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Alessandro Gnoatto, Silvia Lavagnini

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CROSS-CURRENCY HEATH-JARROW-MORTON FRAMEWORK IN THE MULTIPLE-CURVE SETTING

ALESSANDRO GNOATTO AND SILVIA LAVAGNINI

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ABSTRACT. We provide a general HJM framework for forward contracts written on abstract market indices with arbitrary fixing and payment adjustments. We allow for indices on any asset class, featuring collateralization in arbitrary currency denominations. The framework is pivotal for describing portfolios of interest rate products which are denominated in multiple currencies. The benchmark transition has created significant discrepancies among the market conventions of different currency areas: our framework simultaneously covers forward-looking risky IBOR rates, such as EURIBOR, and backward-looking rates based on overnight rates, such as SOFR. In view of this, we provide a thorough study of cross-currency markets in the presence of collateral, where the cash flows of the contract and the margin account can be denominated in arbitrary combinations of currencies. We finally consider cross-currency swap contracts as an example of a contract simultaneously depending on all the risk factors that we describe within our framework.

1. INTRODUCTION

Benchmark reforms have introduced significant discrepancies among interest rate option markets of different currency areas. In the US market, for example, caps and floors are currently written on a compounded version of the secured overnight financing rate (SOFR), whereas in the EUR area the unsecured EURIBOR rate is still the market standard underlying. This poses a significant challenge when considering a portfolio of interest rate derivatives which are denominated in multiple currencies. This is a typical situation arising, for example, when computing risk measures at the portfolio level, or in the context of xVA (x-Value Adjustment) calculations, where all the trades between the bank and the counterparty must be jointly simulated in order to account for netting agreements. A model suitable for these portfolio-wide calculations should then be able to simultaneously describe forward-looking credit-sensitive rates on the one hand, and forward-looking and backward-looking overnight-based interest rates on the other hand.

To solve this issue, in this paper we provide a general HJM framework to describe forward contracts written on abstract market indices. Our setting allows indices with arbitrary fixing and payment adjustments and indices on any asset class, so to accommodate the benchmark transition. Moreover, it allows for multiple currencies, meaning that the cash flows from the contract and the collateralization may be denominated in arbitrary combinations of currencies. Thus we simultaneously extend the literature on multiple-curve valuation with collateral, on interest rate and on cross-currency modelling, and we define the bases for analysing the benchmark transition.

We first provide a sound foundation for our valuation formulas by extending the work of Gnoatto and Seiffert [2021] on cross-currency valuation with collateral. In particular, we consider a general

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setting where the market drivers are Itô semimartingales. Going beyond the diffusive setting, we allow the presence of market incompleteness. In doing so, we extend the martingale approach of Bielecki and Rutkowski [2015] to incomplete markets and reconciled with Piterbarg [2012] and Moreni and Pallavicini [2017], among others. This is achieved by postulating the martingale property of the discounted gains from trading in the contingent claim and without resorting to replication backward stochatic differential equations. Our framework inherits one important feature of Gnoatto and Seiffert [2021] as initially stated in Bielecki and Rutkowski [2015]: the martingale measure of the "domestic" risk-neutral measure, say $\mathbb{Q}^{k_0}$, has not a unique numéraire. On the contrary, under $\mathbb{Q}^{k_0}$ each risky asset, including collateralized derivatives, is discounted by means of an asset-specific cash account. The measure $\mathbb{Q}^{k_0}$ is then a "multi-numéraire" martingale measure. This means in particular that the cash account growing at the overnight rate is not the numéraire of $\mathbb{Q}^{k_0}$, as it is often stated in the existing literature. We obtain valuation formulas for contingent claims denominated under an arbitrary currency k_0 with cash flows paid or received in currency k_2 and with collateral amounts being exchanged in currency k_3. Equipped with these formulas, we systematically treat zero-coupon bonds under arbitrary configurations of the collateral currency and contractual cash flows. This in turns allows to properly define a multitude of forward measures in view of term-structure modeling.

In the second part of the paper, we set the market "in motion" by mean of a Heath-Jarrow-Morton framework Heath et al. [1992]. In doing this, we generalize the existing literature in at least two directions. On the one hand, we extend the general multiple-curve HJM framework of Cuchiero et al. [2016] to a multiple-currency setting, thus setting the cross-currency market "in motion" by means of general Itô semimartingales. On the other hand, we consider abstract indices as the target modeling quantities, in opposition to Cuchiero et al. [2016], where only IBOR rates are modelled. By working with abstract indices, we indeed encompass the case of classical IBOR rates, of new backward-looking indices based on overnight rates, such as SOFR, and other quantities such as inflation or commodity prices. This allows to accommodate the situation where we simultaneously consider a market with standard forward-looking rates (e.g., EURIBOR for the EUR area or TIBOR for the JPY area) and backward-looking rates (e.g., SOFR-based rates for the USD area). This is important in the current market setting since benchmark reforms have introduced a significant level of asymmetry between interest rate markets of different monetary areas. Our framework is then pivotal for the management of large portfolios of interest rates products which are denominated in different currencies and are subject to different market conventions. This is a typical situation which is faced when computing xVA at the portfolio level by Monte Carlo simulations. We also mention that our work extends the cross-currency HJM framework of Fujii et al. [2011] to a general semimartingale setting.

In view of the above-mentioned markets asymmetries, we treat cross-currency swap contracts as test-bed for our framework. These contracts allow us indeed to demonstrate the relevance of all the modeling quantities that we consider in the paper. In particular, we describe cross-currency swap contracts by means of our proposed abstract indices, meaning that we can cover, for example, the situation of a legacy EURUSD cross-currency contract created before the LIBOR transition exchanging USD LIBOR against EURIBOR. According to the US LIBOR act, market participant can indeed choose the LIBOR fallback rate that they deem more appropriate. Hence for the USD leg the agents may agree on the fallback proposed the by International Swaps and Derivatives Association (ISDA) which is based on SOFR, or they may choose an alternative benchmark such as AMERIBOR. Our framework is general enough to cover all the possible situations, so it represents the ideal setup for the valuation of a portfolio that typically combines legacy trades and new positions.

In the remaining part of the introduction we review the most important market features that motivate the present work together with the existing literature.

1.1. **Violations of the covered interest rate parity.** Consider a domestic agent d endowed with an initial capital $\mathcal{N}^d$. At time $t \geq 0$ the agent faces two investment alternatives. A first possibility would be to invest the initial capital for an horizon δ by lending on the d-unsecured market, thus earning the d-IBOR rate $L^d_t(t, t+\delta)$. Alternatively, the agent could enter at time t into a f-foreign exchange forward with length δ and rate $\mathcal{X}^{d,f}_t(t+\delta)$. In this case, at time t he/she would convert the amount $\mathcal{N}^d$ at the spot exchange rate $\mathcal{X}^{d,f}_t$ and lend the amount in foreign currency on the foreign unsecured market, where he/she would earn the unsecured f-IBOR rate $L^f_t(t, t+\delta)$. After the time δ, the agent would then reconvert the amount by means of the foreign exchange forward rate $\mathcal{X}^{d,f}_t(t+\delta)$ agreed at time t. The combination of such an FX spot and an FX forward transaction is termed *FX swap*. We say that the covered interest rate parity holds if the two strategies described deliver the same amount in domestic currency at the end of the period δ. In particular, if the covered interest rate parity were to hold, then we would obtain the classical relation that links the market quote of the FX forward with the unsecured spot rates of the two currencies, namely

$$\mathcal{X}^{d,f}_t(t+\delta) = \mathcal{X}^{d,f}_t \frac{1+\delta L^d_t(t,t+\delta)}{1+\delta L^f_t(t,t+\delta)}. \tag{1.1}$$

Market data on FX swaps and FX forwards show however the systematic violations of (1.1).

We can similarly discuss the valuation of cross-currency swaps. These are long-term transactions which involve an exchange of cash flows between two agents, here denoted with d for *domestic* and f for *foreign*, over a schedule of dates, say $T_0, T_1, \dots, T_n$. Since the cash flows are indexed on the floating rates[1] of two currencies, cross-currency swaps can be seen as a long-short position on two floating-rate bonds denominated in two different currencies. In particular, at time T_0 the two agents lend to each other the notional amounts $\mathcal{N}^d$ and $\mathcal{N}^f$ in domestic and foreign currency, respectively. Then, at each time T_i, $i = 1, \dots, n$, the agents receive floating-rate interests for the notionals lent. In addition, at time T_N the notionals are swapped back. If the covered interest rate parity were to hold, then the sum of the value of the two legs at time $t \leq T_0$ should be zero in the absence of any adjustment. More precisely, taking the perspective of the domestic d agent, we should observe that

$$\begin{aligned} 0 =& \mathcal{N}^d \left(-B^d(t,T_0) + \sum_{i=1}^N (T_i - T_{i-1}) L^d_t(T_{i-1},T_i) B^d(t,T_i) + B^d(t,T_N) \right) \\ &- \mathcal{X}^{d,f}_t \mathcal{N}^f \left(-B^f(t,T_0) + \sum_{i=1}^N (T_i - T_{i-1}) L^f_t(T_{i-1},T_i) B^f(t,T_i) + B^f(t,T_N) \right), \end{aligned} \tag{1.2}$$

where $B^d(t, \cdot)$ and $B^f(t, \cdot)$ denote risk-free zero-coupon bonds in domestic and foreign currency, respectively.

However, when looking at market data, one observes that the covered interest parity is systematically violated. More precisely, the relations (1.1) and (1.2) were approximately satisfied before the 2007 financial crisis. Since then, persistent violations have been observed, with the consequence that cross-currency forward values can not be reconstructed from unsecured funding rates. In particular, the relations (1.1) and (1.2) must be adjusted by introducing the so-called *cross-currency basis swap spread*. For cross-currency swaps against USD, for example, the market practice involves to introduce a spread over the floating rate for the non-USD leg of the contract. If d corresponds to USD, then this means

[1]Depending on the currency pairs involved, the floating rates could be secured or unsecured.

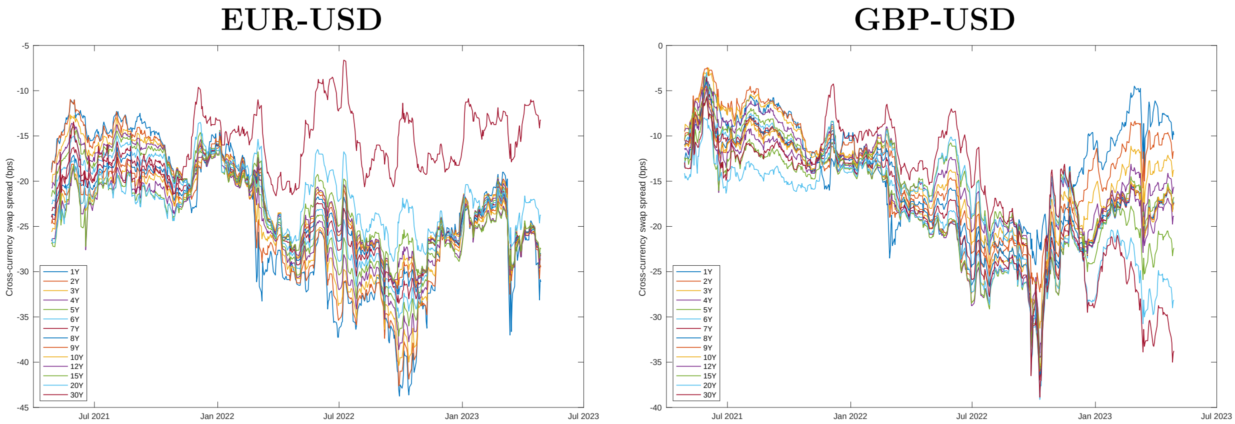

FIGURE 1. Cross-currency basis swap spread time series for the pair EUR-USD (left panel) and for the pair GBP-USD (right panel). Each curve corresponds to a different maturity.

that in order for relation (1.2) to hold, we must substitute $L_t^f(T_{i-1}, T_i)$ with $L_t^f(T_{i-1}, T_i) + \mathcal{S}_0(T_N)$ for all $i = 1, \ldots, N$, where $\mathcal{S}_0(T_N)$ is the cross-currency basis swap spread which is a function of the contract's maturity T_N and is set at the stipulation of the contract. Similarly, for relation (1.1) to hold, we must substitute $L_t^f(t, t+\delta)$ with $L_t^f(t, t+\delta) + \mathcal{S}_0(t+\delta)$. We report in Figure 1 the time series for the cross-currency basis swap spreads $\mathcal{S}_0(T_N)$ for the currencies pairs EUR-USD and GBP-USD, and for several maturities T_N ranging from one to thirty years.

To explain this phenomenon, we need to look into the nature of the contracts under consideration. In particular, the market quotes refer to perfectly collateralized instruments. In other words, the published quotes assume the existence of an ideal collateralization agreement (Credit Support Annex - CSA), in which the two agents exchange margin calls in continuous time so to perfectly annihilate any outstanding credit exposure. On the other hand, the replication strategy involves IBOR rates. Hence, it is subject to (at least) the liquidity risk, since the lending activity is not supported by any guarantee (unsecured lending). In summary, there is a discrepancy between a perfectly collateralized derivative security and a wrongly postulated replication strategy by means of unsecured borrowing/lending. This highlights the importance of studying the cross-currency basis swap spread. This, however, has received a limited coverage in the financial mathematics literature so far: it was analyzed in Fujii and Takahashi [2011], McCloud [2013] and Moreni and Pallavicini [2014]. Moreover, up to our knowledge, the only reference providing a modeling framework for this spread in a HJM setting is Fujii [2013]: our work greatly generalizes this setting.

1.2. **IBOR-OIS spread.** A similar discrepancy is observed in single-currency interest-rate markets when trying to replicate the market quotes of forward-rate agreements (FRA) with unsecured borrowing/lending on IBOR zero-coupon bonds. Let $L_t^{c,k}(T_1, T_2)$ be the (collateralized) FRA rate at time t for the period $[T_1, T_2]$, where c stays for *collateralized* and k denotes a generic currency. If the replication was possible, we should get that

$$L_t^{c,k}(T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{B^k(t, T_1)}{B^k(t, T_2)} - 1\right),$$

where $B^k(t, \cdot)$ denotes the unsecured (IBOR) zero-coupon bonds for the currency k. However, this replication argument fails, as it is shown empirically in Bianchetti and Carlicchi [2013]. Moreover, it is also observed that the forward rate can not be reconstructed from collateralized zero-coupon bonds

$B^{c,k}(t,\cdot)$ in the currency k, namely

$$L_t^{c,k}(T_1,T_2) \neq L_t^{c,k,D}(T_1,T_2) := \frac{1}{T_2-T_1}\left(\frac{B^{c,k}(t,T_1)}{B^{c,k}(t,T_2)}-1\right).$$

This discrepancy has led to the multiple-curve framework initiated by the seminal work by Henrard [2007], and later studied by several authors, e.g. Cuchiero et al. [2016] and Cuchiero et al. [2019], Morini [2013], Kijima et al. [2009], Kijima et al. [2009], Kenyon [2010], Henrard [2010], Mercurio [2010], Mercurio [2013], Mercurio and Xie [2012], Moreni and Pallavicini [2014], Pallavicini and Tarenghi [2010], Crépey et al. [2012], Grbac and Runggaldier [2015], Henrard [2014], Filipović and Trolle [2013], Grasselli and Miglietta [2016], Grbac et al. [2016], Morino and Runggaldier [2014], Grbac et al. [2015], Eberlein et al. [2020].

1.3. **The LIBOR discontinuation is not the end of IBORs.** A further element of complexity in this picture is the ongoing reform of certain interest rate benchmarks. First of all, we need a word of clarity: the term LIBOR refers to the London inter-bank offered rate, which is an unsecured inter-bank rate available for several tenors, maturities and currencies. It has been administrated by the British Bankers Association (BBA) until 2014, and by the Inter Continent Exchange (ICE) afterwards. It is ICE who is managing its discontinuation by publishing selected tenors for the USD and GBP areas via an unrepresentative synthetic methodology. LIBOR, however, is only an example of unsecured inter-bank offered rate subject to a certain jurisdiction. There are indeed several other unsecured inter-bank rates, such as EURIBOR for the EUR area or TIBOR for the JPY area. Hence, we should not take LIBOR as a synonym for inter-bank offered rates in general, and the fact that LIBOR rates are being discontinued does not mean that unsecured inter-bank rates are being discontinued in general. In the following, we clarify the ongoing situation for the EUR and the USD area.

In the EUR area the reform of interest rate benchmarks led to the discontinuation of the unsecured EONIA (Euro Overnight Index Average) which was substituted by ESTR (Euro Short Term Rate). The calculation methodology of the EURIBOR rate has been updated by means of a three step waterfall methodology[2]. There are no plans for a discontinuation of EURIBOR, meaning that for the EUR area a multiple-curve model is still needed in order to properly describe the market of interest rate products[3]. Similarly, in the JPY area there are no plans to discontinue the Tokyo Inter Bank Offered Rate (TIBOR)[4].

The situation in the USD area is more involved. Here the overnight Fed Fund rate has not been discontinued. However, a second overnight rate has been introduced as the central building block of the interest rate market: this is the secured overnight financing rate (SOFR) which is a repo rate where the collateral is given by treasury bills. This means that for the USD area there are two overnight rates, namely an unsecured one (Fed Fund) and a secured one (SOFR). In particular, it is SOFR which is now the market standard for the remuneration of collateral. This means that, for example, a proper valuation of swaps depending on the Fed Fund rate should be performed by means of a two-curve setting in order to account for the spread between the Fed Fund rate and SOFR. Notice that swaps on the Fed Fund rate used to be "old" OIS swaps in the terminology of Cuchiero et al. [2016].

With the demise of USD LIBOR, the market of interest rate swaps and interest rate options mostly moved, in terms of liquidity, to SOFR-based instruments, where the floating rate relevant for a certain coupon is constructed by compounding SOFR over the relevant time window. However, these

[2] `https://www.emmi-benchmarks.eu/benchmarks/euribor/reforms/`

[3] `https://www.esma.europa.eu/sites/default/files/2023-12/ESMA81-1071567537-121_EUR_RFR_WG_Final_Statement.pdf`

[4] `https://www.jbatibor.or.jp/english/reform/`

overnight-based instruments are not suitable for the Asset Liability Management hedging needs of medium and smaller financial institutions. This led on the one hand to criticisms against SOFR, see for example Cooperman et al. [2022], and on the other hand to the introduction of alternative interbank rates such as AMERIBOR T30 or AMERIBOR T90 administrated by the American Financial Exchange[5] with quoted futures on the Chicago Board Options Exchange. As previously mentioned, in the US area, interest rate option markets moved to SOFR-based instruments which have been analyzed in several papers such as Mercurio [2018], Lyashenko and Mercurio [2019], Heitfield and Park [2019], Andersen and Bang [2020], Macrina and Skovmand [2020], Turfus [2020], Willems [2020],Gellert and Schlögl [2021], Rutkowski and Bickersteth [2021], Skov and Skovmand [2021], Backwell and Hayes [2022], Brace et al. [2022], Huggins and Schäller [2022], Schlögl et al. [2023], Fontana [2023], Fontana et al. [2023].

1.4. **Summary of the requirements.** Our objective is to devise a general framework for cross-currency markets that makes it possible to jointly capture all the previously mentioned stylized facts. The paper is structured as follows. In Section 2 we construct the cross-currency basis market which includes general risky assets. In Section 3 we obtain valuation formulas for fully-collateralized contingent claims by extending Gnoatto and Seiffert [2021]. As a preparatory step and application, we thoroughly study zero-coupon bonds (ZCBs) as basic building blocks for term-structure models. Section 4 presents all the measure changes that are relevant for defining our HJM framework. In particular, we define several *extended* forward measures, generalizing the approach of Lyashenko and Mercurio [2019]. In Section 5 we introduce the HJM framework for the multiple discount curves which we need in order to account for the presence of the cross-currency basis spread. In Section 6 we study abstract indices, allowing us to span the whole interest rate market, and, more general, any market with quoted forwards on indices. Finally, Section 7 shows the relevance of the framework in the context of cross-currency swaps valuation.

2. Multi-currency trading in the basic market

We follow the notation of Gnoatto and Seiffert [2021]. Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ satisfying the usual conditions. Here $\mathcal{G}_0$ is assumed to be trivial, and all the processes to be introduced in the sequel are assumed to be $\mathbb{G}$-adapted right-continuous with left limits (RCLL) semimartingales.

We then postulate the existence of $L \in \mathbb{N}$ economies, and we introduce the following indices ranging from 1 to L in order to distinguish between all the possible scenarios:

(i) k_0 denotes the currency of denomination of the portfolio, hence represents the *domestic currency*;
(ii) k_1 denotes the currency of denomination of the risky assets, the associated repo cash accounts, and of the unsecured funding accounts;
(iii) k_2 denotes the currency of denomination for the contractual cash-flows;
(iv) k_3 denotes the currency of denomination of the collateral;
(v) k will be used to denote a general currency.

We set to d_{k_1} the number of risky assets which are traded in terms of the currency with index k_1. Then S^{i,k_1} denotes the ex-dividend price of the i-th risky asset traded in units of currency k_1, and D^{i,k_1} is the corresponding cumulative dividend stream, for every $i = 1, \dots, d_{k_1}$.

[5]The American Financial Exchange is an electronic exchange for direct lending and borrowing for American banks.

The trading desk uses different sources of funding, each being represented by a suitable family of cash accounts. In particular, for each risky asset there is an asset-specific funding account, which we call the *repo account*. We denote by B^{i,k_1} the funding account associated to the asset S^{i,k_1}. Positive or negative dividends from the risky asset S^{i,k_1} are invested in the corresponding funding account B^{i,k_1}. For unsecured funding, we assume that the trading desk can fund its activity by unsecured borrowing and lending in different currencies. We then introduce the cash accounts $B^{k_1} := B^{0,k_1}$ with unsecured rates r^{k_1}, for $k_1 = 1, \dots, L$. Moreover, we use the symbol $\hat{\cdot}$ to denote quantities which are discounted by means of their corresponding repo account, and the symbol $\tilde{\cdot}$ for quantities which are discounted by means of the corresponding unsecured account. For example, for the risky asset S^{i,k_1} we write

$$\hat{S}^{i,k_1} := \frac{S^{i,k_1}}{B^{i,k_1}}, \quad \text{and} \quad \tilde{S}^{i,k_1} := \frac{S^{i,k_1}}{B^{k_1}}.$$

We further denote by $\mathcal{X}^{k_0,k}$ the price of one unit of currency k in terms of currency k_0, for every $k \neq k_0$. Following the usual FORDOM convention, we have, e.g., that $\mathcal{X}^{USD,EUR}$ is the price in USD of 1 EUR.

We work under the following assumptions as in Gnoatto and Seiffert [2021, Assumption 2.1].

Assumption 2.1. *We assume that:*

(i) *For all $i = 1, \dots, d_{k_1}$, and all k_1, the ex-dividend price processes S^{i,k_1} are real-valued RCLL semimartingales;*
(ii) *For all $i = 1, \dots, d_{k_1}$, and all k_1, the cumulative dividend streams D^{i,k_1} are processes of finite variation with $D_0^{i,k_1} = 0$;*
(iii) *For all $j = 0, \dots, d_{k_1}$, and all k_1, the funding accounts B^{j,k_1} are strictly positive and continuous processes of finite variation with $B_0^{j,k_1} = 1$;*
(iv) *For all k_0 and all $k \neq k_0$, the exchange rate processes $\mathcal{X}^{k_0,k}$ are positive-valued RCLL semimartingales.*

Following Gnoatto and Seiffert [2021], we shall first characterize the absence of arbitrage in a market consisting solely of basic traded assets. A trading portfolio in this market is defined as follows.

Definition 2.2. *Let $N_S := \sum_{k_1=1}^{L} d_{k_1}$ be the total number of traded assets in all currencies. A* dynamic portfolio *consisting of risky securities and funding accounts is denoted by $\varphi = (\xi, \psi)$, where:*

(i) *$\xi \in \mathbb{R}^{N_S}$ with $\mathbb{G}$-predictable components, ξ^{i,k_1}, representing the number of shares owned on the risky asset S^{i,k_1}, for $i = 1, \dots, d_{k_1}$, and $k_1 = 1, \dots, L$;*
(ii) *$\psi \in \mathbb{R}^{N_S+L}$ with $\mathbb{G}$-predictable components, ψ^{i,k_1}, representing the units of cash account on B^{i,k_1}, for $i = 0, \dots, d_{k_1}$, and $k_1 = 1, \dots, L$. For $i = 0$ we use the shorthand $\psi^{k_1} := \psi^{0,k_1}$.*

We denote by $V(\varphi)$ the wealth process of the trading strategy φ expressed in currency k_0, where we omit the index k_0 to simplify the notation. From the definition of trading strategy, it is easy to see that

$$V_t(\varphi) = \sum_{k_1=1}^{L} \mathcal{X}_t^{k_0,k_1} \left(\sum_{i=1}^{d_{k_1}} \xi_t^{i,k_1} S_t^{i,k_1} + \sum_{j=0}^{d_{k_1}} \psi_t^{j,k_1} B_t^{j,k_1} \right).$$

The discounted wealth process is $\tilde{V}(\varphi) := \frac{V(\varphi)}{B^{k_0}}$.

