \overline{z}_N = (\overline{t}_1, \overline{t}_2).$$

The solution corresponds to the vertex $A = (5/3, 1/3)$ in Figure 5.

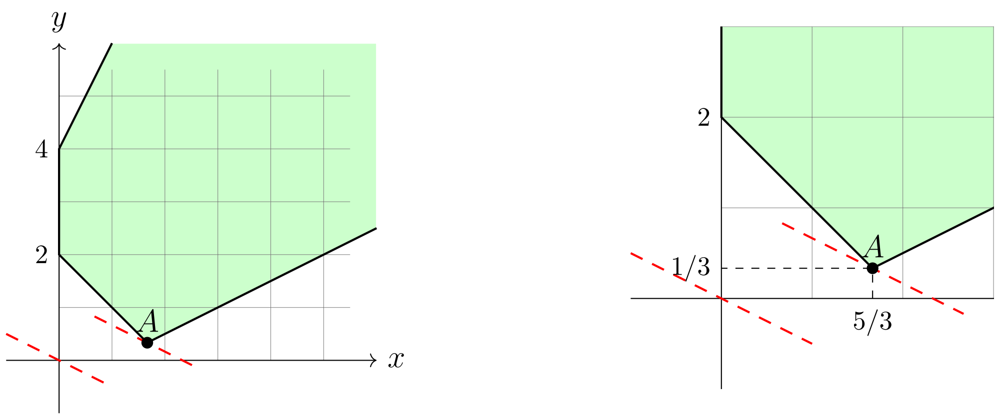


Figure 5

Clearly, the complementarity condition is not fulfilled.

The corresponding simplex tableau is

$$
\begin{array}{ccccc|c}
0 & 0 & 1/3 & 4/3 & 0 & \\
\hline
1 & 0 & 1/3 & -2/3 & 0 & 5/3 \\
0 & 1 & -1/3 & -1/3 & 0 & 1/3 \\
0 & 0 & 1 & -1 & 1 & 7 \\
\hline
x & y & t_1 & t_2 & t_3 &
\end{array}
$$

The application of (1) allows us to determine

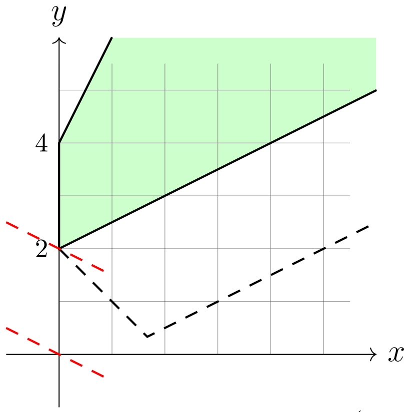
$$z_{t_1}^{sup} = \min\left(\frac{5/3}{1/3}, 7\right) = \min(5,7) = 5 \qquad \text{and} \qquad z_{t_2}^{sup} = +\infty$$

which define the inequality $t_1 \geq 5$ (see (4)).

We obtain the equivalent form in terms of the original variables by using the equation $t_1 = 1 - x + 2y$:

$$-x + 2y \geq 4.$$

The two constraints $-x + 2y \geq -1$ and $x + y \geq 2$ become redundant and the new problem is



$$
\begin{cases}
\min(x + 2y) \\
-x + 2y \geq 4 \\
2x - y \geq -4 \\
x \geq 0, y \geq 0
\end{cases}
$$

The optimal solution is now $(\overline{x}, \overline{y}) = (0, 2)$, that satisfies the complementarity condition and hence is a feasible solution of P, and also the optimal one.

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