We now introduce the concept of self-financing trading strategy.

Definition 2.3. *A trading strategy φ is* self financing *whenever the wealth process $V(\varphi)$ satisfies*

$$\begin{aligned} V_t(\varphi) = \sum_{k_1=1}^{L} \Bigg\{ \sum_{i=1}^{d_{k_1}} \Bigg(\int_{(0,t]} \mathcal{X}_s^{k_0,k_1} \xi_s^{i,k_1} \left(dS_s^{i,k_1} + dD_s^{i,k_1} \right) \\ + \int_{(0,t]} \xi_s^{i,k_1} S_s^{i,k_1} d\mathcal{X}_s^{k_0,k_1} + \int_{(0,t]} \xi_s^{i,k_1} d\left[S^{i,k_1}, \mathcal{X}^{k_0,k_1} \right]_s \Bigg) \\ + \sum_{j=0}^{d_{k_1}} \left(\int_{(0,t]} \mathcal{X}_s^{k_0,k_1} \psi_s^{j,k_1} dB_s^{j,k_1} + \int_{(0,t]} \psi_s^{j,k_1} B_s^{j,k_1} d\mathcal{X}_s^{k_0,k_1} \right) \Bigg\}. \end{aligned}$$

Our first task is to provide conditions guaranteeing absence of arbitrage in the basic market consisting only of risky assets and cash account positions. The concepts of admissibility and of arbitrage opportunity that we consider are the standard ones.

Definition 2.4. *A self-financing trading strategy φ is* admissible *for the trader whenever the discounted wealth $\tilde{V}(\varphi)$ is bounded from below by a constant. An admissible trading strategy φ is an* arbitrage opportunity *whenever $\mathbb{P}\left(\tilde{V}_T(\varphi) \geq 0\right) = 1$ and $\mathbb{P}\left(\tilde{V}_T(\varphi) > 0\right) > 0$, for $T > 0$.*

A classical textbook arbitrage strategy can be constructed in a market with two risk-free assets growing at two different rates. To preclude such trivial arbitrage opportunities, the following repo constraint becomes crucial:

$$\psi_t^{i,k_1} B_t^{i,k_1} + \xi_t^{i,k_1} S_t^{i,k_1} = 0, \quad \text{for every } t \in [0,T],\ i = 1, \dots d_{k_1},\ \text{and } 1 \leq k_1 \leq L. \tag{2.1}$$

The repo constraint reflects the realistic situation where the holdings on every risky asset are financed by a position on the asset-specific cash account, and it is not possible to create long-short positions on different cash accounts to produce risk-less profits.

Lemma 2.5. *Under the repo constraint* (2.1)*, the discounted portfolio dynamics is*

$$d\tilde{V}_t(\varphi) = \frac{1}{B_t^{k_0}} \sum_{k_1=1}^{L} \sum_{i=1}^{d_{k_1}} \xi_t^{i,k_1} \left(dK_t^{i,k_0,k_1} - S_t^{i,k_1} d\mathcal{X}_t^{k_0,k_1} \right) + \sum_{k_1=1, k_1 \neq k_0}^{L} \psi_t^{k_1} d\left(\frac{B^{k_1} \mathcal{X}^{k_0,k_1}}{B^{k_0}} \right)_t,$$

where for every $i = 1, \dots d_{k_1}$, and $k_1 = 1, \dots, L$, the processes

$$K_t^{i,k_0,k_1} := \int_{(0,t]} \left(S_s^{i,k_1} d\mathcal{X}_s^{k_0,k_1} - \frac{S_s^{i,k_1} \mathcal{X}_s^{k_0,k_1}}{B_s^{i,k_1}} dB_s^{i,k_1} + \mathcal{X}_s^{k_0,k_1} dS_s^{i,k_1} + d\left[\mathcal{X}^{k_0,k_1}, S^{i,k_1} \right]_s + \mathcal{X}_s^{k_0,k_1} dD_s^{i,k_1} \right)$$

represent the wealth, denominated in units of currency k_0 and discounted by the funding account B^{i,k_1}, of a self-financing trading strategy that invests in the asset S^{i,k_1}.

Proof. Gnoatto and Seiffert [2021, Corollary 2.15] shows that the portfolio dynamics are of the form

$$\begin{aligned} d\tilde{V}_t(\varphi) = \sum_{k_1=1}^{L} \sum_{i=1}^{d_{k_1}} \frac{1}{B_t^{k_0}} \xi_t^{i,k_1} dK_t^{i,k_0,k_1} + \sum_{k_1=1}^{L} \sum_{i=1}^{d_{k_1}} \frac{1}{B^{i,k_1}} \left(\psi_t^{i,k_1} B_t^{i,k_1} + \xi_t^{i,k_1} S_t^{i,k_1} \right) \mathcal{X}_t^{k_0,k_1} d\left(\frac{B^{i,k_1}}{B^{k_0}} \right)_t \\ + \sum_{k_1=1}^{L} \sum_{i=1}^{d_{k_1}} \frac{B_t^{i,k_1}}{B_t^{k_0}} \psi_t^{i,k_1} d\mathcal{X}_t^{k_0,k_1} + \sum_{k_1=1}^{L} \psi_t^{k_1} d\left(\frac{\mathcal{X}^{k_0,k_1} B^{k_1}}{B^{k_0}} \right)_t. \end{aligned}$$

Hence substituting the repo constraints (2.1) and regrouping terms, we get the result. □

The absence of arbitrage in the basic model is characterized by Gnoatto and Seiffert [2021, Proposition 3.4] as follows.

Proposition 2.6. *Assume that all the strategies available are admissible and satisfy the repo constraint* (2.1)*. Then the multi-currency model is arbitrage free if there exists a probability measure* $\mathbb{Q}^{k_0} \sim \mathbb{P}$ *on* $(\Omega, \mathbb{G})$*, such that the processes*

$$\left(\int_{(0,t]} \left(\mathcal{X}_s^{k_0,k_1} d\left(\frac{S^{i,k_1}}{B^{i,k_1}}\right)_s + \frac{\mathcal{X}_s^{k_0,k_1}}{B_s^{i,k_1}} dD_s^{i,k_1} + d\left[\frac{S^{i,k_1}}{B^{i,k_1}}, \mathcal{X}^{k_0,k_1}\right]_s\right)\right)_{0\le t\le T} \quad \text{and} \tag{2.2}$$

$$\left(\frac{\mathcal{X}_t^{k_0,k_1} B_t^{k_1}}{B_t^{k_0}}\right)_{0\le t\le T} \tag{2.3}$$

are $(\mathbb{Q}^{k_0}, \mathbb{G})$*-local martingales, for all* $i = 1, \ldots, d_{k_1}$*, and all* $k_1 = 1 \ldots, L$*.*

Thanks to Proposition 2.6 we have the recipe to construct arbitrage-free models. In the next section we shall introduce the concept of collateralization and extend the market model with collateralized contracts.

3. Pricing under funding costs and collateralization

The approach of Gnoatto and Seiffert [2021] is based on the assumption that it is possible to replicate contracts with cash flow streams by means of collateralized trading strategies. However, interest rate markets are intrinsically incomplete since interest rates are not traded assets. We then need to adapt the approach of Gnoatto and Seiffert [2021] to the setting of martingale modeling. The idea is that introducing a contingent claim with a given and yet-to-be-determined price process into an arbitrage-free market model does not introduce arbitrage opportunities. Another aspect is that the formulas of Gnoatto and Seiffert [2021] were derived under a diffusive setting, namely without jumps. We consider a more general setting and work with semimartingales as driving processes for the market.

The approach is as follows: starting from the trading portfolio $V(\varphi)$, we include in the market a collateralized contingent claim with dividend flow. The inclusion of this additional asset must be done in a coherent manner, namely without breaking the martingale property of the market. We start with the following assumption.

Assumption 3.1. *The processes* (2.2) *and* (2.3) *and stochastic integrals with respect to* (2.2) *and* (2.3) *are true* $(\mathbb{Q}^{k_0}, \mathbb{G})$*-martingales.*

Thanks to Lemma 2.5, Assumption 3.1 guarantees that $\tilde{V}(\varphi)$ is a true $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingale. As a consequence, the basic market consisting only of the primary assets is free of arbitrage opportunities. We now proceed to extend the market by introducing the contingent claim. In particular, we define the dividend flow of a financial contract as in Gnoatto and Seiffert [2021, Definition 2.5], where we exclude possible cash flows occurring at time zero as these would only represent a shift in the value of the contract.

Definition 3.2. *We define a* financial contract *as an arbitrary RCLL process* A^{k_2} *of finite variation representing the cumulative cash flows paid by the contract in currency* k_2 *from time* 0 *until the maturity date* T*. By convention, we set* $A_0^{k_2} = 0$*.*

We now introduce collateralization. In particular, we consider the cash-collateral convention, meaning that the collateral is exchanged in cash, i.e., in units of an arbitrary currency k_3, and not in terms of units of a risky security. Rehypothecation is also allowed, meaning that the trader can use the cash he/she receives to fund the trading activity. This constitutes the most adopted convention for variation margin.

We represent the collateral by a right-continuous $\mathbb{G}$-adapted process C^{k_3} which is received or posted by the trader in units of currency k_3, with $k_3 = 1, \dots, L$. In particular, for every time instant $t \in [0,T]$, we denote with $C_t^{k_3,+}$ the value of collateral that is received by the trader from the counterparty at time t, and with $C_t^{k_3,-}$ the value of collateral that is posted by the trader to the counterparty at time t. We assume that the collateral account satisfies $C_T^{k_3} = 0$, meaning that the collateral is returned to its legal owner at the terminal time T. We also assume that the agent receives or pays interest contingent on being the poster or the receiver of collateral: the trader receives interest payments based on the rate $r^{c,k_3,l}$ or pays interests based on the rate $r^{c,k_3,b}$.

We then introduce a contingent claim with dividend process A^{k_2} collateralized by means of C^{k_3}, and with price process in domestic currency $S^{k_0}(A^{k_2}, C^{k_3})$. We work under the following assumption.

Assumption 3.3. *The price of the contingent claim $S^{k_0}(A^{k_2}, C^{k_3})$ depends only on the yet-to-be-paid cash-flows, i.e. $S_T^{k_0}(A^{k_2}, C^{k_3}) = 0$, $\mathbb{Q}^{k_0}$-a.s.. Moreover, for every $0 \le t \le T$, we assume that $S_t^{k_0}(A^{k_2}, C^{k_3})$ is integrable with respect to $(\mathbb{Q}^{k_0}, \mathcal{G}_t)$.*

We finally define the full-discounted value process of the claim including the evolution of the mark-to-market and the collateralization procedure.

Definition 3.4. *The* discounted full-value process *of the collateralized contingent claim $S^{k_0}(A^{k_2}, C^{k_3})$ is defined by*

$$\begin{aligned} \tilde{\mathscr{M}}_t :&= \tilde{S}_t^{k_0}(A^{k_2}, C^{k_3}) + \int_{(0,t]} \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{k_0}} dA_s^{k_2} \\ &+ \int_{(0,t]} \left[\left(r_s^{k_0} - r_s^{c,k_3,b} \right) (C_s^{k_3})^+ - \left(r_s^{k_0} - r_s^{c,k_3,l} \right) (C_s^{k_3})^- \right] \frac{\mathcal{X}_s^{k_0,k_3}}{B_s^{k_0}} ds \\ &- \int_{(0,t]} \frac{C_s^{k_3}}{B_s^{k_0}} \mathcal{X}_s^{k_0,k_3} (r_s^{k_0} - r_s^{k_3}) ds. \end{aligned} \tag{3.1}$$

The various terms appearing in (3.1) have the following interpretation: $\tilde{S}_t^{k_0}(A^{k_2}, C^{k_3})$ captures the fluctuations of the mark-to-market of the contract that is held by the trader, who also receives dividends during the lifetime of the contract. These dividends are reinvested in the unsecured cash account which produces $\int_{(0,t]} \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{k_0}} dA_s^{k_2}$. Moreover, the transaction is collateralized. If the trader receives $(C^{k_3})^+$, then he/she reinvests this amount at the unsecured rate r^{k_0} and pays interests to the counterparty at the rate $r^{c,k_3,b}$, thus giving rise to the funding spread $r^{k_0} - r^{c,k_3,b}$. Similar considerations hold for the case when the trader posts the collateral amount $(C^{k_3})^-$. Finally, the last term in (3.1) captures the fluctuations of the collateral amount due to changes in the foreign exchange rate.

The next assumption is crucial in order to preserve absence of arbitrage.

Assumption 3.5. *The process $\tilde{\mathscr{M}}$ and stochastic integral with respect to $\tilde{\mathscr{M}}$ are true $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingales.*

We now denote by $\varphi^{ex} = (\varphi, 1)$ the self-financing portfolio that invests in the basic traded assets according to φ, and invests additionally one unit in the claim with discounted full-value process $\tilde{\mathscr{M}}$. The portfolio is admissible in the sense of Definition 2.4, hence self financing, meaning that

$$d\tilde{V}_t(\varphi^{ex}) = d\tilde{V}_t(\varphi) + d\tilde{\mathscr{M}}_t.$$

Starting from an extended portfolio φ^{ex}, we derive in the following theorem the price formula for the contingent claim $S^{k_0}(A^{k_2}, C^{k_3})$.

Theorem 3.6. *Let Assumption 3.1, 3.3, 3.5 and the repo constraint* (2.1) *hold. Then the price in units of currency k_0 of a contingent claim with cash flows A^{k_2} and with collateral C^{k_3} is*

$$\begin{aligned} S_t^{k_0}(A^{k_2}, C^{k_3}) = B_t^{k_0}\mathbb{E}^{\mathbb{Q}^{k_0}} \Bigg[&\int_{(t,T]} \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{k_0}} dA_s^{k_2} \\ &+ \int_{(t,T]} \left[\left(r_s^{k_0} - r_s^{c,k_3,b}\right)(C_s^{k_3})^+ - \left(r_s^{k_0} - r_s^{c,k_3,l}\right)(C_s^{k_3})^- \right] \frac{\mathcal{X}_s^{k_0,k_3}}{B_s^{k_0}} ds \\ &- \int_{(t,T]} \frac{C_s^{k_3}}{B_s^{k_0}} \mathcal{X}_s^{k_0,k_3}(r_s^{k_0} - r_s^{k_3}) ds \Bigg| \mathcal{G}_t \Bigg]. \end{aligned}$$

Proof. Combining Assumption 3.1 and Assumption 3.5, we deduce that $\tilde{V}_t(\varphi^{ex})$ is a true $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingale. From the martingale property of $\tilde{V}_t(\varphi^{ex})$ we then get that

$$0 = \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\tilde{V}_T(\varphi^{ex}) - \tilde{V}_t(\varphi^{ex}) \middle| \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\tilde{\mathscr{M}}_T - \tilde{\mathscr{M}}_t \middle| \mathcal{G}_t \right],$$

from which, by Assumption 3.3, we deduce the expression for the price of the contingent claim. □

Notice that, modulo the different sign convention, the pricing formula derived in Theorem 3.6 is equivalent to the pricing equation (6.10) in Gnoatto and Seiffert [2021]. We stress however, that the present derivation does not assume a diffusive setting (in fact, it does not rely on any explicit dynamics), nor relies on the concept of replication.

We now simplify the setting by introducing the following assumption.

Assumption 3.7. *We shall assume that $r^{c,k_3,b} = r^{c,k_3,l}$, $\mathbb{Q}^{k_0}$-a.s. for all $k_3 = 1, \dots, L$.*

We then set $r^{c,k_3} := r^{c,k_3,b} = r^{c,k_3,l}$, and let B^{c,k_3} be the collateral cash account with interest rate r^{c,k_3}, namely

$$B_t^{c,k_3} := \exp\left\{ \int_{(0,t]} r_s^{c,k_3} ds \right\}. \tag{3.2}$$

Under Assumption 3.7, we also introduce the following spreads capturing the discrepancy between unsecured rates and collateral rates of different currency denominations.

Definition 3.8. *Let $1 \le k_0 \le L$. We define:*

(i) *The* liquidity spread q^{k_0} *as the difference between the unsecured funding rate r^{k_0} and the collateral rate r^{c,k_0}, namely $q^{k_0} := r^{k_0} - r^{c,k_0}$;*
(ii) *For any $k_3 \neq k_0$, the* cross-currency basis spread q^{k_0,k_3} *as the difference between the liquidity spread for the currency k_0 and the liquidity spread for the currency k_3, namely $q^{k_0,k_3} := q^{k_0} - q^{k_3}$.*

We then introduce the collateral cash account B^{c,k_0,k_3} with interest rate $r^{c,k_0,k_3} := r^{c,k_0} + q^{k_0,k_3}$, namely

$$B_t^{c,k_0,k_3} := \exp\left\{ \int_0^t \left(r_s^{c,k_0} + q_s^{k_0,k_3} \right) ds \right\}. \tag{3.3}$$

We further say that the contingent claim with discounted full-value process (3.1) is perfectly or fully collateralized if

$$C^{k_3} = \frac{S^{k_0}(A^{k_2}, C^{k_3})}{\mathcal{X}^{k_0,k_3}}, \ d\mathbb{P} \otimes dt\text{-a.s.}. \tag{3.4}$$

In this case, the dynamics of $\tilde{\mathscr{M}}$ in (3.1) simplifies to

$$d\tilde{\mathscr{M}}_t = \frac{dS_t^{k_0}(A^{k_2}, C^{k_3})}{B_t^{k_0}} + \frac{\mathcal{X}_t^{k_0,k_2}}{B_t^{k_0}} dA_t^{k_2} - \frac{S_t^{k_0}(A^{k_2}, C^{k_3})}{B_t^{k_0}} \left(r_t^{c,k_0} + q_t^{k_0,k_3} \right) dt, \tag{3.5}$$

and we obtain the following corollary.

Corollary 3.9. *Let Assumption 3.1, 3.3, 3.5 and the repo constraint* (2.1) *hold. Then the price of a fully collateralized contingent claim with cash-flows A^{k_2} and collateral C^{k_3} is*

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = B_t^{c,k_0,k_3} \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{c,k_0,k_3}} dA_s^{k_2} \middle| \mathcal{G}_t \right]. \tag{3.6}$$

Proof. From Assumption 3.5, we have that

$$\begin{aligned}
0 &= \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} \frac{B_s^{k_0}}{B_s^{c,k_0,k_3}} d\tilde{\mathscr{M}}_s \middle| \mathcal{G}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} \frac{B_s^{k_0}}{B_s^{c,k_0,k_3}} \left(\frac{dS_s^{k_0}(A^{k_2}, C^{k_3})}{B_s^{k_0}} + \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{k_0}} dA_s^{k_2} - \frac{S_s^{k_0}(A^{k_2}, C^{k_3})}{B_s^{k_0}} \left(r_s^{c,k_0} + q_s^{k_0,k_3} \right) ds \right) \middle| \mathcal{G}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\frac{S_T^{k_0}(A^{k_2}, C^{k_3})}{B_T^{c,k_0,k_3}} - \frac{S_t^{k_0}(A^{k_2}, C^{k_3})}{B_t^{c,k_0,k_3}} + \int_{(t,T]} \frac{\mathcal{X}_s^{k_0,k_2}}{B_s^{c,k_0,k_3}} dA_s^{k_2} \middle| \mathcal{G}_t \right].
\end{aligned}$$

From Assumption 3.3, we have that $S_T^{k_0}(A^{k_2}, C^{k_3}) = 0$, hence we get the claim. □

The formula obtained in Corollary 3.9 generalizes (6.23) in Gnoatto and Seiffert [2021] to incomplete markets possibly driven by jump-diffusion processes. From (3.6) we can also obtain generalizations of formulas (6.24)-(6.26) of Gnoatto and Seiffert [2021].

Corollary 3.10. *The following pricing formulas can be derived:*

(i) *k_0 cash-flows collateralized in currency k_0: this corresponds to $k_2 = k_3 = k_0$ and we obtain*

$$S_t^{k_0}(A^{k_0}, C^{k_0}) = \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} e^{-\int_t^s r_u^{c,k_0} du} dA_s^{k_0} \middle| \mathcal{G}_t \right],$$

so we discount using the domestic collateral rate. This is the valuation formula employed in the whole literature on single-currency multiple-curve interest rate models.

(ii) *k_0 cash-flows collateralized in a currency k_3: this corresponds to $k_2 = k_0$, $k_3 \neq k_0$ and we obtain*

$$S_t^{k_0}(A^{k_0}, C^{k_3}) = \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} e^{-\int_t^s \left(r_u^{c,k_0} + q_u^{k_0,k_3} \right) du} dA_s^{k_0} \middle| \mathcal{G}_t \right],$$

so that the foreign collateralization results in the appearance of the cross-currency basis in the discount factor.

(iii) *k_2 cash-flows collateralized in currency k_0: this corresponds to $k_2 \neq k_0$, $k_3 = k_0$ and we obtain*

$$S_t^{k_0}(A^{k_2}, C^{k_0}) = \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\int_{(t,T]} e^{-\int_t^s r_u^{c,k_0} du} \mathcal{X}_s^{k_0,k_2} dA_s^{k_2} \middle| \mathcal{G}_t \right].$$

We conclude with a remark.

Remark 3.11. For deriving the pricing formula (3.6), we assumed that $S_T^{k_0}(A^{k_2}, C^{k_3}) = 0$ $\mathbb{P}$-a.s.. If the contract pays a single cash-flow at the terminal time T, then one obtains the same pricing formulas by alternatively postulating that $A^{k_2} = 0$, $d\mathbb{P} \otimes dt$-a.s., and by treating $S_T^{k_0}(A^{k_2}, C^{k_3}) \neq 0$ as the random terminal payoff of the contract. For such a simple instrument the two assumptions are equivalent. This alternative viewpoint will be relevant in the next Section 3.1 when considering forward measures: in this case we will make use of the well-known property of zero-coupon bonds being equal to one at maturity.

3.1. **Pricing of zero-coupon bonds.** We have obtained pricing formulas for contingent claims under arbitrary currency configurations for the promised cash flows and for the collateralization agreement. We proceed now to treat zero-coupon bonds as a special case of this setting. This will serve as basis for term-structure models in Sections 5 and 6. We shall use the shorthand ZCB for zero-coupon bonds.

Remark 3.12. Due to the martingale properties postulated in Section 3, we find it convenient to work directly with the process $\tilde{\mathscr{M}}$ instead of working with the whole extended portfolio. The derivation of equivalent formulas with the extended portfolio is left to the reader.

3.1.1. *Domestic ZCB with domestic collateral.* Let $T \geq 0$ and denote the price process of a domestic ZCB collateralized in domestic currency by $\{B^{k_0,k_0}(t,T),\ 0 \leq t \leq T\}$. In the notation of Section 3, this corresponds to $k_2 = k_3 = k_0$. Moreover,

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = B^{k_0,k_0}(t,T), \quad \text{and} \quad A_t^{k_2} = A_t^{k_0} = 1_{\{t=T\}}.$$

Since $\mathcal{X}^{k_0,k_0} = 1$, $d\mathbb{P} \otimes dt$-a.s., full collateralization takes from equation (3.4) the simpler form

$$C_t^{k_3} = C_t^{k_0} = \frac{B^{k_0,k_0}(t,T)}{\mathcal{X}_t^{k_0,k_0}} = B^{k_0,k_0}(t,T).$$

The process $\tilde{\mathscr{M}}$ in (3.5) simplifies then to

$$d\tilde{\mathscr{M}}_t = \frac{dB^{k_0,k_0}(t,T)}{B_t^{k_0}} + \frac{1}{B_t^{k_0}} d1_{\{t=T\}} - \frac{B^{k_0,k_0}(t,T)}{B_t^{k_0}} r_t^{c,k_0} dt,$$

and from the pricing formula (3.6) we get that

$$B^{k_0,k_0}(t,T) = B_t^{c,k_0} \mathbb{E}^{\mathbb{Q}^{k_0}} \left[\frac{1}{B_T^{c,k_0}} \middle| \mathcal{G}_t \right], \tag{3.7}$$

with B^{c,k_0} the bank account defined in (3.2). Equation (3.7) represents the pricing formula for a so-called *OIS bond* as in, e.g., Cuchiero et al. [2016] and Cuchiero et al. [2019]. Holdings in these bonds are funded by holdings in the asset-specific cash account B^{c,k_0}.

3.1.2. *Domestic ZCB with foreign collateral.* Let $T \geq 0$ and denote the price process of a domestic ZCB collateralized in foreign currency by $\{B^{k_0,k_3}(t,T),\ 0 \leq t \leq T\}$. In the notation of Section 3, this corresponds to $k_2 = k_0$ and $k_3 \neq k_0$. Moreover,

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = B^{k_0,k_3}(t,T), \quad \text{and} \quad A_t^{k_2} = A_t^{k_0} = 1_{\{t=T\}}.$$

Full collateralization from equation (3.4) means that

$$C_t^{k_3} = \frac{B^{k_0,k_3}(t,T)}{\mathcal{X}_t^{k_0,k_3}},$$

and the process $\tilde{\mathscr{M}}$ in (3.5) takes the form

$$d\tilde{\mathscr{M}}_t = \frac{dB^{k_0,k_3}(t,T)}{B_t^{k_0}} + \frac{1}{B_t^{k_0}} d1_{\{t=T\}} - \frac{B^{k_0,k_3}(t,T)}{B_t^{k_0}} (r_t^{c,k_0} + q_t^{k_0,k_3}) dt.$$

From the pricing formula (3.6) we get that

$$B^{k_0,k_3}(t,T) = B_t^{c,k_0,k_3}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{1}{B_T^{c,k_0,k_3}}\right|\mathcal{G}_t\right], \tag{3.8}$$

where B^{c,k_0,k_3} is the bank account defined in (3.3). Notice that the currency dislocation in the collateralization schemes results in the presence of a second term structure of zero-coupon bonds that are funded by means of the newly introduced asset-specific cash account B^{c,k_0,k_3}.

3.1.3. *Foreign ZCB with domestic collateral.* Let $T \geq 0$ and denote the price process of a foreign ZCB collateralized in domestic currency by $\left\{B^{k_2,k_0}(t,T),\ 0 \leq t \leq T\right\}$. In the notation of Section 3, this corresponds to $k_3 = k_0$ and $k_2 \neq k_0$. Moreover,

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = \mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T), \quad \text{and} \quad A_t^{k_2} = 1_{\{t=T\}}.$$

Full collateralization from equation (3.4) means that

$$C_t^{k_3} = \frac{\mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T)}{\mathcal{X}_t^{k_0,k_0}} = \mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T),$$

and the process $\tilde{\mathscr{M}}$ in (3.5) takes the form

$$d\tilde{\mathscr{M}}_t = \frac{d\left(\mathcal{X}_\cdot^{k_0,k_2}B^{k_2,k_0}(\cdot,T)\right)_t}{B_t^{k_0}} + \frac{\mathcal{X}_t^{k_0,k_2}}{B_t^{k_0}}d1_{\{t=T\}} - \frac{\mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T)}{B_t^{k_0}}r_t^{c,k_0}dt.$$

From the pricing formula (3.6) we then get that

$$\mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T) = B_t^{c,k_0}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\mathcal{X}_T^{k_0,k_2}}{B_T^{c,k_0}}\right|\mathcal{G}_t\right], \tag{3.9}$$

with B^{c,k_0} the bank account defined in (3.2).

3.1.4. *Domestic ZCB without collateral.* Let $T \geq 0$ and denote the price of a fully unsecured ZCB in domestic currency by $\left\{B^{k_0}(t,T),\ 0 \leq t \leq T\right\}$. In the notation of Section 3, this corresponds to $k_2 = k_0$ and $C^{k_3} = 0$, $d\mathbb{P}\otimes dt$-a.s.. Then

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = B^{k_0}(t,T), \quad A_t^{k_2} = A_t^{k_0} = 1_{\{t=T\}},$$

and the process $\tilde{\mathscr{M}}$ in (3.5) simplifies significantly to

$$d\tilde{\mathscr{M}}_t = \frac{dB^{k_0}(t,T)}{B_t^{k_0}} + \frac{1}{B_t^{k_0}}d1_{\{t=T\}} - \frac{B^{k_0}(t,T)}{B_t^{k_0}}r_t^{k_0}dt,$$

with r^{k_0} being the unsecured rate. From the pricing formula (3.6) we then get that

$$B^{k_0}(t,T) = B_t^{k_0}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{1}{B_T^{k_0}}\right|\mathcal{G}_t\right].$$

This corresponds to a textbook pre-financial-crisis ZCB linked to the unsecured bank account B^{k_0}.

4. Measure changes

We considered so far the domestic risk-neutral measure $\mathbb{Q}^{k_0}$. This is the measure such that the processes (2.2) and (2.3) are (local) martingales, and we have used $\mathbb{Q}^{k_0}$ as the pricing measure to obtain pricing formulas from the point of view of an agent in the economy k_0. We shall introduce in this sections new measures which naturally arise in our framework and which will be crucial for the

HJM modelling in Sections 5 and 6. More specifically, we shall introduce spot-foreign measures and forward measures.

4.1. **Spot-foreign measures.** Under Assumption 3.1, we introduce spot-foreign risk-neutral measures as follows.

Definition 4.1. *Under Assumption 3.1, let* $1 \leq k_2 \leq L$ *with* $k_2 \neq k_0$. *We define the* $\mathbb{Q}^{k_2}$ (spot)-foreign risk-neutral measure $\mathbb{Q}^{k_2} \sim \mathbb{Q}^{k_0}$ *on* $(\Omega, \mathbb{G})$ *by*

$$\frac{\partial \mathbb{Q}^{k_2}}{\partial \mathbb{Q}^{k_0}} := \frac{B_T^{k_2} \mathcal{X}_T^{k_0,k_2}}{B_T^{k_0}} \frac{B_0^{k_0}}{B_0^{k_2} \mathcal{X}_0^{k_0,k_2}}.$$

Due to the martingale property of (2.3) *we have that*

$$\left.\frac{\partial \mathbb{Q}^{k_2}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\partial \mathbb{Q}^{k_2}}{\partial \mathbb{Q}^{k_0}}\right| \mathcal{G}_t\right] = \frac{B_t^{k_2} \mathcal{X}_t^{k_0,k_2}}{B_t^{k_0}} \frac{B_0^{k_0}}{B_0^{k_2} \mathcal{X}_0^{k_0,k_2}}, \quad \textit{for all } t \leq T. \tag{4.1}$$

This family of measures allows to price ZCBs from different point of views, as illustrated in the following two examples.

4.1.1. *Foreign ZCB with foreign collateral under the domestic measure.* We consider a k_2-ZCB collateralized in currency k_2 from the point of view of the domestic measure $\mathbb{Q}^{k_0}$. Performing a measure change from $\mathbb{Q}^{k_0}$ to $\mathbb{Q}^{k_2}$ allows to obtain the dual formula to (3.7), namely the dual formula to the price of a domestic ZCB with domestic collateral under the domestic measure. In this case, we have $k_3 = k_2$ and $k_0 \neq k_2$. In the notation of Section 3 we then have

$$S_t^{k_0}(A^{k_2}, C^{k_3}) = \mathcal{X}_t^{k_0,k_2} B^{k_2,k_2}(t,T), \quad \text{and} \quad A_t^{k_2} = 1_{\{t=T\}}.$$

Notice that full collateralization in equation (3.4) takes now the form

$$C_t^{k_3} = \frac{\mathcal{X}_t^{k_0,k_2} B^{k_2,k_2}(t,T)}{\mathcal{X}_t^{k_0,k_2}} = B^{k_2,k_2}(t,T),$$

and the process $\tilde{\mathscr{M}}$ in (3.5) satisfies

$$d\tilde{\mathscr{M}}_t = \frac{d\left(\mathcal{X}_\cdot^{k_0,k_2} B^{k_2,k_2}(\cdot,T)\right)_t}{B_t^{k_0}} + \frac{\mathcal{X}_t^{k_0,k_2}}{B_t^{k_0}} dA_t^{k_2} - \frac{\mathcal{X}_t^{k_0,k_2} B^{k_2,k_2}(t,T)}{B_t^{k_0}} \left(r_t^{c,k_0} + q_t^{k_0,k_2}\right) dt.$$

From the pricing formula (3.6) we finally get that

$$\mathcal{X}_t^{k_0,k_2} B^{k_2,k_2}(t,T) = B_t^{c,k_0,k_2} \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\mathcal{X}_T^{k_0,k_2}}{B_T^{c,k_0,k_2}}\right| \mathcal{G}_t\right], \tag{4.2}$$

which provides us with the pricing formula for a k_2-foreign bond collateralized in the currency k_2 from the point of view of a k_0-based agent. By equation (4.1), we now perform a change of measure on the right-hand side of (4.2) and change to the pricing measure $\mathbb{Q}^{k_2}$:

$$\mathcal{X}_t^{k_0,k_2} B^{k_2,k_2}(t,T) = B_t^{c,k_0,k_2} \mathbb{E}^{\mathbb{Q}^{k_2}}\left[\left.\frac{\mathcal{X}_T^{k_0,k_2}}{B_T^{c,k_0,k_2}} \frac{B_T^{k_0}}{\mathcal{X}_T^{k_0,k_2} B_T^{k_2}}\right| \mathcal{G}_t\right] \frac{\mathcal{X}_t^{k_0,k_2} B_t^{k_2}}{B_t^{k_0}}.$$

Notice that

$$\frac{B_T^{k_0}}{B_T^{c,k_0,k_2} B_T^{k_2}} = \exp\left\{\int_0^T \left(r_s^{k_0} - r_s^{k_2} - (r_s^{c,k_0} + q_s^{k_0,k_2})\right) ds\right\} = \exp\left\{-\int_0^T r_s^{c,k_2} ds\right\} = \frac{1}{B_T^{c,k_2}},$$

hence equation (4.2) simplifies to

$$B^{k_2,k_2}(t,T) = B_t^{c,k_2}\mathbb{E}^{\mathbb{Q}^{k_2}}\left[\frac{1}{B_T^{c,k_2}}\middle|\mathcal{G}_t\right].$$

This is the formula for the same contract under the dual measure $\mathbb{Q}^{k_2}$, which is the same as the previously obtained formula (3.7) in Section 3.1.1 for $k_2 = k_0$.

4.1.2. *Foreign ZCB with domestic collateral under the foreign measure.* We consider a k_2-ZCB with domestic collateral as in Section 3.1.3. With a measure change from $\mathbb{Q}^{k_0}$ to $\mathbb{Q}^{k_2}$ we will obtain a dual valuation formula under the foreign measure which is consistent with equation (3.8) of Section 3.1.2, namely with the pricing formula of a domestic ZCB with foreign collateralization.

Starting from equation (3.9), we perform a change of measure to $\mathbb{Q}^{k_2}$ accordingly to equation (4.1):

$$\mathcal{X}_t^{k_0,k_2}B^{k_2,k_0}(t,T) = B_t^{c,k_0}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\frac{\mathcal{X}_T^{k_0,k_2}}{B_T^{c,k_0}}\middle|\mathcal{G}_t\right] = B_t^{c,k_0}\mathbb{E}^{\mathbb{Q}^{k_2}}\left[\frac{\mathcal{X}_T^{k_0,k_2}}{B_T^{c,k_0}}\frac{B_T^{k_0}}{B_T^{k_2}\mathcal{X}_T^{k_0,k_2}}\middle|\mathcal{G}_t\right]\frac{B_t^{k_2}\mathcal{X}_t^{k_0,k_2}}{B_t^{k_0}},$$

which leads to

$$B^{k_2,k_0}(t,T) = \frac{B_t^{c,k_0}B_t^{k_2}}{B_t^{k_0}}\mathbb{E}^{\mathbb{Q}^{k_2}}\left[\frac{B_T^{k_0}}{B_T^{c,k_0}B_T^{k_2}}\middle|\mathcal{G}_t\right]. \tag{4.3}$$

Notice that

$$\frac{B_T^{k_0}}{B_T^{c,k_0}B_T^{k_2}} = \exp\left\{\int_0^T\left(r_s^{k_0} - r_s^{c,k_0} - r_s^{k_2}\right)ds\right\} = \exp\left\{-\int_0^T\left(r_s^{c,k_2} + q_s^{k_2,k_0}\right)ds\right\} = \frac{1}{B_T^{c,k_2,k_0}(t,T)},$$

hence (4.3) becomes

$$B^{k_2,k_0}(t,T) = B_t^{c,k_2,k_0}\mathbb{E}^{\mathbb{Q}^{k_2}}\left[\frac{1}{B_T^{c,k_2,k_0}}\middle|\mathcal{G}_t\right].$$

This corresponds to (3.8) with flipped currency indices.

4.2. **Forward measures.** Recall from Remark 3.11, that ZCBs are contracts with zero dividend process and a terminal price of one unit of currency. Then, from Section 3.1, we obtain that, under Assumption 3.1, 3.3, 3.5 and under the repo constraint (2.1), the processes

$$\left(\frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0}}\right)_{0\le t\le T},\quad \left(\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}\right)_{0\le t\le T},\quad \text{and}\quad \left(\frac{B^{k_0}(t,T)}{B_t^{k_0}}\right)_{0\le t\le T}, \tag{4.4}$$

are martingales for every choice of the indices k_1, k_2, and k_3. This shows that including ZCBs with different funding strategies corresponds to including new risky assets together with their asset-specific cash-accounts: each new asset is funded by an associated asset-specific cash account, hence giving rise to further repo constraints of the form (2.1).

Furthermore, the martingales in (4.4) may serve as density processes for new probability measures. We shall introduce some of them in the following definition: the list is not exhaustive, but covers all the essential tools that are needed for cross-currency term-structure modeling. In particular, notice that classical forward measure are defined up to the maturity of the corresponding ZCB. For later use, we adopt the approach of Lyashenko and Mercurio [2019] and extend the concept of forward measure by considering as numéraire the self-financing strategy that after the maturity of the corresponding ZCB, say T, reinvests the notional into the corresponding cash account.

Definition 4.2. *Let $T \ge 0$ be fixed. We define the following forward measures:*

(i) *The* domestic-collateralized domestic T-forward measure $\mathbb{Q}^{T,k_0,k_0} \sim \mathbb{Q}^{k_0}$ *on* $(\Omega, \mathbb{G})$ *is defined via the Radon-Nikodym derivative*

$$\frac{\partial \mathbb{Q}^{T,k_0,k_0}}{\partial \mathbb{Q}^{k_0}} := \frac{B^{k_0,k_0}(T,T)}{B_T^{c,k_0}} \frac{B_0^{c,k_0}}{B^{k_0,k_0}(0,T)}.$$

In particular:

(a) For $t \leq T$ we have

$$\left.\frac{\partial \mathbb{Q}^{T,k_0,k_0}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\partial \mathbb{Q}^{T,k_0,k_0}}{\partial \mathbb{Q}^{k_0}}\right| \mathcal{G}_t\right] = \frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0}} \frac{B_0^{c,k_0}}{B^{k_0,k_0}(0,T)};$$

(b) For $t > T$, since $B^{k_0,k_0}(t,T) = \frac{B_t^{c,k_0}}{B_T^{c,k_0}}$, we define

$$\left.\frac{\partial \mathbb{Q}^{T,k_0,k_0}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} := \frac{1}{B_T^{c,k_0}} \frac{B_0^{c,k_0}}{B^{k_0,k_0}(0,T)},$$

hence $\mathbb{Q}^{T,k_0,k_0} \equiv \mathbb{Q}^{k_0}$ on $t > T$.

(ii) *The* k_3-collateralized domestic T-forward measure $\mathbb{Q}^{T,k_0,k_3} \sim \mathbb{Q}^{k_0}$ *on* $(\Omega, \mathbb{G})$ *is defined via the Radon-Nikodym derivative*

$$\frac{\partial \mathbb{Q}^{T,k_0,k_3}}{\partial \mathbb{Q}^{k_0}} := \frac{B^{k_0,k_3}(T,T)}{B_T^{c,k_0,k_3}} \frac{B_0^{c,k_0,k_3}}{B^{k_0,k_3}(0,T)}. \tag{4.5}$$

In particular:

(a) For $t \leq T$ we have

$$\left.\frac{\partial \mathbb{Q}^{T,k_0,k_3}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\partial \mathbb{Q}^{T,k_0,k_3}}{\partial \mathbb{Q}^{k_0}}\right| \mathcal{G}_t\right] = \frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}} \frac{B_0^{c,k_0,k_3}}{B^{k_0,k_3}(0,T)};$$

(b) For $t > T$, since $B^{k_0,k_3}(t,T) = \frac{B_t^{c,k_0,k_3}}{B_T^{c,k_0,k_3}}$, we define

$$\left.\frac{\partial \mathbb{Q}^{T,k_0,k_3}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} := \frac{1}{B_T^{c,k_0,k_3}} \frac{B_0^{c,k_0,k_3}}{B^{k_0,k_3}(0,T)},$$

hence $\mathbb{Q}^{T,k_0,k_3} \equiv \mathbb{Q}^{k_0}$ on $t > T$.

(iii) *The* uncollateralized/unsecured domestic T-forward measure $\mathbb{Q}^{T,k_0} \sim \mathbb{Q}^{k_0}$ *on* $(\Omega, \mathbb{G})$ *is defined via the Radon-Nikodym derivative*

$$\frac{\partial \mathbb{Q}^{T,k_0}}{\partial \mathbb{Q}^{k_0}} := \frac{B^{k_0}(T,T)}{B_T^{k_0}} \frac{B_0^{k_0}}{B^{k_0}(0,T)}.$$

In particular:

(a) For $t \leq T$ we have

$$\left.\frac{\partial \mathbb{Q}^{T,k_0}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\partial \mathbb{Q}^{T,k_0}}{\partial \mathbb{Q}^{k_0}}\right| \mathcal{G}_t\right] = \frac{B^{k_0}(t,T)}{B_t^{k_0}} \frac{B_0^{k_0}}{B^{k_0}(0,T)};$$

(b) For $t > T$, since $B^{k_0}(t,T) = \frac{B_t^{k_0}}{B_T^{k_0}}$, we define

$$\left.\frac{\partial \mathbb{Q}^{T,k_0}}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} := \frac{1}{B_T^{k_0}} \frac{B_0^{k_0}}{B^{k_0}(0,T)},$$

hence $\mathbb{Q}^{T,k_0} \equiv \mathbb{Q}^{k_0}$ on $t > T$.

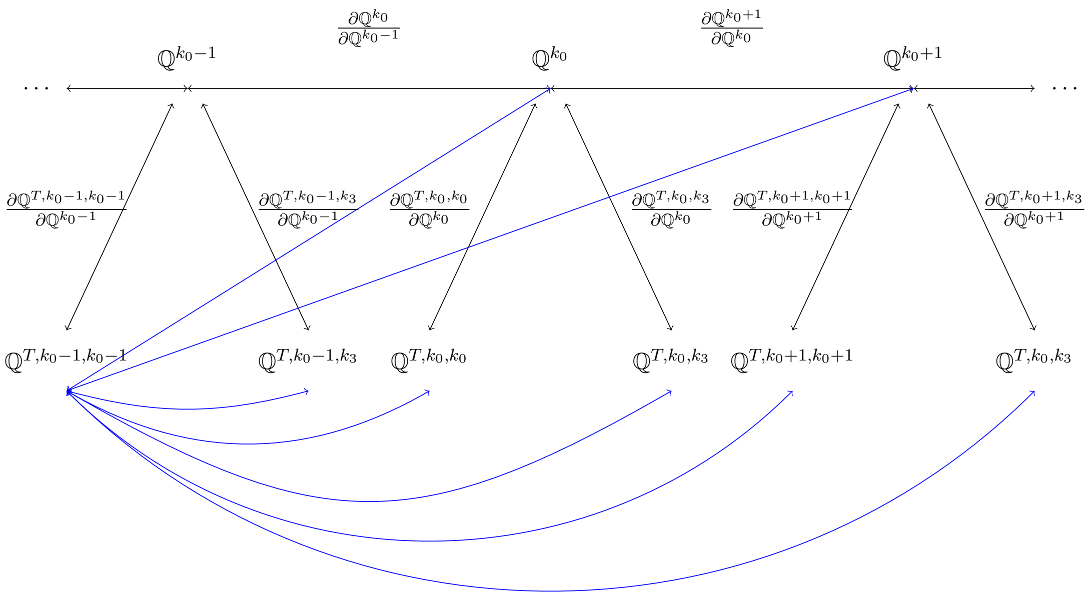

FIGURE 2. This graph summarizes (a part of) the relations between the different pricing measures. Each node in the graph is a pricing measure and each edge represents the link between two probability measures via a suitable Radon-Nikodym derivative. For the sake of readability, we only plot the Radon-Nikodym derivatives with respect to the spot measures. However, each node of the graph can be linked to all the other nodes. For illustrative purposes, we represent these relations only for the forward measure $\mathbb{Q}^{T,k_0-1,k_0-1}$ (blue arrows).

Figure 2 summarizes some of the relations between the different pricing measures.

Remark 4.3. We observe that the domestic-collateralized domestic T-forward measure that uses T-OIS bonds as numéraire is the one typically employed in the literature on single-currency multiple-curve models, such as Cuchiero et al. [2016] and Cuchiero et al. [2019]. Our general setting, however, highlights the fact that there is no need to assume (as in the references above) that the OIS bank account is the numéraire of $\mathbb{Q}^{k_0}$ (in fact, it is not): under $\mathbb{Q}^{k_0}$, as we have seen, we have that multiple assets with different funding strategies are simultaneously martingales with no cash account playing the role of universal numéraire for all the risky assets.

5. Cross-currency HJM framework

The previous sections served the purpose of introducing a general valuation setup in the multi-currency setting, where in each currency we have a multitude of interest rates and risky assets. The results were formulated in an (almost) model-free setting: the only assumption is that certain processes are martingales. The aim of this section is to set some of these processes "in motion" by introducing a general HJM framework that accounts for a multitude of features of the cross-currency multiple-curve market.

So far, we mainly adopted the point of view of a k_0-based investor with associated risk-neutral measure $\mathbb{Q}^{k_0}$. Here, we will instead consider a generic k_0 currency, $1 \leq k_0 \leq L$, in order to provide a symmetric treatment of the different currency areas. We aim to construct a modeling framework that takes into account the intimate links existing among the different curves. In view of this, instead of directly modeling all the families of ZCBs, we choose to fix a *reference curve* for each currency area, and to model the remaining curves by means of (either positive or real-valued) spreads with respect to the reference curve. This is intuitive from a financial point of view and in line with the modeling philosophy of Cuchiero et al. [2016].

In the following, for $d_X \in \mathbb{N}$, let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^{d_X}$-valued Itô semimartingale with differential characteristics (b, c, K) with respect to a truncation function χ. We recall the notion of local exponent (see also Kallsen and Krühner [2013, Definition A.6]), which we state under the probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$.

Definition 5.1. *Let $\beta = (\beta_t)_{t\geq 0}$ an $\mathbb{R}^{d_X}$-valued predictable and X-integrable process. The* local exponent *of X at β under the measure $\mathbb{P}$ is a predictable real-valued process $(\Psi_t^{\mathbb{P},X}(\beta_t))_{t\geq 0}$ such that $\left(\exp(\int_0^t \beta_s dX_s - \int_0^t \Psi_s^{\mathbb{P},X}(\beta_s)ds)\right)_{t\geq 0}$ is a local $(\mathbb{P}, \mathbb{G})$-martingale. We denote by $\mathcal{U}^{\mathbb{P},X}$ the set of processes β such that $\Psi^{\mathbb{P},X}(\beta)$ exists.*

Moreover, with the following lemma we can express the local exponent in Lévy-Kintchine form.

Lemma 5.2. *For any $\beta \in \mathcal{U}^{\mathbb{P},X}$, outside some $d\mathbb{P} \otimes dt$-nullset, it holds that*

$$\Psi_t^{\mathbb{P},X}(\beta_t) = \beta_t^\top b_t + \frac{1}{2}\beta_t^\top c_t \beta_t + \int \left(e^{\beta_t^\top \xi} - 1 - \beta_t^\top \chi(\xi)\right) K_t(d\xi).$$

Moreover, the gradient $\nabla_{\beta_t}\Psi_t^{\mathbb{P},X}(\beta_t)$ of $\Psi_t^{\mathbb{P},X}(\beta_t)$ in the direction of β_t is the $\mathbb{R}^{d_X}$-valued vector given by

$$\nabla_{\beta_t}\Psi_t^{\mathbb{P},X}(\beta_t) = b_t + c_t\beta_t + \int \left(e^{\beta_t^\top \xi}\xi - \chi(\xi)\right) K_t(d\xi). \tag{5.1}$$

We shall use the shorthand $\nabla\Psi_t^{\mathbb{P},X}(\beta_t) := \nabla_{\beta_t}\Psi_t^{\mathbb{P},X}(\beta_t)$.

Proof. For the first part, see Cuchiero et al. [2016, Proposition 3.3]. Equation (5.1) is obtained by simple computations. □

We first focus on ZCBs prices. Starting from the particular configuration of the process $\tilde{\mathscr{M}}$ in Section 3, we know that

$$\left(\frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0}}\right)_{0\leq t\leq T}, \quad \text{and} \quad \left(\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}\right)_{0\leq t\leq T}$$

are $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingales. We now specify suitable HJM frameworks which allow to model these two families of ZCBs.

Remark 5.3. The results of the present paper can be suitably adjusted to include the case of stochastic discontinuities as in Fontana et al. [2023]. The estimation of term-structure models in the presence of stochastic discontinuities due to the action of the central bank is delicate: in the aftermath of the financial crisis, most central banks adopted a near-zero interest rate policy for several years without significant changes. Hence the statistical basis for the estimation of models with stochastic discontinuities, in particular for the estimation of the jumps size of the distribution, is rather limited. Backwell and Hayes [2022], recognizing the above challenge, postulate a distribution for the jumps size without performing a statistical estimation of the jumps linked to the monetary policy. In view of this, we choose not to pursue such a direction.

5.1. **HJM framework for collateral discount curves.** We first concentrate on the domestic bond with domestic collateral. This is the case that is considered by the whole literature on multiple-curve models. The terminology OIS bond is common for this process, since the market-observed initial term structure of such ZCBs is obtained from a bootstrap procedure applied to overnight indexed swaps (OIS).

We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}^1, \ldots, \mathbb{Q}^L)$ with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ satisfying the usual assumptions. On the probability space we postulate the existence of multiple probability measures $\mathbb{Q}^{k_0}$, for $1 \leq k_0 \leq L$. Following Cuchiero et al. [2016], we introduce a term-structure model for $\{(B^{k_0,k_0}(t,T))_{t\in[0,T]},\ T \geq 0\}$ for each currency k_0. Let $\mathcal{P}$ denote the predictable σ-algebra.

Definition 5.4. *We say that the triple (f, α, σ) satisfies the* HJM-basic condition, *if:*

(i) *The map $f : \mathbb{R}_+ \to \mathbb{R}$ is measurable with $\int_0^T |f(u)|du < \infty$, $\mathbb{Q}^{k_0}$-a.s. for all $T \in \mathbb{R}_+$;*
(ii) *The map $(\omega, t, T) \mapsto \alpha_t(T)(\omega)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable $\mathbb{R}$-valued process such that $\int_0^t \int_0^T |\alpha_s(u)|duds < \infty$, $\mathbb{Q}^{k_0}$-a.s. for all $t, T \in \mathbb{R}_+$;*
(iii) *The map $(\omega, t, T) \mapsto \sigma_t(T)(\omega)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable $\mathbb{R}^{d_X}$-valued process such that $\int_0^T \sigma_t(u)^\top \sigma_t(u)du < \infty$, $\mathbb{Q}^{k_0}$-a.s. for all $t, T \in \mathbb{R}_+$, and the process $\left(\left(\int_0^T |\sigma_{t,j}(u)|^2 du\right)^{\frac{1}{2}}\right)_{t\geq 0}$ is integrable with respect to the j-th component of the semimartingale X.*

Definition 5.5. *For any $1 \leq k_0 \leq L$, a* bond-price model *for the currency k_0 is a quintuple $(B^{c,k_0}, X, f_0^{c,k_0}, \alpha^{c,k_0}, \sigma^{c,k_0})$ where*

(i) *The collateral cash account B^{c,k_0} is absolutely continuous with respect to the Lebesgue measure, i.e. $B_t^{c,k_0} = e^{\int_0^t r_s^{c,k_0} ds}$ with collateral short rate $r^{c,k_0} = (r_t^{c,k_0})_{t\geq 0}$;*
(ii) *X is an $\mathbb{R}^{d_X}$-valued Itô semimartingale;*
(iii) *The triple $(f_0^{c,k_0}, \alpha^{c,k_0}, \sigma^{c,k_0})$ satisfies the HJM-basic condition in Definition 5.4;*
(iv) *For every $T \in \mathbb{R}_+$, the instantaneous collateral forward rate $(f_t^{c,k_0}(T))_{t\in[0,T]}$ is given by*

$$f_t^{c,k_0}(T) = f_0^{c,k_0}(T) + \int_0^t \alpha_s^{c,k_0}(T)ds + \int_0^t \sigma_s^{c,k_0}(T)dX_s; \tag{5.2}$$

(v) *The k_0-collateralized k_0-ZCB prices $\{(B^{k_0,k_0}(t,T))_{t\in[0,T],\ T\geq 0}\}$ satisfy*

$$B^{k_0,k_0}(t,T) = e^{-\int_t^T f_t^{c,k_0}(u)du}, \tag{5.3}$$

for all $t \leq T$ and $T \geq 0$. Moreover $B^{k_0,k_0}(t,t) = 1$ for all $t \geq 0$.

Next, we characterize the measures $\mathbb{Q}^{k_0}$ as the "domestic" risk-neutral measure for each economy k_0. This is in line with the previously introduced measure $\mathbb{Q}^{k_0}$. The next definition is in line with Cuchiero et al. [2016, Definition 3.9].

Definition 5.6. *Let $1 \leq k_0 \leq L$. We say that the bond price model for the k_0-collateralized k_0-ZCBs is* risk-neutral *if the processes*

$$\left\{ \left(\frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0}} \right)_{t\in[0,T]},\ T \geq 0 \right\} \tag{5.4}$$

are $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingales.

We stress the fact that (5.4) being martingales does not mean that the cash account B^{c,k_0} is the numéraire of the measure $\mathbb{Q}^{k_0}$. In fact, it is not. As we have seen in Section 3, B^{c,k_0} is the specific funding account of the k_0-collateralized k_0-ZCB. When considering multiple risky assets in the k_0 economy as in Section 2, it becomes clear that B^{c,k_0} is only the funding account for one of the several risky assets in the economy. We also recall that, in order to prevent arbitrage opportunities, it is sufficient to impose the repo constraint (2.1) together with the assumption that both (2.2) and (2.3) are $\mathbb{Q}^{k_0}$-martingales.

We shall now characterize the martingale property for the family of processes in (5.4). Let

$$\Sigma_t^{c,k_0}(T) := \int_t^T \sigma_t^{c,k_0}(u)du.$$

We state the following result.

Proposition 5.7. *Let $1 \le k_0 \le L$ and $0 \le t \le T$. The followings are equivalent:*

(i) *The bond-price model for the currency k_0 is risk neutral;*

(ii) *The conditional expectation hypothesis holds, i.e.*

$$\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\frac{B_t^{c,k_0}}{B_T^{c,k_0}}\middle|\mathcal{G}_t\right] = e^{-\int_t^T f_t^{c,k_0}(u)du};$$

(iii) *The process $-\Sigma_t^{c,k_0}(T) \in \mathcal{U}^{\mathbb{Q}^{k_0},X}$ and the following conditions are satisfied:*

(a) *The process*

$$\left(\exp\left\{-\int_0^t \Sigma_s^{c,k_0}(T)dX_s - \int_0^t \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T))ds\right\}\right)_{t\in[0,T]}$$

is a $(\mathbb{Q}^{k_0},\mathbb{G})$-martingale;

(b) *The consistency condition holds, meaning that*

$$\Psi_t^{\mathbb{Q}^{k_0},-\int_0^\cdot r_s^{c,k_0}ds}(1) = -r_{t-}^{c,k_0} = -f_{t-}^{c,k_0}(t); \tag{5.5}$$

(c) *The HJM drift condition*

$$\int_t^T \alpha_t^{c,k_0}(u)du = \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)) \tag{5.6}$$

holds.

Proof. See the proof of Cuchiero et al. [2016, Proposition 3.9]. □

Corollary 5.8. *If the bond-price model for the currency k_0 is risk neutral, then:*

(i) *For every $T > 0$, the instantaneous collateral forward rate $(f_t^{c,k_0}(T))_{t\in[0,T]}$ is given by*

$$f_t^{c,k_0}(T) = f_0^{c,k_0}(T) - \int_0^t \sigma_s^{c,k_0}(T)\nabla\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T))ds + \int_0^t \sigma_s^{c,k_0}(T)dX_s; \tag{5.7}$$

(ii) *For every $t \ge 0$, the collateral short rate r_t^{c,k_0} at time t is given by*

$$r_t^{c,k_0} = f_0^{c,k_0}(t) - \int_0^t \sigma_s^{c,k_0}(t)\nabla\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(t))ds + \int_0^t \sigma_s^{c,k_0}(t)dX_s.$$

Proof. By taking the derivative with respect to $T > 0$ on both sides of equation (5.6) we get

$$\alpha_t^{c,k_0}(T) = \frac{\partial\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T))}{\partial T} = -\sigma_t^{c,k_0}(T)\,\nabla\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)). \tag{5.8}$$

By substituting equation (5.8) into (5.2) we get (i). By letting $t \to T$ we then obtain (ii). □

Remark 5.9. We remark that if X is a standard Brownian motion, then the differential characteristics of X are $(0, I_{d_X}, 0)$, where I_{d_X} denotes the identity matrix of dimension d_X. Hence

$$\nabla\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)) = -\Sigma_t^{c,k_0}(T),$$

and we recover the classical formulation for the instantaneous collateral forward rate, namely

$$f_t^{c,k_0}(T) = f_0^{c,k_0}(T) + \int_0^t \sigma_s^{c,k_0}(T)\left(\int_s^T \sigma_s^{c,k_0}(u)du\right)ds + \int_0^t \sigma_s^{c,k_0}(T)dX_s,$$

as found, e.g, in Filipović [2009, Theorem 6.1].

5.2. **HJM framework for cross-currency basis curves.** The previous results conveniently summarize the HJM methodology applied to the domestic-collateralized domestic-ZCBs, $\left(B^{k_0,k_0}(t,T)\right)_{t\in[0,T]}$, for every currency denomination $1 \le k_0 \le L$. The next step is to model foreign-collateralized domestic-ZCBs, $\left(B^{k_0,k_3}(t,T)\right)_{t\in[0,T]}$, for every $T \ge 0$. The starting point is the result in Section 3.1.2 accordingly to which the processes

$$\left(\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}\right)_{t\in[0,T]}$$

should be $(\mathbb{Q}^{k_0},\mathbb{G})$-martingales. We formulate this requirement in the following assumption.

Assumption 5.10. *For all $1 \le k_0, k_3 \le L$ with $k_0 \neq k_3$, we assume that the processes*

$$\left\{\left(\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}\right)_{t\in[0,T]}, \ T \ge 0\right\}$$

are $(\mathbb{Q}^{k_0},\mathbb{G})$-martingales.

One possible approach is to introduce a specific bond-price model for $B^{k_0,k_3}(t,T)$ in the spirit of Definition 5.5. This, however, would not exploit the link between $B^{k_0,k_0}(t,T)$ and $B^{k_0,k_3}(t,T)$, and would lead to some redundancy. With a similar approach to Cuchiero et al. [2016], one can instead model the multiplicative spread between $B^{k_0,k_0}(t,T)$ and $B^{k_0,k_3}(t,T)$. By doing this, the previously studied domestic-collateralized domestic-ZCB price models serve as reference curves, while the remaining curves are obtained by means of spreads with respect to these reference curves. In particular, this will lead us to construct a HJM framework for the instantaneous cross-currency basis spreads $\{(q_t^{k_0,k_3}(T))_{t\in[0,T]},\ T \ge 0\}$.

We first define the modeling quantities.

Definition 5.11. *Let $t \in [0,T]$ with $T \ge 0$, and $1 \le k_0, k_3 \le L$ such that $k_0 \neq k_3$. We define the* k_0-k_3 cross-currency spread bond *via*

$$Q^{k_0,k_3}(t,T) := \frac{B^{k_0,k_3}(t,T)}{B^{k_0,k_0}(t,T)}, \tag{5.9}$$

and the k_0-k_3 cross-currency spread cash account $Q^{k_0,k_3} = (Q_t^{k_0,k_3})_{t\ge 0}$ *by setting*

$$Q_t^{k_0,k_3} := e^{\int_0^t q_s^{k_0,k_3}ds},$$

with q^{k_0,k_3} being the k_0-k_3 cross-currency basis spreads introduced in Definition 3.8.

The following lemma states the relevant martingale property for our purposes.

Lemma 5.12. *Let $1 \le k_0, k_3 \le L$ with $k_0 \neq k_3$ and $T \ge 0$. Assume that the bond-price model for the k_0-collateralized k_0-ZCB is risk neutral. Then Assumption 5.10 holds if and only if the processes*

$$\left\{\left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}}\right)_{t\in[0,T]}, \ T \ge 0\right\}$$

are $(\mathbb{Q}^{T,k_0,k_0},\mathbb{G})$-martingales.

Proof. We use the Bayes's formula for conditional expectations. For some fixed $1 \le k_0, k_3 \le L$ with $k_0 \ne k_3$ and $T \ge 0$, the process $\left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}}\right)_{t\in[0,T]}$ is a $(\mathbb{Q}^{T,k_0,k_0}, \mathbb{G})$-martingale if and only if the process

$$\left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}} \frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0} B^{k_0,k_0}(0,T)}\right)_{t\in[0,T]}$$

is a $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingale. Since $\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}} \frac{B^{k_0,k_0}(t,T)}{B_t^{c,k_0}} = \frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}$, we then conclude thanks to Assumption 5.10. $\square$

Remark 5.13. We understand from Lemma 5.12 that cross-currency spread models should satisfy the martingale property under the forward measure $\mathbb{Q}^{T,k_0,k_0}$. Looking at the time series of cross-currency basis swaps in Figure 1, we also observe that cross-currency spreads could be positive or negative and could exhibit a term structure of different shapes (either increasing or decreasing). This is different from the IBOR-OIS basis considered in Cuchiero et al. [2016]: the multiplicative spot spreads between OIS and IBORs is indeed greater than one and is increasing with respect to the tenor length. This poses us in a more flexible modelling setting.

For any given currency k_0, we shall now put together a bond price model for the k_0-collateralized k_0-ZCBs with a family of $L-1$ models for the k_0-k_3 cross-currency bond spreads. We call such a combination an *extended bond price model.*

Definition 5.14. *Let $1 \le k_0 \le L$ be fixed. We call a model consisting of*

I. The $\mathbb{R}^{d_X+L}$-valued Itô semimartingale $(X, Q^{k_0,1}, \dots, Q^{k_0,k_0-1}, Q^{k_0,k_0+1} \dots, Q^{k_0,L}, B^{c,k_0})$;
II. The functions $f_0^{c,k_0}, q_0^{k_0,1}, \dots, q_0^{k_0,k_0-1}, q_0^{k_0,k_0+1}, \dots, q_0^{k_0,L}$;
III. The processes

$$\alpha^{c,k_0}, \alpha^{k_0,1}, \dots, \alpha^{k_0,k_0-1}, \alpha^{k_0,k_0+1}, \dots, \alpha^{k_0,L},$$

and

$$\sigma^{c,k_0}, \sigma^{k_0,1}, \dots, \sigma^{k_0,k_0-1}, \sigma^{k_0,k_0+1}, \dots, \sigma^{k_0,L};$$

an extended bond-price model *for the currency k_0, if for every $k_3 \ne k_0$ the following conditions are satisfied:*

(i) *The quintuple $(B^{c,k_0}, X, f_0^{c,k_0}, \alpha^{c,k_0}, \sigma^{c,k_0})$ is a bond-price model in the sense of Definition 5.5.*
(ii) *The k_0-k_3 cross-currency spread cash account Q^{k_0,k_3} is absolutely continuous with respect to the Lebesgue measure, i.e. $Q_t^{k_0,k_3} = e^{\int_0^t q_s^{k_0,k_3} ds}$ with cross-currency basis rate $q^{k_0,k_3} = (q_t^{k_0,k_3})_{t\ge 0}$;*
(iii) *The triple $(q_0^{k_0,k_3}, \alpha^{k_0,k_3}, \sigma^{k_0,k_3})$ satisfies the HJM-basic condition in Definition 5.4;*
(iv) *For $T \in \mathbb{R}_+$, the instantaneous cross-currency basis spread $(q_t^{k_0,k_3}(T))_{t\in[0,T]}$ is given by*

$$q_t^{k_0,k_3}(T) = q_0^{k_0,k_3}(T) + \int_0^t \alpha_s^{k_0,k_3}(T) ds + \int_0^t \sigma_s^{k_0,k_3}(T) dX_s;$$

vi) The k_0-k_3 cross-currency spread bonds $\{(Q^{k_0,k_3}(t,T))_{t\in[0,T]},\ T \ge 0\}$ satisfy

$$Q^{k_0,k_3}(t,T) = e^{-\int_t^T q_t^{k_0,k_3}(u)du} \tag{5.10}$$

for all $t \le T$ and $T \ge 0$. Moreover $Q^{k_0,k_3}(t,t) = 1$ for all $t \ge 0$.

Remark 5.15. We point out that for each currency k_0, the dynamics of the instantaneous cross-currency basis spreads $q_t^{k_0,k_3}(T)$ are defined under the measure $\mathbb{Q}^{k_0}$.

The next definition naturally collects the martingale conditions that are relevant in the current setting.

Definition 5.16. *Let $1 \leq k_0 \leq L$. We say that the extended bond-price model for the currency k_0 is* risk neutral *if the following conditions hold:*

(i) *The bond-price model for the k_0-collateralized k_0-ZCB $(B^{c,k_0}, X, f_0^{c,k_0}, \alpha^{c,k_0}, \sigma^{c,k_0})$ is risk neutral in the sense of Definition 5.6;*
(ii) *For each $1 \leq k_3 \leq L$ with $k_3 \neq k_0$ and $T \geq 0$, the processes*

$$\left\{ \left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}} \right)_{t\in[0,T]}, \ T \geq 0 \right\}$$

are $(\mathbb{Q}^{T,k_0,k_0}, \mathbb{G})$-martingales.

The next result characterizes condition (ii) of Definition 5.16. We define

$$\Sigma_t^{k_0,k_3}(T) := \int_t^T \sigma_t^{k_0,k_3}(u)du.$$

Theorem 5.17. *Let $1 \leq k_0 \leq L$ and $0 \leq t \leq T$. For an extended bond-price model for the currency k_0 satisfying condition* (i) *of Definition 5.16, the followings are equivalent:*

(i) *The extended bond-price model satisfies condition* (ii) *of Definition 5.16;*
(ii) *For every $k_3 \neq k_0$, the conditional expectation hypothesis holds, i.e.*

$$\mathbb{E}^{\mathbb{Q}^{T,k_0,k_0}} \left[\frac{Q_t^{k_0,k_3}}{Q_T^{k_0,k_3}} \middle| \mathcal{G}_t \right] = e^{-\int_t^T q_t^{k_0,k_3}(u)du};$$

(iii) *For every $k_3 \neq k_0$, the process $-\left(\Sigma^{c,k_0}(T) + \Sigma^{k_0,k_3}(T)\right) \in \mathcal{U}^{\mathbb{Q}^{k_0},X}$ and the following conditions are satisfied:*
   (a) The process

$$\left(\exp\left\{ -\int_0^t \left(\Sigma_s^{c,k_0}(T) + \Sigma_s^{k_0,k_3}(T)\right) dX_s - \int_0^t \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T) - \Sigma_s^{k_0,k_3}(T))ds \right\} \right)_{t\in[0,T]} \tag{5.11}$$

   is a $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingale;
   (b) The consistency condition holds, meaning that

$$\Psi_t^{\mathbb{Q}^{k_0}, -\int_0^\cdot q_s^{k_0,k_3}ds}(1) = -q_{t-}^{k_0,k_3} = -q_{t-}^{k_0,k_3}(t); \tag{5.12}$$

   (c) The HJM drift condition

$$\int_t^T \alpha_t^{k_0,k_3}(u)du = \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T) - \Sigma_t^{k_0,k_3}(T)) - \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)) \tag{5.13}$$

   holds for every $k_3 \neq k_0$.

Proof. Let $T > 0$ and $1 \leq k_3 \leq L$ with $k_3 \neq k_0$ be fixed.

(i) $\Rightarrow$ (ii) Since the process $\left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}}\right)_{t\in[0,T]}$ is a $(\mathbb{Q}^{T,k_0,k_0}, \mathbb{G})$-martingale, it follows that

$$\mathbb{E}^{\mathbb{Q}^{T,k_0,k_0}} \left[\frac{1}{Q_T^{k_0,k_3}} \middle| \mathcal{G}_t \right] = \frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}},$$

which, rearranged, gives (ii).

(i) $\Rightarrow$ (iii) Since the process $\left(\frac{Q^{k_0,k_3}(t,T)}{Q_t^{k_0,k_3}}\right)_{t\in[0,T]}$ is a $(\mathbb{Q}^{T,k_0,k_0},\mathbb{G})$-martingale, we know from Lemma 5.12 that the process

$$\left(\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}=\frac{B^{k_0,k_0}(t,T)Q^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}=\frac{e^{-\int_t^T(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u))du}}{e^{\int_0^t(r_s^{c,k_0}+q_s^{k_0,k_3})ds}}\right)_{t\in[0,T]} \tag{5.14}$$

is a $(\mathbb{Q}^{k_0},\mathbb{G})$-martingale. Let $R_t:=-\int_t^T(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u))du-\int_0^t(r_s^{c,k_0}+q_s^{k_0,k_3})ds$. Then the martingale property of (5.14) is equivalent to the martingale property of $\exp(R)$, which implies that $1\in\mathcal{U}^{\mathbb{Q}^{k_0},R}$ and $\Psi_t^{\mathbb{Q}^{k_0},R}(1)=0$. Due to the integrability conditions on α^{c,k_0} and σ^{c,k_0} in Definition 5.5, and on α^{k_0,k_3} and σ^{k_0,k_3} in Definition 5.14, we can apply the classical and the stochastic Fubini theorem, which yield

$$\begin{aligned}
&\int_t^T(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u))du\\
&=\int_0^T(f_0^{c,k_0}(u)+q_0^{k_0,k_3}(u))du+\int_0^t\int_s^T(\alpha_s^{c,k_0}(u)+\alpha_s^{k_0,k_3}(u))duds\\
&+\int_0^t(\Sigma_s^{c,k_0}(T)+\Sigma_s^{k_0,k_3}(T))dX_s-\int_0^t(f_u^{c,k_0}(u)+q_u^{k_0,k_3}(u))du.
\end{aligned} \tag{5.15}$$

By applying Kallsen and Krühner [2013, Lemma A.13] we then obtain that

$$\begin{aligned}
0=\Psi_t^{\mathbb{Q}^{k_0},R}(1)=\Psi_t^{\mathbb{Q}^{k_0},(-\int_0^\cdot(r_s^{c,k_0}+q_s^{k_0,k_3})ds,X)}&\left(\left(1,-(\Sigma_t^{c,k_0}(T)+\Sigma_t^{k_0,k_3}(T))^\top\right)^\top\right)\\
&-\int_t^T(\alpha_t^{c,k_0}(u)+\alpha_t^{k_0,k_3}(u))du+f_{t-}^{c,k_0}(t)+q_{t-}^{k_0,k_3}(t).
\end{aligned}$$

Set now $T=t$. Since $\Sigma_t^{c,k_0}(t)=\Sigma_t^{k_0,k_3}(t)=0$, we get

$$\Psi_t^{\mathbb{Q}^{k_0},-\int_0^\cdot r_s^{c,k_0}ds}(1)+\Psi_t^{\mathbb{Q}^{k_0},-\int_0^\cdot q_s^{k_0,k_3}ds}(1)=\Psi_t^{\mathbb{Q}^{k_0},-\int_0^\cdot(r_s^{c,k_0}+q_s^{k_0,k_3})ds}(1)=-f_{t-}^{c,k_0}(t)-q_{t-}^{k_0,k_3}(t), \tag{5.16}$$

hence (5.12) due to the consistency condition (5.5). Moreover, substituting (5.16) into (5.15) yields the following drift condition:

$$-\int_t^T\left(\alpha_t^{c,k_0}(u)+\alpha_t^{k_0,k_3}(u)\right)du=-\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)-\Sigma_t^{k_0,k_3}(T)),$$

hence

$$\begin{aligned}
\int_t^T\alpha_t^{k_0,k_3}(u)du&=\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)-\Sigma_t^{k_0,k_3}(T))-\int_t^T\alpha_t^{c,k_0}(u)du\\
&=\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)-\Sigma_t^{k_0,k_3}(T))-\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)),
\end{aligned}$$

where the last equality is due to the drift condition for the bond-price model in Proposition 5.7. We now have both the consistency condition and the drift condition. By substituting them into (5.15), together with (5.14) we can write that

$$\begin{aligned}
\frac{B^{k_0,k_3}(t,T)}{B_t^{c,k_0,k_3}}&=\exp\left\{-\int_t^T(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u))du-\int_0^t(r_s^{c,k_0}+q_s^{k_0,k_3})ds\right\}\\
&=\exp\left\{-\int_0^T(f_0^{c,k_0}(u)+q_0^{k_0,k_3}(u))du-\int_0^t(\Sigma_s^{c,k_0}(T)+\Sigma_s^{k_0,k_3}(T))dX_s\right.\\
&\qquad\left.-\int_0^t\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T)-\Sigma_s^{k_0,k_3}(T))dX_s\right\},
\end{aligned} \tag{5.17}$$

from which we deduce that the process (5.11) is a $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingale for every $T \geq 0$.
(iii) $\Rightarrow$ (i) The consistency condition and the drift condition yield again equation (5.17). The martingale property of (5.11) together with Lemma 5.12 implies then the last statement.

□

Remark 5.18. We emphasise that the HJM drift condition (5.13) is given in terms of the local exponent $\Psi^{\mathbb{Q}^{k_0},X}$ under the measure $\mathbb{Q}^{k_0}$. However, one can show that the local exponent $\Psi^{\mathbb{Q}^{T,k_0,k_0},X}$ under the measure $\mathbb{Q}^{T,k_0,k_0}$ is obtained from $\Psi^{\mathbb{Q}^{k_0},X}$ through the following relation:

$$\Psi^{\mathbb{Q}^{T,k_0,k_0},X}(\beta) = \Psi^{\mathbb{Q}^{k_0},X}(\beta - \Sigma^{c,k_0}(T)) - \Psi^{\mathbb{Q}^{k_0},X}(-\Sigma^{c,k_0}(T)), \tag{5.18}$$

for any $\beta \in \mathcal{U}^{\mathbb{Q}^{k_0},X} \cap \mathcal{U}^{\mathbb{Q}^{T,k_0,k_0},X}$. Hence the HJM drift condition (5.13) can be rewritten as

$$\int_t^T \alpha_t^{k_0,k_3}(u)du = \Psi_t^{\mathbb{Q}^{T,k_0,k_0},X}(-\Sigma_t^{k_0,k_3}(T)).$$

Notice also that relation (5.18) could be used to formulate an alternative proof for Theorem 5.17 by working under the measure $\mathbb{Q}^{T,k_0,k_0}$ instead of under the measure $\mathbb{Q}^{k_0}$.

Corollary 5.19. *If the extended bond-price model for the currency k_0 is risk neutral, then for every $1 \leq k_3 \leq L$ with $k_3 \neq k_0$, we have that:*

(i) *For every $T > 0$, the instantaneous cross-currency basis spread $(q_t^{k_0,k_3}(T))_{t\in[0,T]}$ is given by*

$$\begin{aligned} q_t^{k_0,k_3}(T) &= q_0^{k_0,k_3}(T) \\ &- \int_0^t \Big((\sigma_s^{c,k_0}(T) + \sigma_s^{k_0,k_3}(T)) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T) - \Sigma_s^{k_0,k_3}(T)) \\ &- \sigma_s^{c,k_0}(T) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T)) \Big) ds + \int_0^t \sigma_s^{k_0,k_3}(T) dX_s; \end{aligned} \tag{5.19}$$

(ii) *For every $t \geq 0$, the k_0-k_3 cross-currency basis rate $q_t^{k_0,k_3}$ at time t is given by*

$$\begin{aligned} q_t^{k_0,k_3} &= q_0^{k_0,k_3} \\ &- \int_0^t \Big((\sigma_s^{c,k_0}(t) + \sigma_s^{k_0,k_3}(t)) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(t) - \Sigma_s^{k_0,k_3}(t)) - \sigma_s^{c,k_0}(t) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(t)) \Big) ds \\ &+ \int_0^t \sigma_s^{k_0,k_3}(t) dX_s. \end{aligned}$$

Proof. The proof proceeds similarly to the proof of Corollary 5.8. □

Remark 5.20. We remark that if X is a standard Brownian motion, then the differential characteristics of X are $(0, I_{d_X}, 0)$, hence

$$\nabla \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T)) = -\Sigma_t^{c,k_0}(T),$$

and

$$\nabla \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T) - \Sigma_t^{k_0,k_3}(T)) = -\Sigma_t^{c,k_0}(T) - \Sigma_t^{k_0,k_3}(T).$$

Then, after some simplifications, the dynamics for the instantaneous cross-currency basis spread in equation (5.19) becomes

$$\begin{aligned} q_t^{k_0,k_3}(T) &= q_0^{k_0,k_3}(T) \\ &+ \int_0^t \left(\sigma_s^{k_0,k_3}(T) \left(\int_s^T (\sigma_s^{c,k_0}(u) + \sigma_s^{k_0,k_3}(u)) du \right) + \sigma_s^{c,k_0}(T) \left(\int_s^T \sigma_s^{k_0,k_3}(u) du \right) \right) ds \\ &+ \int_0^t \sigma_s^{k_0,k_3}(T) dX_s. \end{aligned}$$

This is in line with the results found in Piterbarg [2012].

5.3. **HJM framework for foreign-collateral discount curves.** We consider now the domestic bond with foreign collateral. More precisely, for a fixed currency $1 \leq k_0 \leq L$, we consider $L-1$ bonds $\{(B^{k_0,k_3}(t,T))_{t\in[0,T]}, T \geq 0\}$, one for each currency $k_3 \neq k_0$. From equation (5.9), $B^{k_0,k_3}(t,T)$ is obtained as the product between the domestic bond with domestic collateral, $B^{k_0,k_0}(t,T)$, and the k_0-k_3 cross-currency bond spread, $Q^{k_0,k_3}(t,T)$, namely

$$B^{k_0,k_3}(t,T) = B^{k_0,k_0}(t,T)Q^{k_0,k_3}(t,T). \tag{5.20}$$

Moreover, by combining equations (5.3), (5.10) and (5.20), we can rewrite $B^{k_0,k_3}(t,T)$ in terms of the instantaneous collateral forward rate and of the instantaneous cross-currency basis spread, namely

$$B^{k_0,k_3}(t,T) = e^{-\int_t^T (f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u))du}. \tag{5.21}$$

Remember also that, by definition, the k_0-k_3 collateral cash account at time t is given by

$$B_t^{c,k_0,k_3} = e^{\int_0^t (r_s^{c,k_0}+q_s^{k_0,k_3})ds}. \tag{5.22}$$

In other words, given an extended bond-price model as introduced in the previous section, the foreign collateral discount curve is also implicitly modelled. In particular, for every $T \in \mathbb{R}_+$, the instantaneous foreign-collateral forward rate $(f_t^{c,k_0,k_3}(T))_{t\in[0,T]}$ follows the HJM dynamics

$$\begin{aligned} f_t^{c,k_0,k_3}(T) &= f_t^{c,k_0}(T) + q_t^{k_0,k_3}(T) \\ &= f_0^{c,k_0}(T) + q_0^{k_0,k_3}(T) + \int_0^t (\alpha_s^{c,k_0}(T) + \alpha_s^{k_0,k_3}(T))ds + \int_0^t (\sigma_s^{c,k_0}(T) + \sigma_s^{k_0,k_3}(T))dX_s, \end{aligned} \tag{5.23}$$

and the k_0-k_3 collateral short rate r_t^{c,k_0,k_3} at time t is obtained by

$$\begin{aligned} r_t^{c,k_0,k_3} &= r_t^{c,k_0} + q_t^{k_0,k_3} \\ &= r_0^{c,k_0} + q_0^{k_0,k_3} + \int_0^t (\alpha_s^{c,k_0}(t) + \alpha_s^{k_0,k_3}(t))ds + \int_0^t (\sigma_s^{c,k_0}(t) + \sigma_s^{k_0,k_3}(t))dX_s. \end{aligned} \tag{5.24}$$

We conclude with the following results.

Lemma 5.21. *Let $1 \leq k_0 \leq L$ and $0 \leq t \leq T$. For an extended bond-price model for the currency k_0, the followings are equivalent:*

(i) *The extended bond-price model is risk neutral;*

(ii) *For every $k_3 \neq k_0$, the conditional expectation hypothesis holds, i.e.*

$$\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0,k_3}}{B_T^{c,k_0,k_3}}\right| \mathcal{G}_t\right] = e^{-\int_t^T f_t^{c,k_0,k_3}(u)du};$$

(iii) *For every $k_3 \neq k_0$, the following conditions are satisfied:*

(a) The consistency condition holds, meaning that

$$\Psi_t^{\mathbb{Q}^{k_0}, -\int_0^\cdot r_s^{c,k_0,k_3}ds}(1) = -r_{t-}^{c,k_0,k_3} = -f_{t-}^{c,k_0,k_3}(t); \tag{5.25}$$

(b) The HJM drift condition

$$\int_t^T (\alpha_t^{c,k_0}(u) + \alpha_t^{k_0,k_3}(u))du = \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T) - \Sigma_t^{k_0,k_3}(T)) \tag{5.26}$$

holds.

Proof. These results are implicitly obtained in the proof of Theorem 5.17. □

Corollary 5.22. *If the extended bond-price model for the currency k_0 is risk neutral, then for every $k_3 \neq k_0$ we have that:*

(i) *For every $T > 0$, the instantaneous foreign-collateral forward rate $(f_t^{c,k_0,k_3}(T))_{t\in[0,T]}$ is given by*

$$\begin{aligned} f_t^{c,k_0,k_3}(T) &= f_0^{c,k_0}(T) + q_0^{k_0,k_3}(T) \\ &\quad - \int_0^t (\sigma_s^{c,k_0}(T) + \sigma_s^{k_0,k_3}(T)) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T) - \Sigma_s^{k_0,k_3}(T))ds \\ &\quad + \int_0^t (\sigma_s^{c,k_0}(T) + \sigma_s^{k_0,k_3}(T))dX_s; \end{aligned}$$

(ii) *For every $t \geq 0$, the k_0-k_3 collateral short rate r_t^{c,k_0,k_3} at time t is given by*

$$\begin{aligned} r_t^{c,k_0,k_3} &= r_0^{c,k_0} + q_0^{k_0,k_3} \\ &\quad - \int_0^t (\sigma_s^{c,k_0}(t) + \sigma_s^{k_0,k_3}(t)) \nabla \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(t) - \Sigma_s^{k_0,k_3}(t))ds \\ &\quad + \int_0^t (\sigma_s^{c,k_0}(t) + \sigma_s^{k_0,k_3}(t))dX_s. \end{aligned}$$

Proof. The two results are obtained by inserting the drift condition (5.26) into equation (5.23) and equation (5.24), respectively. □

5.4. **Foreign exchange rate models and changes of measure.** We considered so far HJM models for the collateral discount curves, for the cross-currency basis curves, and for the foreign collateral discount curves. All these models have been presented under a domestic currency measure $\mathbb{Q}^{k_0}$, with the index k_0 ranging from 1 to L. In this section, we derive the corresponding dynamics under a different measure $\mathbb{Q}^k$ for $k \neq k_0$. This allows to specify the term-structure models of every economy under a single probability measure, which is essential when performing Monte Carlo simulations.

The first step is to link the different economies by means of general foreign exchange (FX) rate processes. From Proposition 2.6 we learned that, to guarantee absence of arbitrage in the unextended market, we need the $(\mathbb{Q}^{k_0}, \mathbb{G})$-local martingale property of (2.3), namely, we require that, given a fixed currency k_0, for any choice of $k \neq k_0$, the processes

$$\left(\frac{\mathcal{X}_t^{k_0,k} B_t^k}{B_t^{k_0}} \right)_{0 \leq t \leq T} \tag{5.27}$$

are $(\mathbb{Q}^{k_0}, \mathbb{G})$-local martingales. Moreover, in Section 3 we worked under Assumption 3.1, stating that the processes (5.27) are true $(\mathbb{Q}^{k_0}, \mathbb{G})$-martingales. We now proceed to construct models which are consistent with this setting.

Definition 5.23. *Let $1 \leq k_0 \leq L$. We call a model consisting of*

I. The $\mathbb{R}^{d_X+2L-1}$-valued Itô semimartingale

$$(X, B^{c,1}, \dots, B^{c,L}, Q^{k_0,1}, \dots, Q^{k_0,k_0-1}, Q^{k_0,k_0+1}, \dots, Q^{k_0,L});$$

II. The initial conditions $(\mathcal{X}_0^{k_0,1}, \dots, \mathcal{X}_0^{k_0,k_0-1}, \mathcal{X}_0^{k_0,k_0+1}, \dots, \mathcal{X}_0^{k_0,L}) \in \mathbb{R}_+^{L-1}$;

III. The processes $(\sigma^{\mathcal{X},k_0,1}, \dots, \sigma^{\mathcal{X},k_0,k_0-1}, \sigma^{\mathcal{X},k_0,k_0+1}, \dots, \sigma^{\mathcal{X},k_0,L})$;

an FX market model *for the currency k_0, if for every $k \neq k_0$ the following conditions are satisfied:*

(i) *The collateral cash accounts $B^{c,k}$ satisfies condition* (i) *of Definition 5.5;*

(ii) *The k_0-k cross-currency spread cash accounts $Q^{k_0,k}$ satisfies condition* (ii) *of Definition 5.14;*

(iii) *The maps* $(\omega, t) \mapsto \sigma_t^{\mathcal{X},k_0,k}$ *are* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$*-measurable* $\mathbb{R}^{d_X}$*-valued processes such that:*
   (a) $\sigma_{t,j}^{\mathcal{X},k_0,k}$ *is integrable with respect to the* j*-th component of the semimartingale* X*;*
   (b) $\sigma_t^{\mathcal{X},k_0,k} \in \mathcal{U}^{\mathbb{Q}^{k_0},X}$.

We then postulate the following dynamics for the FX rates:

$$\mathcal{X}_t^{k_0,k} = \mathcal{X}_0^{k_0,k} \frac{B_t^{c,k_0} Q_t^{k_0,k}}{B_t^{c,k}} \exp\left\{ -\int_0^t \Psi_s^{\mathbb{Q}^{k_0},X}(\sigma_s^{\mathcal{X},k_0,k}) ds + \int_0^t \sigma_s^{\mathcal{X},k_0,k} dX_s \right\}, \tag{5.28}$$

or, analogously,

$$\mathcal{X}_t^{k_0,k} = \mathcal{X}_0^{k_0,k} + \int_0^t \mathcal{X}_s^{k_0,k}(r_s^{c,k_0} - r_s^{c,k} + q_s^{k_0,k}) ds + \int_0^t \mathcal{X}_{s-}^{k_0,k} \sigma_s^{\mathcal{X},k_0,k} dX_s.$$

The dynamics of the FX rates (5.28) immediately implies that the processes (5.27) are $(\mathbb{Q}^{k_0}, \mathbb{G})$-local martingales. In line with Assumption 3.1, we must further impose the assumption that the processes (5.28) are such that (5.27) are $(\mathbb{Q}^{k_0}, \mathbb{G})$-true martingales. Concretely, for general Itô semimartingales, this can be achieved by imposing the conditions of Kallsen and Shiryaev [2002], see also Criens et al. [2017]. This will allow us to introduce changes between different spot martingale measures via the processes (5.27).

But before defining the change of measure starting from (5.27), we introduce some assumptions on the differential characteristics of the Itô semimartingale X. This is in line with Criens et al. [2017, Condition (B2)] and leads us to a concrete version of Assumption 3.1 for the process (5.27) in terms of the semimartingale characteristics.

Assumption 5.24. *Let* $1 \leq k_0 \leq L$ *be fixed. For the* $\mathbb{R}^{d_X}$*-valued Itô semimartingale* X *with* $\mathbb{Q}^{k_0}$*-differential characteristics* $(b^{\mathbb{Q}^{k_0}}, c, K^{\mathbb{Q}^{k_0}})$ *with respect to the truncation function* χ*, we assume that:*

(i) *For all* $t \geq 0$ *and all* $k \neq k_0$*,*

$$\int_0^t \int_{\left|\left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi\right| > 1} \exp\left\{ \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi \right\} \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi \, K_s^{\mathbb{Q}^{k_0}}(d\xi) ds < \infty, \quad \mathbb{Q}^{k_0}\text{-a.s.;}$$

(ii) *For all* $T \geq 0$ *and all* $k \neq k_0$*,*

$$\sup_{t \leq T} \mathbb{E}^{\mathbb{Q}^{k_0}} \Bigg[\exp\Bigg\{ \frac{1}{2} \int_0^t \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top c_s \sigma_s^{\mathcal{X},k_0,k} ds + \int_0^t \int_{\mathbb{R}^{d_X}} \left(\exp\left\{ \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi \right\} \left(\left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi - 1 \right) + 1 \right) K_s^{\mathbb{Q}^{k_0}}(d\xi) ds \Bigg\} \Bigg] < \infty, \quad \mathbb{Q}^{k_0}\text{-a.s..}$$

Assumption 5.24 allows us to derive the following representation for the semimartingale X.

Lemma 5.25. *Under Assumption 5.24, the semimartingale* X *admits the following representation under the measure* $\mathbb{Q}^{k_0}$*:*

$$X_t = X_0 + \int_0^t b_s^{\mathbb{Q}^{k_0}} ds + \int_0^t \sqrt{c_s} dW_s^{\mathbb{Q}^{k_0}} + \int_0^t \int_{\mathbb{R}^{d_X}} \xi \left(\mu^X - K_s^{\mathbb{Q}^{k_0}} \right)(d\xi, ds), \tag{5.29}$$

where $W^{\mathbb{Q}^{k_0}}$ *is an* $\mathbb{R}^{d_X}$*-valued* $(\mathbb{Q}^{k_0}, \mathbb{G})$*-Brownian motion,* $\sqrt{\cdot}$ *denotes the matrix-square root for symmetric positive semidefinite matrices, and* μ^X *is the random measure for the jump components of* X *with compensator* $K^{\mathbb{Q}^{k_0}}$ *under the measure* $\mathbb{Q}^{k_0}$*.*

Proof. The canonical decomposition of X under $\mathbb{Q}^{k_0}$ is

$$X_t = X_0 + \int_0^t b_s^{\mathbb{Q}^{k_0}} ds + \int_0^t \sqrt{c_s} dW_s^{\mathbb{Q}^{k_0}}$$

$$+\int_0^t \int_{\mathbb{R}^{d_X}} \chi(\xi)\left(\mu^X - K_s^{\mathbb{Q}^{k_0}}\right)(d\xi, ds) + \int_0^t \int_{\mathbb{R}^{d_X}} (\xi - \chi(\xi))\, \mu^X(d\xi, ds).$$

Because of Assumption 5.24, for the last term we notice that we can add and subtract

$$\int_0^t \int_{\mathbb{R}^{d_X}} (\xi - \chi(\xi))\, \mu^X(d\xi, ds) < \infty,$$

hence we obtain the decomposition (5.29) with

$$\tilde{b}_s^{\mathbb{Q}^{k_0}} := b_s^{\mathbb{Q}^{k_0}} + \int_{\mathbb{R}^{d_X}} (\xi - \chi(\xi))\, K_s^{\mathbb{Q}^{k_0}}(d\xi),$$

which, with an abuse of notation, we denoted again by $b^{\mathbb{Q}^{k_0}}$. □

Measure changes between different spot martingale measures are now defined via the processes (5.27). We detail the measure transformation in the following result.

Lemma 5.26. *Under Assumption 5.24, for any $k \neq k_0$, we introduce the risk-neutral measure $\mathbb{Q}^k \sim \mathbb{Q}^{k_0}$ on $\mathcal{G}_T$ by*

$$\frac{\partial \mathbb{Q}^k}{\partial \mathbb{Q}^{k_0}} := \frac{\mathcal{X}_T^{k_0,k} B_T^k}{B_T^{k_0}} \frac{B_0^{k_0}}{\mathcal{X}_0^{k_0,k} B_0^k},$$

such that

$$\left.\frac{\partial \mathbb{Q}^k}{\partial \mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{\partial \mathbb{Q}^k}{\partial \mathbb{Q}^{k_0}}\right| \mathcal{G}_t\right] = \frac{\mathcal{X}_t^{k_0,k} B_t^k}{B_t^{k_0}} \frac{B_0^{k_0}}{\mathcal{X}_0^{k_0,k} B_0^k}. \tag{5.30}$$

Then

$$W^{\mathbb{Q}^k} := W^{\mathbb{Q}^{k_0}} - \int_0^{\cdot} \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \sqrt{c_s} ds, \quad \textit{and} \tag{5.31}$$

$$K^{\mathbb{Q}^k}(d\xi) := \exp\left\{\left(\sigma^{\mathcal{X},k_0,k}\right)^\top \xi\right\} K^{\mathbb{Q}^{k_0}}(d\xi), \tag{5.32}$$

represent, respectively, a $(\mathbb{Q}^k, \mathbb{G})$-Brownian motion and the compensator of μ^X under $\mathbb{Q}^k$. Finally, the semimartingale X admits the following representation under $\mathbb{Q}^k$:

$$X_t = X_0 + \int_0^t b_s^{\mathbb{Q}^k} ds + \int_0^t \sqrt{c_s} dW_s^{\mathbb{Q}^k} + \int_0^t \int_{\mathbb{R}^{d_X}} \xi \left(\mu^X - K_s^{\mathbb{Q}^k}\right)(d\xi, ds), \tag{5.33}$$

where

$$b_s^{\mathbb{Q}^k} := b_s^{\mathbb{Q}^{k_0}} + \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top c_s - \int_{\mathbb{R}^{d_X}} \xi \left(1 - \exp\left\{\left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi\right\}\right) K_s^{\mathbb{Q}^{k_0}}(d\xi).$$

Proof. Equality (5.30) is easily satisfied since the processes (5.27) are $(\mathbb{Q}^{k_0}, \mathbb{G})$-local martingales, while the transformations (5.31) and (5.32) are due to the Girsanov transform, see, e.g., Jacod and Shiryaev [2003]. It is left to show (5.33).

Combining (5.29) with (5.31) and (5.32), and by adding and subtracting $\int_0^t \int_{\mathbb{R}^{d_X}} \xi K_s^{\mathbb{Q}^k}(d\xi, ds)$, we get

$$\begin{aligned} X_t = X_0 &+ \int_0^t b_s^{\mathbb{Q}^{k_0}} ds + \int_0^t \sqrt{c_s}\left(dW_s^{\mathbb{Q}^k} + \left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \sqrt{c_s} ds\right) \\ &+ \int_0^t \int_{\mathbb{R}^{d_X}} \xi \left(\mu^X - K_s^{\mathbb{Q}^k}\right)(d\xi, ds) + \int_0^t \int_{\mathbb{R}^{d_X}} \xi \left(e^{\left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top \xi} - 1\right) K_s^{\mathbb{Q}^{k_0}}(d\xi, ds). \end{aligned}$$

Regrouping the terms concludes the proof. □

We provide an application of the previous computations. Consider two currency areas k_0, k. To underline the fact that the characteristics of the semimartingale X are specified under a certain probability measure, we write $X^{\mathbb{Q}^{k_0}}$ and $X^{\mathbb{Q}^k}$, respectively. We consider the following hybrid model resulting from the combination of (5.7) for the two currency areas, (5.19) and (5.28):

$$
\begin{aligned}
f_t^{c,k_0}(T) &= f_0^{c,k_0}(T) - \int_0^t \sigma_s^{c,k_0}(T)\nabla\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T))ds + \int_0^t \sigma_s^{c,k_0}(T)dX_s^{\mathbb{Q}^{k_0}};\\
f_t^{c,k}(T) &= f_0^{c,k}(T) - \int_0^t \sigma_s^{c,k}(T)\nabla\Psi_s^{\mathbb{Q}^{k},X}(-\Sigma_s^{c,k}(T))ds + \int_0^t \sigma_s^{c,k}(T)dX_s^{\mathbb{Q}^{k}};\\
q_t^{k_0,k}(T) &= q_0^{k_0,k}(T) - \int_0^t \Big((\sigma_s^{c,k_0}(T)+\sigma_s^{k_0,k}(T))\nabla\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T)-\Sigma_s^{k_0,k}(T))\\
&\qquad -\sigma_s^{c,k_0}(T)\nabla\Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{c,k_0}(T))\Big)\,ds + \int_0^t \sigma_s^{k_0,k}(T)dX_s^{\mathbb{Q}^{k_0}};\\
\mathcal{X}_t^{k_0,k} &= \mathcal{X}_0^{k_0,k}\frac{B_t^{c,k_0}Q_t^{k_0,k}}{B_t^{c,k}}\exp\left\{-\int_0^t \Psi_s^{\mathbb{Q}^{k_0},X}(\sigma_s^{\mathcal{X},k_0,k})ds + \int_0^t \sigma_s^{\mathcal{X},k_0,k}dX_s^{\mathbb{Q}^{k_0}}\right\}.
\end{aligned}
$$

All the quantities are modelled under the measure $\mathbb{Q}^{k_0}$, except for the k-instantaneous collateral forward rate $\{(f_t^{c,k}(T))_{t\in[0,T]},\ T\geq 0\}$. By Lemma 5.26, we can express the whole system under a unique measure $\mathbb{Q}^{k_0}$ as follows:

$$
\begin{aligned}
f_t^{c,k}(T) &= f_0^{c,k}(T) - \int_0^t \sigma_s^{c,k}(T)\nabla\Psi_s^{\mathbb{Q}^{k},X}(-\Sigma_s^{c,k}(T))ds\\
&+\int_0^t \sigma_s^{c,k}(T)\Bigg(b_s^{\mathbb{Q}^k}ds - \big(\sigma_s^{\mathcal{X},k_0,k}\big)^\top c_s ds + \int_{\mathbb{R}^{d_X}}\xi\big(1-e^{\left(\sigma_s^{\mathcal{X},k_0,k}\right)^\top\xi}\big)K_s^{\mathbb{Q}^{k_0}}(d\xi)ds\\
&+\sqrt{c_s}dW_s^{\mathbb{Q}^{k_0}} + \int_{\mathbb{R}^{d_X}}\xi\left(\mu^X - K_s^{\mathbb{Q}^{k_0}}\right)(d\xi,ds)\Bigg),
\end{aligned}
$$

where we recognize additional drift terms capturing the so-called *quanto adjustments*.

6. Forwards of indices

The aim of the present section is to analyze the financial concept of a forward contract in a general way. For this reason, we first introduce on the usual filtered probability space $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\geq0},\mathbb{Q}^1,\dots,\mathbb{Q}^L)$ an abstract index associated to a generic currency area k_2. Then, our definition of a forward price is that of a fixed rate that makes a forward contract fair, in the sense that the price of the contract at initiation time is zero. This principle is not new, but was first used by Mercurio [2009] in order to introduce the concept of a forward rate agreement (FRA). The same principle has then been virtually employed in all the subsequent literature on multiple curves, and was also specialized in Fries [2016]. However, the definitions currently existing in the literature usually require to perform changes of measure, since the pricing measure is linked to the cash-account numéraire. We have seen instead that under the measure $\mathbb{Q}^{k_0}$ there is a multitude of martingales, namely each risky asset discounted by means of its own asset-specific cash account. We can then define all the forwards under a unique spot pricing measure $\mathbb{Q}^{k_0}$: this is a consequence of our fully coherent model of funding from the previous sections.

We denote by $I^{k_2}(T^s,T^f,T^e,T^p)$ the index referring to the period $[T^s,T^e]$ which is fixed in T^f for a payment in T^p, with $T^s\leq T^f\leq T^e$ and $T^p\geq T^f$, see Figure 3 for an illustration of the structure of the index. This means that the index is treated as a $\mathcal{G}_{T^f}$-measurable random variable, while T^p fixes the time horizon for discounting. In particular, this definition includes both the case $T^p\geq T^e$ and the case $T^p\leq T^e$, meaning that the payment may happen before, at, or after the end of the period, depending on the payment adjustment. In some cases, the index may be observed and paid simultaneously, namely $T^s=T^e=T^f=T^p$. Notice also that this extends the classical notation

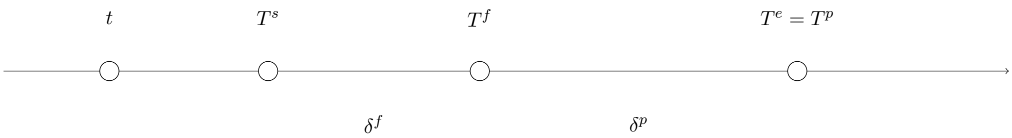

FIGURE 3. Illustration of an abstract index. The period starts at T^s and ends at T^e. The fixing and payment times are adjusted versions of T^s and T^e. In this case, we set $T^s < T^f < T^e$ and $T^e = T^p$.

where $T^f = T^s = T$ and $T^p = T^e = T + \delta$. In this case, indeed, the index $I^{k_2}(T, T, T + \delta, T + \delta)$ corresponds to $I^{k_2}(T, T + \delta)$ in the usual notation, with the additional superscript k_2 for the currency denomination of the index.

We then denote by $I^{k_0,k_2,k_3}(T^s, T^f, T^e, T^p)$ the value for a k_0-based agent of a forward, written on the index $I^{k_2}(T^s, T^f, T^e, T^p)$, collateralized in currency k_3, and with payment date T^p. We define $I_t^{k_0,k_2,k_3}(T^s, T^f, T^e, T^p)$ as the $\mathcal{G}_t$-measurable random variable that solves the following equation

$$\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\frac{B_t^{c,k_0,k_3}}{B_{T^p}^{c,k_0,k_3}}\left(I_{T^f}^{k_2}(T^s, T^f, T^e, T^p)\mathcal{X}_{T^p}^{k_0,k_2} - I_t^{k_0,k_2,k_3}(T^s, T^f, T^e, T^p)\right)\middle|\, \mathcal{G}_t\right] = 0, \tag{6.1}$$

where we assume that the conditional expectation is finite. We notice however that, from the mathematical point of view, the date T^e does not play any role in the forward definition (6.1). Indeed, the dates that really matter for the pricing of the forward are the start of the period T^s, which sets the start of the measuring of the index, the fixing date T^f, which sets the $\mathcal{G}_{T^f}$-measurability of the index, and the payment date T^p, which sets the discounting. For this reason, without loss of generality, we shall drop the dependency on T^e and work with $I^{k_2}(T^s, T^f, T^p)$ and $I^{k_0,k_2,k_3}(T^s, T^f, T^p)$ for the index and the associated forward, respectively.

In view of specifying a dynamic model for the forward $I^{k_0,k_2,k_3}(T^s, T^f, T^p)$, we need however to consider a running version of the index $I^{k_2}(T^s, T^f, T^p)$. For this, let us define $\delta^f := T^f - T^s$ and $\delta^p := T^p - T^f$ being, respectively, the fixing and the payment adjustments with respect to the fixing date. Then, at a given time $t \geq \delta^f$, we observe the spot index fixed in t, i.e. $T^f = t$, and with fixing window which has started in the past in $T^s = t - \delta^f$ for the payment horizon $T^p = t + \delta^p$. In other words, for the spot index with fixing in t, we shall adopt the notation $I_t^{k_2}(t - \delta^f, t, t + \delta^p)$ instead of $I_t^{k_2}(T^s, t, T^p)$. Notice that for every time instant t, the spot index $I_t^{k_2}(t - \delta^f, t, t + \delta^p)$ refers to a measurement period which has started in $t - \delta^f$, hence, with this new notation, the running time must start in δ^f, namely $t \geq \delta^f$, since we need enough past information to evaluate the index.

By working under this new notation, we now formalize the definition of a forward of an index.

Definition 6.1. *For a fixed $\delta^f \geq 0$ and a fixed $\delta^p \geq 0$, let $\left(I_t^{k_2}(t - \delta^f, t, t + \delta^p)\right)_{t \geq \delta^f}$ be a generic index which at every time instant $t \geq \delta^f$ has reference period starting in $t - \delta^f$ and payment date $t + \delta^p$. For any $T \geq \delta^f$, we denote by*

$$I_t^{k_0,k_2,k_3}(T - \delta^f, T, T + \delta^p), \quad \text{for } \delta^f \leq t \leq T + \delta^p,$$

the value at time t for a k_0-based agent of a forward, written on the index $\left(I_t^{k_2}(t - \delta^f, t, t + \delta^p)\right)_{t \geq \delta^f}$, collateralized in currency k_3, with period starting in $T - \delta^f$, fixing in T and payment date $T + \delta^p$. We define $I_t^{k_0,k_2,k_3}(T - \delta^f, T, T + \delta^p)$ as the $\mathcal{G}_t$-measurable random variable that solves the following equation:

$$\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\frac{B_t^{c,k_0,k_3}}{B_{T+\delta^p}^{c,k_0,k_3}}\left(I_T^{k_2}(T - \delta^f, T, T + \delta^p)\mathcal{X}_{T+\delta^p}^{k_0,k_2} - I_t^{k_0,k_2,k_3}(T - \delta^f, T, T + \delta^p)\right)\middle|\, \mathcal{G}_t\right] = 0, \tag{6.2}$$

where we assume that the conditional expectation is finite. We set $B^{c,k_0,k_3} = B^{c,k_0}$ *whenever* $k_3 = k_0$.

From the above definition of a forward, we immediately obtain the relation

$$I_t^{k_0,k_2,k_3}(T-\delta^f,T,T+\delta^p) = \frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0,k_3}}{B_{T+\delta^p}^{c,k_0,k_3}} I_T^{k_2}(T-\delta^f,T,T+\delta^p)\mathcal{X}_{T+\delta^p}^{k_0,k_2}\right| \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0,k_3}}{B_{T+\delta^p}^{c,k_0,k_3}}\right| \mathcal{G}_t\right]}. \tag{6.3}$$

It is important to notice that, for a fixed index $\left(I_t^{k_2}(t-\delta^f,t,t+\delta^p)\right)_{t\geq\delta^f}$, there are as many forward prices as there are possible funding (collateralization) policies. This is captured by the cardinality of the index $1 \leq k_3 \leq L$. In particular, by means of the change of measure (4.5), from (6.3) we obtain that

$$I_t^{k_0,k_2,k_3}(T-\delta^f,T,T+\delta^p) = \mathbb{E}^{\mathbb{Q}^{T+\delta^p,k_0,k_3}}\left[\left. I_T^{k_2}(T-\delta^f,T,T+\delta^p)\mathcal{X}_{T+\delta^p}^{k_0,k_2}\right| \mathcal{G}_t\right], \tag{6.4}$$

hence the forward index is a $(\mathbb{Q}^{T+\delta^p,k_0,k_3},\mathbb{G})$-martingale.

Remark 6.2. We point out that in Definition 6.1, after that the the spot index $I_T^{k_2}(T-\delta^f,T,T+\delta^p)$ has been fixed, namely for any $T < t \leq T+\delta^p$, the only varying components are the rate $\mathcal{X}^{k_0,k_2}$ and the collateral cash account B^{c,k_0,k_3}, which are fixed at time $T+\delta^p$, see equation (6.2). Hence, strictly speaking, the forward $I_t^{k_0,k_2,k_3}(T-\delta^f,T,T+\delta^p)$ is a genuine forward only for $\delta^f \leq t \leq T$, but it keeps varying for $T < t \leq T+\delta^p$ because of the fluctuations of $\mathcal{X}^{k_0,k_2}$ and B^{c,k_0,k_3}.

6.1. **Some examples.** We present in this section some examples which serve as motivation for our general framework which allows for the fixing date of the index to vary within the observation interval $[T^s,T^e]$. In particular, the formalism that we introduce can be used to recover various types of interest rates linked to the SOFR rate as introduced by Lyashenko and Mercurio [2019]. In what follows, for $M \in \mathbb{N}$, let $0 = T_0, T_1, \dots, T_M$ be a schedule of times, with $\delta_m := T_m - T_{m-1}$ the year fraction for the interval $[T_{m-1},T_m)$, for $1 \leq m \leq M$.

6.1.1. *Backward-looking rate.* An example of index is the backward-looking rate. In the notation of Lyashenko and Mercurio [2019], this is denoted by $R(T_{m-1},T_m)$. In our notation, $T^s = T_{m-1}$, $T^f = T^p = T_m$ and $t = T_m$. We then write $I_{T_m}^{k_0}(T_{m-1},T_m,T_m) = R(T_{m-1},T_m)$ for the backward-looking rate in currency $k_2 = k_0$, which is given by

$$I_{T_m}^{k_0}(T_{m-1},T_m,T_m) = \frac{1}{\delta_m}\left(e^{\int_{T_{m-1}}^{T_m} r_u^{c,k_0}du} - 1\right) = \frac{1}{\delta_m}\left(\frac{B_{T_m}^{c,k_0}}{B_{T_{m-1}}^{c,k_0}} - 1\right), \quad m = 1,\dots,M.$$

Clearly $I_{T_m}^{k_0}(T_{m-1},T_m,T_m)$ is a $\mathcal{G}_{T_m}$-measurable spot index.

6.1.2. *Forward-looking rate.* The forward-looking rate, in the notation of Lyashenko and Mercurio [2019] $F(T_{m-1},T_m)$, is the T_{m-1}-time value of the fair FRA rate K_F in the swaplet with payoff $\delta_m\left(R(T_{m-1},T_m) - K_F\right)$. This is a spot interest rate at time T_{m-1}, which we can however map in our general formalism of forwards. In this case, we have $T^s = T_{m-1}$, $T^f = T^p = T_m$ and $t = T_{m-1}$. Let then $I_{T_{m-1}}^{k_0}(T_{m-1},T_m,T_m) = F(T_{m-1},T_m)$ be the forward-looking rate in currency $k_2 = k_0$, which, from (6.3), is given by

$$I_{T_{m-1}}^{k_0}(T_{m-1},T_m,T_m) = \frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_{T_{m-1}}^{c,k_0}}{B_{T_m}^{c,k_0}} I_{T_m}^{k_0}(T_{m-1},T_m,T_m)\right| \mathcal{G}_{T_{m-1}}\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_{T_{m-1}}^{c,k_0}}{B_{T_m}^{c,k_0}}\right| \mathcal{G}_{T_{m-1}}\right]} = \frac{1}{\delta_m}\left(\frac{1}{B^{k_0,k_0}(T_{m-1},T_m)} - 1\right).$$

This is a $\mathcal{G}_{T_{m-1}}$-measurable spot index.

6.1.3. *Backward-looking in-arrears forward rate.* The backward-looking in-arrears forward rate, in the notation of Lyashenko and Mercurio [2019] $R_m(t)$, is the t-time fixed fair value K_R of a FRA with payoff $\delta_m\left(R(T_{m-1},T_m)-K_R\right)$. In this case, we have a genuine forward rate which can be mapped in the general definition. In particular, we have $T^s=T_{m-1}$, $T^f=T_m$ and $T^p=T_m$. We then identify this forward rate as a general forward in currency $k_2=k_0$ with collateralization in currency $k_3=k_0$, by setting $I_t^{k_0,k_0,k_0}(T_{m-1},T_m,T_m)=R_m(t)$. Equation (6.3) takes the form

$$I_t^{k_0,k_0,k_0}(T_{m-1},T_m,T_m)=\frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}I_{T_m}^{k_0}(T_{m-1},T_m,T_m)\right|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}\right|\mathcal{G}_t\right]}=\frac{1}{\delta_m}\left(\frac{B^{k_0,k_0}(t,T_{m-1})}{B^{k_0,k_0}(t,T_m)}-1\right).$$

It is clear that $R_m(T_{m-1})=F(T_{m-1},T_m)$.

6.1.4. *Forward-looking forward rate.* The forward-looking forward rate, $F_m(t)$ in the notation of Lyashenko and Mercurio [2019], is the t-time fixed fair value K_F of a FRA with payoff $\delta_m\left(F(T_{m-1},T_m)-K_F\right)$. Also in this case we have a genuine forward rate which can be mapped in our general definition by setting $T^s=T_{m-1}$, $T^f=T_{m-1}$ and $T^p=T_m$. We then identify this forward rate as a general forward in currency $k_2=k_0$ with collateralization in currency $k_3=k_0$, by setting $I_t^{k_0,k_0,k_0}(T_{m-1},T_{m-1},T_m)=F_m(t)$. Equation (6.3) takes then the form

$$I_t^{k_0,k_0,k_0}(T_{m-1},T_{m-1},T_m)=\frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}I_{T_{m-1}}^{k_0}(T_{m-1},T_m,T_m)\right|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}\right|\mathcal{G}_t\right]}=\frac{1}{\delta_m}\left(\frac{B^{k_0,k_0}(t,T_{m-1})}{B^{k_0,k_0}(t,T_m)}-1\right).$$

As already observed by Lyashenko and Mercurio [2019], for $t>T_{m-1}$, we have $F_m(t)=F\left(T_{m-1},T_m\right)$. Moreover, for $t\leq T_{m-1}$, $R_m(t)=F_m(t)$, $d\mathbb{P}\otimes dt$-a.s., and for $t=T_{m-1}$, we have $R_m\left(T_{m-1}\right)=F_m\left(T_{m-1}\right)=F\left(T_{m-1},T_m\right)$.

6.1.5. *Forward-looking inter-bank offered rate.* An example of forward is a contract written on a forward-looking inter-bank offered rate (IBOR), such as LIBOR, EURIBOR, TIBOR or AMERIBOR. We denote by $\mathcal{I}^{k_0}(T^s,T^p)$ the value of the IBOR index for the currency k_0. The IBOR index is fixed at the beginning of the period, namely $T^f=T^s$, hence for $t=T^s$, the index $\mathcal{I}^{k_0}_{T^s}(T^s,T^p)$ is $\mathcal{G}_{T^s}$-measurable. For $k_2=k_0$, we then set $T^f=T^s=T_{m-1}$ and $T^p=T_m$, and equation (6.3) takes the form

$$\begin{aligned}I_t^{k_0,k_0,k_0}(T_{m-1},T_{m-1},T_m)&=\frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}\mathcal{I}_{T_{m-1}}^{k_0}(T_{m-1},T_m)\right|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_m}^{c,k_0}}\right|\mathcal{G}_t\right]}\\&=\mathbb{E}^{\mathbb{Q}^{T_m,k_0,k_0}}\left[\left.\mathcal{I}_{T_{m-1}}^{k_0}(T_{m-1},T_m)\right|\mathcal{G}_t\right]=:\mathcal{I}_t^{k_0,k_0}(T_{m-1},T_{m-1},T_m),\end{aligned}$$

for $m=1,\dots,M$, which serves a definition for the IBOR forward rate for currency k_0 collateralized in units of currency k_0. This coincides with the definition of forward LIBOR rate originally introduced in Mercurio [2009] and then employed in the literature on interest rate modeling in the multiple curve framework.

6.1.6. *Commodity forwards.* Definition 6.1 covers also examples in the commodity markets. Let $I^{k_0}(T_1,T_2)$ be an index whose values depend on an underlying process observed over the interval $[T_1,T_2]$

with $T_1 \leq T_2$, hence in this case $T^s = T_1$ and $T^f = T^p = T_2$. The underlying process could be, e.g., the spot price of electricity or a temperature index, such as the cumulative average temperature (CAT) index. In the notation of Benth et al. [2008], we would have $I^{k_0}(T_1, T_2) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathcal{S}_u du$ for the electricity index, and $I^{k_0}(T_1, T_2) = \int_{T_1}^{T_2} \mathcal{T}_u du$ for the temperature index, with $\{\mathcal{S}_u\}_{u \geq 0}$ and $\{\mathcal{T}_u\}_{u \geq 0}$, respectively, the spot price of electricity and the instantaneous temperature. For $k_2 = k_3 = k_0$, the forward written on $I^{k_0}(T_1, T_2)$ is then, in the notation of Benth et al. [2008], $I_t^{k_0,k_0,k_0}(T_1, T_2, T_2) = F(t; T_1, T_2)$. Equation (6.3) becomes

$$I_t^{k_0,k_0,k_0}(T_1, T_2, T_2) = \frac{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_2}^{c,k_0}} I_{T_2}^{k_0}(T_1, T_2)\right| \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_0}}\left[\left.\frac{B_t^{c,k_0}}{B_{T_2}^{c,k_0}}\right| \mathcal{G}_t\right]} = \mathbb{E}^{\mathbb{Q}^{T_2,k_0,k_0}}\left[\left. I_{T_2}^{k_0}(T_1, T_2)\right| \mathcal{G}_t\right].$$

This is a $\mathcal{G}_{T_2}$-measurable contract which is referred to as a forward with delivery period $[T_1, T_2]$. Here $\mathbb{Q}^{T_2,k_0,k_0}$ is the pricing measure and $I_t^{k_0,k_0,k_0}(T_1, T_2, T_2)$ is a $(\mathbb{Q}^{T_2,k_0,k_0}, \mathbb{G})$-martingale. This is a typical kind of contracts in the electricity markets where the underlying (the electricity) must be delivered over a period of time. For those types of commodities with instantaneous delivery, we have $T_1 = T_2$ instead, and we may simply write $I^{k_0}(T_1) = I^{k_0}(T_1, T_1)$ for the index and $I_t^{k_0,k_0,k_0}(T_1) = I_t^{k_0,k_0,k_0}(T_1, T_1, T_1) = \mathbb{E}^{\mathbb{Q}^{T_1,k_0,k_0}}\left[\left. I_{T_1}^{k_0}(T_1)\right| \mathcal{G}_t\right]$ for the forward contract.

6.2. **HJM framework for abstract indices.** The aim of the present section is to study an HJM-type of framework for forward contracts on the abstract index from Definition 6.1. As for discount curves, we can choose between modeling directly the forward contracts, or we can introduce appropriate spread models. We shall follow the approach of Cuchiero et al. [2016], where the authors modelled the multiplicative spreads between LIBOR FRA rates and OIS FRA rates. In particular, we shall extend their approach in at least two directions. In fact, OIS FRA rates correspond to the forwards on the future performance of the collateral account. However, we have seen that in our setting we have multiple collateral accounts corresponding to multiple collateral currencies. Hence, on one hand, we will allow for multiple collateral currencies. On the other hand, we will work in a general setting allowing for both forward- and backward-looking rates, namely we will extend the definition of OIS FRA rates to include also backward-looking rates.

Before doing that, let us discuss the role of the native currency of denomination k_2 for the spot index $\left(I_t^{k_2}(t - \delta^f, t, t + \delta^p)\right)_{t \geq \delta^f}$ which appears in Definition 6.1. Starting from equation (6.3) and switching to the $\mathbb{Q}^{k_2}$ spot measure as in Definition 4.1, we obtain that

$$\begin{aligned} I_t^{k_0,k_2,k_3}(T - \delta^f, T, T + \delta^p) &= \frac{\frac{\mathcal{X}_t^{k_0,k_2} B_t^{k_2}}{B_t^{k_0}} \mathbb{E}^{\mathbb{Q}^{k_2}}\left[\left.\frac{B_{T+\delta^p}^{k_0}}{\mathcal{X}_{T+\delta^p}^{k_0,k_2} B_{T+\delta^p}^{k_2}} \frac{B_t^{c,k_0,k_3}}{B_{T+\delta^p}^{c,k_0,k_3}} I_T^{k_2}(T - \delta^f, T, T + \delta^p) \mathcal{X}_{T+\delta^p}^{k_0,k_2}\right| \mathcal{G}_t\right]}{\frac{\mathcal{X}_t^{k_0,k_2} B_t^{k_2}}{B_t^{k_0}} \mathbb{E}^{\mathbb{Q}^{k_2}}\left[\left.\frac{B_{T+\delta^p}^{k_0}}{\mathcal{X}_{T+\delta^p}^{k_0,k_2} B_{T+\delta^p}^{k_2}} \frac{B_t^{c,k_0,k_3}}{B_{T+\delta^p}^{c,k_0,k_3}}\right| \mathcal{G}_t\right]} \\ &= \frac{\mathcal{X}_t^{k_0,k_2} \mathbb{E}^{\mathbb{Q}^{k_2}}\left[\left.\frac{B_t^{c,k_2,k_3}}{B_{T+\delta^p}^{c,k_2,k_3}} I_T^{k_2}(T - \delta^f, T, T + \delta^p)\right| \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{Q}^{k_2}}\left[\left.\frac{\mathcal{X}_t^{k_0,k_2} B_t^{c,k_2,k_3}}{\mathcal{X}_{T+\delta^p}^{k_0,k_2} B_{T+\delta^p}^{c,k_2,k_3}}\right| \mathcal{G}_t\right]}, \end{aligned} \tag{6.5}$$

where we used that $\frac{B^{c,k_0,k_3} B^{k_2}}{B^{k_0}} = B^{c,k_2,k_3}$ due to the relations

$$r^{c,k_0} + q^{k_0,k_3} + r^{k_2} - r^{k_0} = r^{k_2} - r^{k_3} + r^{c,k_3} = r^{c,k_2} + q^{k_2,k_3}.$$

Equation (6.5) shows that we can conveniently postulate a model for the generic forward under the native currency of denomination of the index k_2 and then perform a measure change to recover the formulation of Definition 6.1. For this reason, we will limit ourselves to model $I^{k_0,k_2,k_3}(T-\delta^f,T,T+\delta^p)$ for $k_2=k_0$, which will allow us to further simplify the notation and write $I^{k_0,k_3}(T-\delta^f,T,T+\delta^p) := I^{k_0,k_0,k_3}(T-\delta^f,T,T+\delta^p)$.

Our proposed generalization of OIS FRA rate, or, equivalently, of forward performance of the collateral rate, is the following:

Definition 6.3. *For any $1\leq k_0,k_3\leq L$, any $\delta^f,\delta^p\geq 0$ and $T\geq\delta^f$, we define the k_0-k_3* simple forward collateral rate $\left(I_t^{k_0,k_3,D}(T-\delta^f,T,T+\delta^p)\right)_{\delta^f\leq t\leq T+\delta^p}$ *related to the term structure of discount factors $\{(B^{k_0,k_3}(t,\tau))_{t\in[0,\tau]},\ \tau\geq 0\}$ by*

$$
\begin{aligned}
&I_t^{k_0,k_3,D}(T-\delta^f,T,T+\delta^p)\\
&:=\begin{cases}
\frac{1}{\delta^f}\left(\frac{1}{B^{k_0,k_3}(t,T+\delta^p)}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[e^{-\int_t^{T-\delta^f} r_u^{c,k_0,k_3}du-\int_T^{T+\delta^p} r_u^{c,k_0,k_3}du}\middle|\mathcal{G}_t\right]-1\right), & \delta^f\leq t\leq T-\delta^f,\\
\frac{1}{\delta^f}\left(\frac{B_t^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_3}}\frac{1}{B^{k_0,k_3}(t,T+\delta^p)}\mathbb{E}^{\mathbb{Q}^{k_0}}\left[e^{-\int_T^{T+\delta^p} r_u^{c,k_0,k_3}du}\middle|\mathcal{G}_t\right]-1\right), & T-\delta^f<t<\leq T,\\
\frac{1}{\delta^f}\left(\frac{B_T^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_3}}-1\right), & T<t\leq T+\delta^p.
\end{cases}
\end{aligned}
$$

The definition above is very general since it does not restrict the fixing time T to coincide with the start of the period or with the payment date. In particular, this latter case is obtained by letting $\delta^p=0$. This leads to

$$
I_t^{k_0,k_3,D}(T-\delta^f,T,T)=\begin{cases}
\frac{1}{\delta^f}\left(\frac{B^{k_0,k_3}(t,T-\delta^f)}{B^{k_0,k_3}(t,T)}-1\right), & \delta^f\leq t\leq T-\delta^f,\\
\frac{1}{\delta^f}\left(\frac{B_t^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_3}}\frac{1}{B^{k_0,k_3}(t,T)}-1\right), & T-\delta^f<t\leq T.
\end{cases}
\tag{6.6}
$$

Notice that, in the single-currency setting, for $\delta^f\leq t\leq T-\delta^f$, this corresponds to the OIS FRA rate of Cuchiero et al. [2016] with $k_3=k_0$, namely $I_t^{k_0,k_0,D}(T-\delta^f,T,T)=L_t^D(T-\delta^f,T)$ in the notation of Cuchiero et al. [2016]. Moreover, in Section 5.3 we derived the HJM dynamics of $B^{k_0,k_3}(t,\cdot)$ as a consequence of the frameworks postulated for $B^{k_0,k_0}(t,\cdot)$ and $Q^{k_0,k_3}(t,\cdot)$. It is then immediate to link the k_0-k_3 simple forward collateral rate with the k_0-k_3 cross-currency spread bond since, by definition, we have

$$
I_t^{k_0,k_3,D}(T-\delta^f,T,T)=\begin{cases}
\frac{1}{\delta^f}\left(\frac{B^{k_0,k_0}(t,T-\delta^f)Q^{k_0,k_3}(t,T-\delta^f)}{B^{k_0,k_0}(t,T)Q^{k_0,k_3}(t,T)}-1\right), & \delta^f\leq t\leq T-\delta^f,\\
\frac{1}{\delta^f}\left(\frac{B_t^{c,k_0,k_0}Q_t^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_0}Q_{T-\delta^f}^{c,k_0,k_3}}\frac{1}{B^{k_0,k_0}(t,T)Q^{k_0,k_3}(t,T)}-1\right), & T-\delta^f<t\leq T.
\end{cases}
$$

We then see that the dynamics model for $I_t^{k_0,k_3,D}(T-\delta^f,T,T)$ (but also the one for $I_t^{k_0,k_3,D}(T-\delta^f,T,T+\delta^p)$) is fully characterized by the HJM models studied in Section 5.

The next step is to introduce appropriate HJM frameworks for the spreads with respect to the forward of a generic index. It is obvious that for modelling the multiplicative spread between the forward I^{k_0,k_3} on a generic index and the discount curve $I^{k_0,k_3,D}$, the start of the period for I^{k_0,k_3} must coincide with the start of the period for $I^{k_0,k_3,D}$, say $T-\delta^f$ for some $T\geq\delta^f\geq 0$. Similarly, the payment date for I^{k_0,k_3} must coincide with the payment date for $I^{k_0,k_3,D}$, say $T+\delta^p$ for a certain $\delta^p\geq 0$. However, we observe at this point that for any fixed start of period $T-\delta^f$ and any fixed payment date $T+\delta^p$, the two quantities $I^{k_0,k_3}(T-\delta^f,\cdot,T+\delta^p)$ and $I^{k_0,k_3,D}(T-\delta^f,\cdot,T+\delta^p)$ may have different fixing dates, both varying between $T-\delta^f$ and $T+\delta^p$. As this would lead to work with four different date indices, we shall let free the fixing date in $I^{k_0,k_3}(T-\delta^f,\cdot,T+\delta^p)$, namely we set it to T, and we shall set the fixing date in $I^{k_0,k_3,D}(T-\delta^f,\cdot,T+\delta^p)$ to coincide with the payment date, namely $T+\delta^p$, similarly as in (6.6). Under this simplifying assumption for the discount curve index, we generalize Cuchiero et al. [2016] by introducing the following.

Definition 6.4. *Fix $1 \le k_0, k_3 \le L$ and $\delta^f, \delta^p \ge 0$, and let $\delta := \delta^f + \delta^p$. For every $T \ge \delta^f$, the* forward index spread *at time $\delta^f \le t \le T+\delta^p$ for the time period $[T-\delta^f, T+\delta^p]$ over the index $I^{k_0}(T-\delta^f,T,T+\delta^p)$ collateralized according to B^{c,k_0,k_3} and payed at time $T+\delta^p$ is defined by*

$$\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) := \frac{1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)}{1+\delta I_t^{k_0,k_3,D}(T-\delta^f,T+\delta^p,T+\delta^p)}. \tag{6.7}$$

Notice that, strictly speaking, the quantities δ^f and δ^p in (6.7) refer to the fixing and payment adjustments for the numerator $I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)$. Indeed, the fixing and payment adjustments for the denominator $I_t^{k_0,k_3,D}(T-\delta^f,T+\delta^p,T+\delta^p)$ are $\delta=\delta^f+\delta^p$ and 0, respectively. Hence (6.6) in this case becomes

$$I_t^{k_0,k_3,D}(T-\delta^f,T+\delta^p,T+\delta^p) = \begin{cases} \frac{1}{\delta}\left(\frac{B^{k_0,k_3}(t,T-\delta^f)}{B^{k_0,k_3}(t,T+\delta^p)}-1\right), & \delta^f \le t \le T-\delta^f, \\ \frac{1}{\delta}\left(\frac{B_t^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_3}}\frac{1}{B^{k_0,k_3}(t,T+\delta^p)}-1\right), & T-\delta^f < t \le T+\delta^p, \end{cases}$$

and the forward spread in (6.7) can be rewritten by

$$\begin{aligned} &\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) \\ &= \begin{cases} \left(1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right)\frac{B^{k_0,k_3}(t,T+\delta^p)}{B^{k_0,k_3}(t,T-\delta^f)}, & \delta^f \le t \le T-\delta^f, \\ \left(1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right)\frac{B_{T-\delta^f}^{c,k_0,k_3}}{B_t^{c,k_0,k_3}}B^{k_0,k_3}(t,T+\delta^p), & T-\delta^f < t \le T, \\ \left(1+\delta I_T^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right)\frac{B_{T-\delta^f}^{c,k_0,k_3}}{B_t^{c,k_0,k_3}}B^{k_0,k_3}(t,T+\delta^p), & T < t \le T+\delta^p. \end{cases} \end{aligned} \tag{6.8}$$

This definition of spread, explicitly featuring the k_0-k_3 cross-currency spread bonds, highlights the role that cross-currency convexity adjustments will play in the dynamics of the generalized forward.

In particular, we observe that for $\delta^f \le t \le T$, both numerator and denominator in the definition of spread (6.7) are random quantities. However, for $T-\delta^f < t \le T$ we are in the monitoring period of both numerator and denominator, hence we expect the volatility to be decreasing. Finally, for $T < t \le T+\delta^p$ the numerator is no longer random since it has been fixed in $t=T$, while we still observe fluctuations of the denominator with decreasing volatility up to time $t=T+\delta^p$, where the volatility becomes zero. From (6.8), we then deduce that for $T<t\le T+\delta^p$ the multiplicative spread is of the form

$$\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) = \mathcal{C}\,\frac{B^{k_0,k_3}(t,T+\delta^p)}{B_t^{c,k_0,k_3}},$$

with $\mathcal{C} := B_{T-\delta^f}^{c,k_0,k_3}\left(1+\delta I_T^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right) \in \mathbb{R}$ being a $\mathcal{G}_T$-measurable random variable. Hence for $T<t\le T+\delta^p$ the model for the multiplicative spread is given by the HJM framework for the foreign-collateral discount curves in Section 5.3, since the only fluctuating components in the spread are $B^{k_0,k_3}(\cdot,T+\delta^p)$ and $B_\cdot^{c,k_0,k_3}$.

Notice further that for $T=t$, we have that $I_t^{k_0,k_3}(t-\delta^f,t,t+\delta^p) = I_t^{k_0}(t-\delta^f,t,t+\delta^p)$, hence from (6.7) the spot index spread is given by

$$\mathcal{S}_t^{k_0,k_3}(t-\delta^f,t,t+\delta^p) = \frac{1+\delta I_t^{k_0}(t-\delta^f,t,t+\delta^p)}{1+\delta I_t^{k_0,k_3,D}(t-\delta^f,t+\delta^p,t+\delta^p)}, \quad \text{for any } t \ge \delta^f.$$

Remark 6.5. In some applications, it is desirable to guarantee that the forward index spread is larger than one. These situations arise for example when modeling a forward risky inter-bank rate. This property can be achieved by imposing restrictions on the dynamics of the spread as in Cuchiero et al. [2016] and Cuchiero et al. [2019]. Alternatively, one can define the forward index spread by first modelling

$$1+\delta\tilde{\mathcal{S}}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) := \frac{1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)}{1+\delta I_t^{k_0,k_3,D}(T-\delta^f,T+\delta^p,T+\delta^p)},$$

so that $\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)=1+\delta\tilde{\mathcal{S}}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)$ is larger than one as soon as $\tilde{\mathcal{S}}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)$ is positive. The drawback of this approach is that it typically leads to more complicated pricing formulas. For example, for a caplet written on the IBOR rate, this approach leads in general to the pricing formula of a two-dimensional basket option.

The next lemma characterizes the martingale property of the forward index spreads.

Lemma 6.6. *Let $1\le k_0,k_3\le L$ and $\delta^f,\delta^p\ge 0$. Then the forward index spread $(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p))_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T-\delta^f,k_0,k_3},\mathbb{G})$-martingale for every $T\ge\delta^f$.*

Proof. The proof generalises Lemma 3.11 of Cuchiero et al. [2016]. Fix any schedule $\delta^f,\delta^p\ge 0$. We know from (6.4) that $(I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p))_{t\in[\delta^f,T+\delta^p]}$ is a martingale under $\mathbb{Q}^{T+\delta^p,k_0,k_3}$. Moreover, using Bayes' formula, the spread process $(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p))_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T-\delta^f,k_0,k_3},\mathbb{G})$-martingale if and only if the process

$$\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\left.\frac{\partial\mathbb{Q}^{T-\delta^f,k_0,k_3}}{\partial\mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t}$$

is a $(\mathbb{Q}^{k_0},\mathbb{G})$-martingale. From Definition 4.2, we can compactly write that

$$\left.\frac{\partial\mathbb{Q}^{T-\delta^f,k_0,k_3}}{\partial\mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t} = \frac{B^{c,k_0,k_3}_{\delta^f}}{B^{c,k_0,k_3}_{t\wedge(T-\delta^f)}}\frac{B^{k_0,k_3}(t\wedge(T-\delta^f),T-\delta^f)}{B^{k_0,k_3}(\delta^f,T-\delta^f)}, \qquad \text{for all } \delta^f\le t\le T+\delta^p,$$

where $t\wedge(T-\delta^f)$ is the minimum between t and $T-\delta^f$, and we consider δ^f as the start of the running time. Similarly, from (6.8) we can rewrite compactly the spread as

$$\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) = \Big(1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\Big)\frac{B^{c,k_0,k_3}_{t\wedge(T-\delta^f)}}{B^{c,k_0,k_3}_t}\frac{B^{k_0,k_3}(t,T+\delta^p)}{B^{k_0,k_3}(t\wedge(T-\delta^f),T-\delta^f)}.$$

By combining the last three equations, we then obtain

$$\Big(1+\delta I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\Big)\frac{B^{c,k_0,k_3}_{\delta^f}B^{k_0,k_3}(t,T+\delta^p)}{B^{c,k_0,k_3}_t B^{k_0,k_3}(\delta^f,T+\delta^p)}\frac{B^{k_0,k_3}(\delta^f,T+\delta^p)}{B^{k_0,k_3}(\delta^f,T-\delta^f)},$$

which is indeed a $(\mathbb{Q}^{k_0},\mathbb{G})$-martingale because $(I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p))_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T+\delta^p,k_0,k_3},\mathbb{G})$-martingale and $\left.\frac{\partial\mathbb{Q}^{T+\delta^p,k_0,k_3}}{\partial\mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t}=\frac{B^{c,k_0,k_3}_{\delta^f}B^{k_0,k_3}(t,T+\delta^p)}{B^{c,k_0,k_3}_t B^{k_0,k_3}(\delta^f,T+\delta^p)}$, for all $t\ge\delta^f$. ☐

We point out that Lemma 6.6 generalises Lemma 3.11 of Cuchiero et al. [2016] in the sense that in our setting the fixing date T may differ from the start-of-period date, $T-\delta^f$. Moreover, since we model the spread up to the payment date, $T+\delta^p$, we obtain a more general result, stating that the multiplicative forward index spread is a martingale under the forward measure $\mathbb{Q}^{T-\delta^f,k_0,k_3}$ for any $\delta^f\le t\le T+\delta^p$. Notice, however, that for $t\ge T-\delta^f$ the Radon-Nikodym derivative $\left.\frac{\partial\mathbb{Q}^{T-\delta^f,k_0,k_3}}{\partial\mathbb{Q}^{k_0}}\right|_{\mathcal{G}_t}$ is a known quantity, hence the two measures $\mathbb{Q}^{T-\delta^f,k_0,k_3}$ and $\mathbb{Q}^{k_0}$ are equivalent for any $t\ge T-\delta^f$ up to a multiplicative factor. This means that, in practice, the multiplicative spread is a $\mathbb{Q}^{k_0}$-martingale after the start of the monitoring period, $T-\delta^f$.

We proceed now to introduce the modeling framework for the forward index spreads. The following definition captures the complexity given by the fact that the forward on the index, for any given schedule $\delta^f, \delta^p \geq 0$ and any currency k_0, can be collateralized in any currency $k_3 = 1, \dots, L$.

Definition 6.7. *Let $1 \leq k_0 \leq L$ and $\delta^f, \delta^p \geq 0$ be fixed. We call a model consisting of*

I. An extended bond-price model for the currency k_0

$$\Big(X, Q^{k_0,1}, \dots, Q^{k_0,k_0-1}, Q^{k_0,k_0+1} \dots, Q^{k_0,L}, B^{c,k_0}, f^{c,k_0}_{\delta^f}, q^{k_0,1}_{\delta^f}, \dots, q^{k_0,k_0-1}_{\delta^f}, q^{k_0,k_0+1}_{\delta^f}, \dots, q^{k_0,L}_{\delta^f},$$
$$\alpha^{c,k_0}, \alpha^{k_0,1}, \dots, \alpha^{k_0,k_0-1}, \alpha^{k_0,k_0+1}, \dots, \alpha^{k_0,L}, \sigma^{c,k_0}, \sigma^{k_0,1}, \dots, \sigma^{k_0,k_0-1}, \sigma^{k_0,k_0+1}, \dots, \sigma^{k_0,L}\Big)$$

in the sense of Definition 5.14;

II. The $\mathbb{R}^L$-valued Itô semimartingale $\Big(\mathcal{S}^{k_0,1}_t(t-\delta^f, t, t+\delta^p), \dots, \mathcal{S}^{k_0,L}_t(t-\delta^f, t, t+\delta^p)\Big)_{t\geq\delta^f}$;

III. The functions $h^{\delta^f,\delta^p,k_0,1}_{\delta^f}, \dots, h^{\delta^f,\delta^p,k_0,L}_{\delta^f}$;

IV. The processes

$$\alpha^{\delta^f,\delta^p,k_0,1}, \dots, \alpha^{\delta^f,\delta^p,k_0,L},$$

and

$$\sigma^{\delta^f,\delta^p,k_0,1}, \dots, \sigma^{\delta^f,\delta^p,k_0,L};$$

a multiplicative spread model *for the currency k_0, if for every $1 \leq k_3 \leq L$ the following conditions are satisfied:*

(i) *The spot spread index $\mathcal{S}^{k_0,k_3}_t(t-\delta^f, t, t+\delta^p)$ is absolutely continuous with respect to the Lebsegue measure and satisfies $\mathcal{S}^{k_0,k_3}_t(t-\delta^f, t, t+\delta^p) = e^{-\int_{\delta^f}^t h^{\delta^f,\delta^p,k_0,k_3}_s ds}$ with multiplicative spread short rate $h^{\delta^f,\delta^p,k_0,k_3} = (h^{\delta^f,\delta^p,k_0,k_3}_t)_{t\geq\delta^f}$;*

(ii) *The triple $(h^{\delta^f,\delta^p,k_0,k_3}_{\delta^f}, \alpha^{\delta^f,\delta^p,k_0,k_3}, \sigma^{\delta^f,\delta^p,k_0,k_3})$ satisfies the HJM-basic condition in Definition 5.4;*

(iii) *For every $\tau \geq \delta^f$, the instantaneous multiplicative spread forward rate $(h^{\delta^f,\delta^p,k_0,k_3}_t(\tau))_{t\in[\delta^f,\tau]}$ is given by*

$$h^{\delta^f,\delta^p,k_0,k_3}_t(\tau) = h^{\delta^f,\delta^p,k_0,k_3}_{\delta^f}(\tau) + \int_{\delta^f}^t \alpha^{\delta^f,\delta^p,k_0,k_3}_s(\tau) ds + \int_{\delta^f}^t \sigma^{\delta^f,\delta^p,k_0,k_3}_s(\tau) dX_s; \tag{6.9}$$

(iv) *For every $T \geq \delta^f$, the instantaneous multiplicative spread forward rate satisfies*

$$h^{\delta^f,\delta^p,k_0,k_3}_t(\tau) = f^{c,k_0,k_3}_t(\tau), \qquad \text{for every } T < t \leq \tau \leq T+\delta^p;$$

(v) *The forward index spread $(\mathcal{S}^{k_0,k_3}_t(T-\delta^f, T, T+\delta^p))_{t\in[\delta^f, T+\delta^p]}$ satisfies*

$$\mathcal{S}^{k_0,k_3}_t(T-\delta^f, T, T+\delta^p) = e^{-\int_{\delta^f}^t h^{\delta^f,\delta^p,k_0,k_3}_s ds - \int_t^{T+\delta^p} h^{\delta^f,\delta^p,k_0,k_3}_t(u) du}. \tag{6.10}$$

The next definition naturally collects the martingale conditions that are relevant in the current setting.

Definition 6.8. *Let $1 \leq k_0 \leq L$ and $\delta^f, \delta^p \geq 0$. We say that the multiplicative spread model for the currency k_0 is* risk neutral *if the following conditions hold:*

(i) *The extended bond-price model for the currency k_0*

$$\Big(X, Q^{k_0,1}, \dots, Q^{k_0,k_0-1}, Q^{k_0,k_0+1} \dots, Q^{k_0,L}, B^{c,k_0}, f^{c,k_0}_{\delta^f}, q^{k_0,1}_{\delta^f}, \dots, q^{k_0,k_0-1}_{\delta^f}, q^{k_0,k_0+1}_{\delta^f}, \dots, q^{k_0,L}_{\delta^f},$$
$$\alpha^{c,k_0}, \alpha^{k_0,1}, \dots, \alpha^{k_0,k_0-1}, \alpha^{k_0,k_0+1}, \dots, \alpha^{k_0,L}, \sigma^{c,k_0}, \sigma^{k_0,1}, \dots, \sigma^{k_0,k_0-1}, \sigma^{k_0,k_0+1}, \dots, \sigma^{k_0,L}\Big)$$

is risk neutral in the sense of Definition 5.16;

(ii) *For each $1 \leq k_3 \leq L$, the forward index spreads*

$$\left\{ \left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f, T, T+\delta^p \right)_{t\in[\delta^f, T+\delta^p]}, \ T \geq \delta^f \right\}$$

are $(\mathbb{Q}^{T-\delta^f,k_0,k_3}, \mathbb{G})$-martingales.

For every $\tau \geq \delta^f$, we further define

$$\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(\tau) := \int_t^\tau \sigma_t^{\delta^f,\delta^p,k_0,k_3}(u)du,$$

and state the following result characterizing condition (ii) of Definition 6.8.

Theorem 6.9. *Let $1 \leq k_0 \leq L$ and $\delta^f, \delta^p \geq 0$. For a multiplicative spread model for the currency k_0, the followings are equivalent:*

(i) *The multiplicative spread model satisfies condition* (ii) *of Definition 6.8;*

(ii) *For every $T \geq \delta^f$ and every k_3, the conditional expectation hypothesis holds, namely*

$$\mathbb{E}^{\mathbb{Q}^{T-\delta^f,k_0,k_3}}\left[\mathcal{S}_T^{k_0,k_3}(T-\delta^f,T,T+\delta^p) \middle| \mathcal{G}_t \right] = e^{-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds - \int_t^{T+\delta^p} h_t^{\delta^f,\delta^p,k_0,k_3}(u)du},$$

for every $t \in [\delta^f, T+\delta^p]$;

(iii) *For every $T \geq \delta^f$ and every k_3, $-\Sigma^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) \in \mathcal{U}^{\mathbb{Q}^{k_0},X}$ and*

$$-\left(\Sigma^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) + \Sigma^{c,k_0}(T-\delta^f) + \Sigma^{k_0,k_3}(T-\delta^f) \right) \in \mathcal{U}^{\mathbb{Q}^{k_0},X},$$

and the following conditions are satisfied:

(a) The process

$$\begin{aligned}(6.11)\quad &\left(\exp\left\{ -\int_{\delta^f}^t \left(\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) + \Sigma_{s\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) + \Sigma_{s\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f) \right) dX_s \right.\right. \\ &\left.\left. - \int_{\delta^f}^t \Psi_s^{\mathbb{Q}^{k_0},X}(-\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{s\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{s\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f))ds \right\} \right)_{t\in[\delta^f,T+\delta^p]}\end{aligned}$$

is a $\mathbb{Q}^{k_0}$-martingale;

(b) The consistency condition holds, meaning that

$$\Psi_t^{\mathbb{Q}^{k_0}, -\int_{\delta^f}^{\cdot} h_s^{\delta^f,\delta^p,k_0,k_3}ds}(1) = -h_{t-}^{\delta^f,\delta^p,k_0,k_3} = -h_{t-}^{\delta^f,\delta^p,k_0,k_3}(t) \tag{6.12}$$

for all $t \geq \delta^f$;

(c) For every $T \geq \delta^f$ and every k_3, the HJM drift condition

$$\begin{aligned}&\int_t^{T+\delta^p} \alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du \\ &= \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)) \\ &- \Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f))\end{aligned} \tag{6.13}$$

holds for every $t \in [\delta^f, T+\delta^p]$.

Proof. The proof goes in parallel to the proof of Theorem 5.17. In the following, let $T \geq \delta^f$ and $1 \leq k_3 \leq L$ be fixed.

(i) $\Rightarrow$ (ii) Since the process $\left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right)_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T-\delta^f,k_0,k_3},\mathbb{G})$-martingale, it follows that

$$\mathbb{E}^{\mathbb{Q}^{T-\delta^f,k_0,k_3}}\left[\mathcal{S}_T^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\middle|\mathcal{G}_t\right]=\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)$$
$$=e^{-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds-\int_t^{T+\delta^p}h_t^{\delta^f,\delta^p,k_0,k_3}(u)du}.$$

(i) $\Rightarrow$ (iii) Since the process $\left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\right)_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T-\delta^f,k_0,k_3},\mathbb{G})$-martingale, by Bayes' formula, the process

$$\begin{aligned}&\left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\frac{B_{\delta^f}^{c,k_0,k_3}}{B_{t\wedge(T-\delta^f)}^{c,k_0,k_3}}\frac{B^{k_0,k_3}(t\wedge(T-\delta^f),T-\delta^f)}{B^{k_0,k_3}(\delta^f,T-\delta^f)}\right)_{t\in[\delta^f,T+\delta^p]}\\ &=\left(\frac{e^{-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds-\int_t^{T+\delta^p}h_t^{\delta^f,\delta^p,k_0,k_3}(u)du-\int_{t\wedge(T-\delta^f)}^{T-\delta^f}f_{t\wedge(T-\delta^f)}^{c,k_0,k_3}(u)du}}{e^{\int_{\delta^f}^{t\wedge(T-\delta^f)}r_s^{c,k_0,k_3}ds-\int_{\delta^f}^{T-\delta^f}f_{\delta^f}^{c,k_0,k_3}(u)du}}\right)_{t\in[\delta^f,T+\delta^p]}\end{aligned}\tag{6.14}$$

is a $(\mathbb{Q}^{k_0},\mathbb{G})$-martingale. Let

$$\begin{aligned}R_t:=&-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds-\int_t^{T+\delta^p}h_t^{\delta^f,\delta^p,k_0,k_3}(u)du-\int_{t\wedge(T-\delta^f)}^{T-\delta^f}f_{t\wedge(T-\delta^f)}^{c,k_0,k_3}(u)du\\&-\int_{\delta^f}^{t\wedge(T-\delta^f)}r_s^{c,k_0,k_3}ds+\int_{\delta^f}^{T-\delta^f}f_{\delta^f}^{c,k_0,k_3}(u)du.\end{aligned}$$

Then the martingale property of (6.14) is equivalent to the martingale property of $\exp(R)$, which implies that $1\in\mathcal{U}^{\mathbb{Q}^{k_0},R}$ and $\Psi_t^{\mathbb{Q}^{k_0},R}(1)=0$. Due to the integrability conditions on α^{c,k_0} and σ^{c,k_0} in Definition 5.5, on α^{k_0,k_3} and σ^{k_0,k_3} in Definition 5.14, and $\alpha^{\delta^f,\delta^p,k_0,k_3}$ and $\sigma^{\delta^f,\delta^p,k_0,k_3}$ in Definition 6.7, we can apply the classical and the stochastic Fubini theorem, which yield

$$\begin{aligned}\int_t^{T+\delta^p}h_t^{\delta^f,\delta^p,k_0,k_3}(u)du=&\int_{\delta^f}^{T+\delta^p}h_{\delta^f}^{\delta^f,\delta^p,k_0,k_3}(u)du+\int_{\delta^f}^t\int_s^{T+\delta^p}\alpha_s^{\delta^f,\delta^p,k_0,k_3}(u)duds\\&+\int_{\delta^f}^t\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p)dX_s-\int_{\delta^f}^t h_u^{\delta^f,\delta^p,k_0,k_3}(u)du,\end{aligned}\tag{6.15}$$

and, similarly, starting from (5.23),

$$\begin{aligned}&\int_{t\wedge(T-\delta^f)}^{T-\delta^f}f_{t\wedge(T-\delta^f)}^{c,k_0,k_3}(u)du\\&=\int_{\delta^f}^{T-\delta^f}f_{\delta^f}^{c,k_0,k_3}(u)du+\int_{\delta^f}^{t\wedge(T-\delta^f)}\int_s^{T-\delta^f}(\alpha_s^{c,k_0}(u)+\alpha_s^{k_0,k_3}(u))duds\\&+\int_{\delta^f}^{t\wedge(T-\delta^f)}(\Sigma_s^{c,k_0}(T-\delta^f)+\Sigma_s^{k_0,k_3}(T-\delta^f))dX_s-\int_{\delta^f}^{t\wedge(T-\delta^f)}f_u^{c,k_0,k_3}(u)du.\end{aligned}\tag{6.16}$$

By applying Kallsen and Krühner [2013, Lemma A.13] and using the equality

$$\Psi_t^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot\wedge(T-\delta^f)}r_s^{c,k_0,k_3}ds}(1)=\Psi_{t\wedge(T-\delta^f)}^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot\wedge(T-\delta^f)}r_s^{c,k_0,k_3}ds}(1),$$

we then obtain that

$$
\begin{aligned}
0 &= \Psi_t^{\mathbb{Q}^{k_0},R}(1) \\
&= \Psi_t^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot} h_s^{\delta^f,\delta^p,k_0,k_3}ds}(1) + \Psi_{t\wedge(T-\delta^f)}^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot\wedge(T-\delta^f)} r_s^{c,k_0,k_3}ds}(1) \\
&+ \Psi_t^{\mathbb{Q}^{k_0},X}\left(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)\right) \\
&- \int_t^{T+\delta^p} \alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du - \int_{t\wedge(T-\delta^f)}^{T-\delta^f} (\alpha_{t\wedge(T-\delta^f)}^{c,k_0}(u) + \alpha_{t\wedge(T-\delta^f)}^{k_0,k_3}(u))du \\
&+ h_{t-}^{\delta^f,\delta^p,k_0,k_3}(t) + f_{(t\wedge(T-\delta^f))-}^{c,k_0,k_3}(t\wedge(T-\delta^f)),
\end{aligned} \tag{6.17}
$$

where $\Psi_{t\wedge(T-\delta^f)}^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot\wedge(T-\delta^f)} r_s^{c,k_0,k_3}ds}(1) + f_{(t\wedge(T-\delta^f))-}^{c,k_0,k_3}(t\wedge(T-\delta^f)) = 0$ because of the consistency condition (5.25). Set now $t = T+\delta^p$ in (6.17). Since $\Sigma_{T+\delta^p}^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) = \Sigma_{T-\delta^f}^{c,k_0}(T-\delta^f) = \Sigma_{T-\delta^f}^{k_0,k_3}(T-\delta^f) = 0$, we get

$$
0 = \Psi_t^{\mathbb{Q}^{k_0},-\int_{\delta^f}^{\cdot} h_s^{\delta^f,\delta^p,k_0,k_3}ds}(1) + h_{t-}^{\delta^f,\delta^p,k_0,k_3}(t),
$$

hence (6.12). Moreover, substituting the two consistency conditions into (6.17) yields the following drift condition:

$$
\begin{aligned}
&\int_t^{T+\delta^p} \alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du + \int_{t\wedge(T-\delta^f)}^{T-\delta^f} (\alpha_{t\wedge(T-\delta^f)}^{c,k_0}(u) + \alpha_{t\wedge(T-\delta^f)}^{k_0,k_3}(u))du \\
&= \Psi_t^{\mathbb{Q}^{k_0},X}\left(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)\right),
\end{aligned}
$$

hence, by the drift condition (5.26),

$$
\begin{aligned}
&\int_t^{T+\delta^p} \alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du \\
&= \Psi_t^{\mathbb{Q}^{k_0},X}\left(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)\right) \\
&- \Psi_{t\wedge(T-\delta^f)}^{\mathbb{Q}^{k_0},X}(-\Sigma_{t\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{t\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)).
\end{aligned}
$$

We now have both the consistency condition and the drift condition. By substituting them into (6.15), and then together with (6.16) into (6.14) we write that

$$
\begin{aligned}
&\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\frac{B_{\delta^f}^{c,k_0,k_3}}{B_{t\wedge(T-\delta^f)}^{c,k_0,k_3}}\frac{B^{k_0,k_3}(t\wedge(T-\delta^f),T-\delta^f)}{B^{k_0,k_3}(\delta^f,T-\delta^f)} \\
&= \exp\Bigg\{-\int_{\delta^f}^{T+\delta^p} h_{\delta^f}^{\delta^f,\delta^p,k_0,k_3}(u)du \\
&\quad - \int_{\delta^f}^t \Psi_s^{\mathbb{Q}^{k_0},X}\left(-\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) - \Sigma_{s\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) - \Sigma_{s\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)\right)ds \\
&\quad - \int_{\delta^f}^t \left(\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p) + \Sigma_{s\wedge(T-\delta^f)}^{c,k_0}(T-\delta^f) + \Sigma_{s\wedge(T-\delta^f)}^{k_0,k_3}(T-\delta^f)\right)dX_s\Bigg\},
\end{aligned} \tag{6.18}
$$

from which we deduce that the process (6.11) is a $\mathbb{Q}^{k_0}$-martingale for every $T \geq \delta^f$.
(iii) $\Rightarrow$ (i) The consistency condition and the drift condition yield again equation (6.18). The martingale property of (6.11) implies then that the forward index spread $(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p))_{t\in[\delta^f,T+\delta^p]}$ is a $(\mathbb{Q}^{T-\delta^f,k_0,k_3},\mathbb{G})$-martingale.

□

The drift condition (6.13) allows us to explicitly observe the interplay between the different risk factors that drive the multiplicative spread. In particular, we observe that the drift depends of course on the integrated volatility of the spread $\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p)$ up to the payment date $T+\delta^p$, but also on the integrated volatility of the instantaneous collateral forward rate $\Sigma_t^{c,k_0}(T-\delta^f)$ and on that of the instantaneous cross-currency basis spread $\Sigma_t^{k_0,k_3}(T-\delta^f)$, which is an interesting feature from an economic perspective.

Remark 6.10. Similarly as in Remark 5.18, we notice that the HJM drift condition (6.13) is given in terms of the local exponent $\Psi^{\mathbb{Q}^{k_0},X}$. However, one can show that for any $\delta^f \le t \le T-\delta^f$ the local exponent $\Psi^{\mathbb{Q}^{T-\delta^f,k_0,k_3},X}$ under the measure $\mathbb{Q}^{T-\delta^f,k_0,k_3}$ is obtained from $\Psi^{\mathbb{Q}^{k_0},X}$ by

$$\Psi^{\mathbb{Q}^{T-\delta^f,k_0,k_3},X}(\beta)=\Psi^{\mathbb{Q}^{k_0},X}(\beta-\Sigma^{c,k_0}(T-\delta^f)-\Sigma^{k_0,k_3}(T-\delta^f))-\Psi^{\mathbb{Q}^{k_0},X}(-\Sigma^{c,k_0}(T-\delta^f)-\Sigma^{k_0,k_3}(T-\delta^f)),$$

for any $\mathbb{R}^d$-valued predictable and X-integrable process $\beta=(\beta_t)_{t\ge 0}$. We further observe from Definition 4.2, that for $T-\delta^f<t\le T+\delta^p$ the two measures $\mathbb{Q}^{k_0}$ and $\mathbb{Q}^{T-\delta^f,k_0,k_3}$ are equivalent up to a multiplicative constant. Hence, basically, we have that

$$\Psi^{\mathbb{Q}^{T-\delta^f,k_0,k_3},X}(\beta)=\Psi^{\mathbb{Q}^{k_0},X}(\beta).$$

The HJM drift condition (6.13) can then be rewritten under the forward measure $\mathbb{Q}^{T-\delta^f,k_0,k_3}$ as

$$\int_t^{T+\delta^p}\alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du=\Psi_t^{\mathbb{Q}^{T-\delta^f,k_0,k_3},X}(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p)),$$

for every $\delta^f \le t \le T+\delta^p$. We further notice that, since $h_t^{\delta^f,\delta^p,k_0,k_3}(\tau)=f_t^{c,k_0,k_3}(\tau)$ for every $T<t\le\tau\le T+\delta^p$ by definition, then $\alpha_t^{\delta^f,\delta^p,k_0,k_3}(\tau)=\alpha_t^{c,k_0}(\tau)+\alpha_t^{k_0,k_3}(\tau)$ and $\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(\tau)=\Sigma_t^{c,k_0}(\tau)+\Sigma_t^{k_0,k_3}(\tau)$ for every $T<t\le\tau\le T+\delta^p$, and the drift condition (6.13) coincides with the drift condition (5.26), namely

$$\begin{aligned}\int_t^{T+\delta^p}(\alpha_t^{c,k_0}(u)+\alpha_t^{k_0,k_3}(u))du&=\int_t^{T+\delta^p}\alpha_t^{\delta^f,\delta^p,k_0,k_3}(u)du\\&=\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{\delta^f,\delta^p,k_0,k_3}(T+\delta^p))\\&=\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{c,k_0}(T+\delta^p)-\Sigma_t^{k_0,k_3}(T+\delta^p)),\end{aligned}$$

for every $T<t\le T+\delta^p$.

Remark 6.11. For $k_3=k_0$, $\delta^f=0$ and $\delta^p=\delta$, the drift condition (6.13) becomes

$$\int_t^{T+\delta}\alpha_t^{0,\delta,k_0,k_0}(u)du=\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_t^{0,\delta,k_0,k_0}(T+\delta)-\Sigma_{t\wedge T}^{c,k_0}(T))-\Psi_t^{\mathbb{Q}^{k_0},X}(-\Sigma_{t\wedge T}^{c,k_0}(T)). \tag{6.19}$$

Notice that this extends the results of Cuchiero et al. [2016], because we model the spread beyond the fixing date, in this case T, namely we model the spread up to the payment date, $T+\delta$. This is important since, as discussed before, despite for $T<t\le T+\delta^p$ the index has already been fixed, one still observes fluctuations in the spread due to fluctuations in the foreign-collateral discount curves. Instead, in Cuchiero et al. [2016], these fluctuations are implicitly ignored since the spread is only modelled up to the fixing date T. In particular, this would mean to set

$$\int_T^{T+\delta^p}\alpha_t^{0,\delta,k_0,k_0}(u)du=\int_T^{T+\delta^p}\sigma_t^{0,\delta,k_0,k_0}(u)du=0,$$

hence $\Sigma_t^{0,\delta,k_0,k_0}(T+\delta) = \Sigma_t^{0,\delta,k_0,k_0}(T)$. Under these assumptions, we observe that the drift condition (6.19) coincides indeed with the drift condition found in Cuchiero et al. [2016, Theorem 3.15].

Corollary 6.12. *Let $1 \leq k_0 \leq L$ and $\delta^f, \delta^p \geq 0$. If the multiplicative spread model for the currency k_0 is risk neutral, then for every $1 \leq k_3 \leq L$ we have that:*

(i) *For every $t \geq \delta^f$ and $\tau \geq \delta = \delta^f + \delta^p$, the instantaneous multiplicative spread forward rate is given by*

$$\begin{aligned}
&h_t^{\delta^f,\delta^p,k_0,k_3}(\tau) = h_{\delta^f}^{\delta^f,\delta^p,k_0,k_3}(\tau) \\
&- \int_{\delta^f}^t \Bigg(\left(\sigma_s^{\delta^f,\delta^p,k_0,k_3}(\tau) + \sigma_{s\wedge(\tau-\delta)}^{c,k_0}(\tau-\delta) + \sigma_{s\wedge(\tau-\delta)}^{k_0,k_3}(\tau-\delta) \right) \cdot \\
&\quad \cdot \nabla\Psi_s^{\mathbb{Q}^{k_0},X} \left(-\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(\tau) - \Sigma_{s\wedge(\tau-\delta)}^{c,k_0}(\tau-\delta) - \Sigma_{s\wedge(\tau-\delta)}^{k_0,k_3}(\tau-\delta) \right) \\
&\quad - \left(\sigma_{s\wedge(\tau-\delta)}^{c,k_0}(\tau-\delta) + \sigma_{s\wedge(\tau-\delta)}^{k_0,k_3}(\tau-\delta) \right) \nabla\Psi_s^{\mathbb{Q}^{k_0},X} \left(-\Sigma_{s\wedge(\tau-\delta)}^{c,k_0}(\tau-\delta) - \Sigma_{s\wedge(\tau-\delta)}^{k_0,k_3}(\tau-\delta) \right) \Bigg) ds \\
&+ \int_{\delta^f}^t \sigma_s^{\delta^f,\delta^p,k_0,k_3}(\tau) dX_s;
\end{aligned}$$

(ii) *For every $t \geq \delta$, the multiplicative spread short rate $h_t^{\delta^f,\delta^p,k_0,k_3}$ at time t is given by*

$$\begin{aligned}
&h_t^{\delta^f,\delta^p,k_0,k_3} = h_t^{\delta^f,\delta^p,k_0,k_3}(t) = h_{\delta^f}^{\delta^f,\delta^p,k_0,k_3}(t) \\
&- \int_{\delta^f}^t \Bigg(\left(\sigma_s^{\delta^f,\delta^p,k_0,k_3}(t) + \sigma_{s\wedge(t-\delta)}^{c,k_0}(t-\delta) + \sigma_{s\wedge(t-\delta)}^{k_0,k_3}(t-\delta) \right) \cdot \\
&\quad \cdot \nabla\Psi_s^{\mathbb{Q}^{k_0},X} \left(-\Sigma_s^{\delta^f,\delta^p,k_0,k_3}(t) - \Sigma_{s\wedge(t-\delta)}^{c,k_0}(t-\delta) - \Sigma_{s\wedge(t-\delta)}^{k_0,k_3}(t-\delta) \right) \\
&\quad - \left(\sigma_{s\wedge(t-\delta)}^{c,k_0}(t-\delta) + \sigma_{s\wedge(t-\delta)}^{k_0,k_3}(t-\delta) \right) \nabla\Psi_s^{\mathbb{Q}^{k_0},X} \left(-\Sigma_{s\wedge(\tau-\delta)}^{c,k_0}(t-\delta) - \Sigma_{s\wedge(t-\delta)}^{k_0,k_3}(t-\delta) \right) \Bigg) ds \\
&+ \int_{\delta^f}^t \sigma_s^{\delta^f,\delta^p,k_0,k_3}(t) dX_s;
\end{aligned}$$

Proof. The proof proceeds similarly to the proof of Corollary 5.8, starting from (6.9) and using the drift condition (6.13) for $\tau := T + \delta^p$. For the short rate, we let $t \to \tau$. □

We conclude this section by deriving the HJM framework for an abstract index by combining the multiplicative spread model for the currency k_0 with the HJM framework for the extended bond-price model in Section 5.3. In particular, from Definition 6.4, for any fixed $1 \leq k_0, k_3 \leq L$, any $\delta^f, \delta^p \geq 0$ and for any $T \geq \delta^f$, the forward on the abstract index $I^{k_0}(T-\delta^f, T, T+\delta^p)$ at time $\delta^f \leq t \leq T + \delta^p$ for the time period $[T-\delta^f, T+\delta^p]$ collateralized according to B^{c,k_0,k_3} and payed at time $T+\delta^p$ is the product between the multiplicative spread and the discount curve index, namely

$$I_t^{k_0,k_3}(T-\delta^f, T, T+\delta^p) = \frac{1}{\delta} \left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f, T, T+\delta^p)\big(1 + \delta I_t^{k_0,k_3,D}(T-\delta^f, T+\delta^p, T+\delta^p)\big) - 1 \right),$$

where $\delta = \delta^f + \delta^p$. From (6.8) we further write that

$$I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) = \begin{cases} \frac{1}{\delta}\left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\frac{B^{k_0,k_3}(t,T-\delta^f)}{B^{k_0,k_3}(t,T+\delta^p)}-1\right), & \delta^f \le t \le T-\delta^f, \\ \frac{1}{\delta}\left(\mathcal{S}_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p)\frac{B_t^{c,k_0,k_3}}{B_{T-\delta^f}^{c,k_0,k_3}}\frac{1}{B^{k_0,k_3}(t,T+\delta^p)}-1\right), & T-\delta^f < t \le T, \\ I_T^{k_0}(T-\delta^f,T,T+\delta^p), & T < t \le T+\delta^p. \end{cases} \tag{6.20}$$

By combining equations (5.20), (5.21), (5.22) and (6.10), we can also rewrite the index in terms of the instantaneous collateral forward rate, the cross-currency basis spread and of the instantaneous multiplicative spread forward rate, namely

$$I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) = \frac{1}{\delta}\left(e^{-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds-\int_t^{T+\delta^p} h_t^{\delta^f,\delta^p,k_0,k_3}(u)du+\int_{T-\delta^f}^{T+\delta^p}\left(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u)\right)du}-1\right),$$

for $\delta^f \le t \le T-\delta^f$, and

$$\begin{aligned} &I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) \\ &= \frac{1}{\delta}\left(e^{-\int_{\delta^f}^t h_s^{\delta^f,\delta^p,k_0,k_3}ds-\int_t^{T+\delta^p} h_t^{\delta^f,\delta^p,k_0,k_3}(u)du+\int_{T-\delta^f}^{t}\left(r_s^{c,k_0}+q_s^{k_0,k_3}\right)ds-\int_t^{T+\delta^p}\left(f_t^{c,k_0}(u)+q_t^{k_0,k_3}(u)\right)du}-1\right), \end{aligned}$$

for $T-\delta^f < t \le T$. Notice that for $T < t \le T+\delta^p$ the forward equals the index at time T, namely $I_t^{k_0,k_3}(T-\delta^f,T,T+\delta^p) = I_T^{k_0}(T-\delta^f,T,T+\delta^p)$. Hence, in particular, it is constant.

7. Application: cross-currency swaps pricing

We provide an in-depth study of cross-currency swap contracts. The motivation for studying these instruments is twofold: on the one hand, cross-currency swap contracts offer the perfect example of instruments which depend on all the sources of risk that we have described in the previous sections. On the other hand, studying cross-currency swap contracts serves as a starting point for analysing the benchmark transition, which is still not treated in the literature. In particular, in view of the benchmark reform, we shall describe the legs of the contracts in terms of some abstract indices, in line with the approach adopted in Section 6.2. We proceed as follows: we consider first the case of constant-notional cross-currency swaps and provide the corresponding pricing formulas under different collateral currencies. Then, we consider resetting cross-currency swaps, and, finally, we study potential leg asymmetries which are originated by the LIBOR transition.

We consider two generic currencies $1 \le k_0, k \le L$ and a time interval $[\tau^s, \tau^e]$, with $\tau^e \ge \tau^s \ge 0$, representing the monitoring period of the cross-currency swap contract which is stipulated by two agents at time 0. In general, the number of legs for a swap can be arbitrary, most typically for bespoke over-the-counter (OTC) contracts. However, for simplicity, we consider contracts with only two legs and we use the index k_0 and the index k to denote, respectively, the domestic and the foreign leg. Let then $N^{k_0} \in \mathbb{N}$ and $N^k \in \mathbb{N}$ be the numbers of time intervals in the two schedules of cash flows happening within the interval $[\tau^s, \tau^e]$ and referring, respectively, to k_0 and to k. Notice that, in general, the number of cash flows can be different among different legs, typically due to different payment frequencies. We then denote the time instants for the domestic leg by $t_n^{k_0}$, with $0 \le n \le N^{k_0}$, and the time instants for the foreign leg by t_n^k, with $0 \le n \le N^k$. In particular, we set $t_0^{k_0} = t_0^k = \tau^s$ and $t_{N^{k_0}}^{k_0} = t_{N^k}^k = \tau^e$. For each interval of the form $(t_{n-1}^{k_0}, t_n^{k_0}]$, we define $\delta_n^{k_0} := t_n^{k_0} - t_{n-1}^{k_0}$, for

$1 \le n \le N^{k_0}$. Similarly, we define $\delta_n^k := t_n^k - t_{n-1}^k$, for $1 \le n \le N^k$. We then introduce $\delta_n^{k_0,f}$, $\delta_n^{k_0,p}$, $\delta_n^{k,f}$ and $\delta_n^{k,p}$ such that $\delta_n^{k_0,f} + \delta_n^{k_0,p} = \delta_n^{k_0}$ and $\delta_n^{k,f} + \delta_n^{k,p} = \delta_n^k$, respectively.

We further introduce $\phi \in \{+1,-1\}$ as the indicator for long and short positions, respectively, and $\mathcal{N}^{k_0}$ and $\mathcal{N}^k$ for the constant notional of the contract in domestic and foreign currency, respectively. Notice that a schedule of time-varying notional could be included by writing $\mathcal{N}^{k_0}_{t_n^{k_0}}$ and $\mathcal{N}^k_{t_n^k}$, instead. Spreads can also be included in either leg by means of the quantities $\mathcal{S}_0^{k_0}(\tau^e)$ and $\mathcal{S}_0^k(\tau^e)$. The standard market practice is to quote cross-currency swaps against USD and to add the spread to the non-USD leg. Notice also that the two spreads are fixed at the moment of the stipulation of the contract, namely in 0, and depend on the final horizon of the monitoring period, namely τ^e. As in the previous sections, we denote with k_3 the currency of denomination of the collaterals.

7.1. **Constant-notional cross-currency swaps.** The simplest form of cross-currency swap involves two agents who lend to each other notional amounts in two different currencies. The notional is swapped at the initial time, τ^s, and then swapped back at the terminal time, τ^e. We denote by CCS^{k_0,k_3} the value in domestic currency of the constant-notional cross-currency swap collateralized in currency k_3, which at time $t \ge 0$ is given by

$$CCS_t^{k_0,k_3} = \phi\left(S_t^{k_0}(A^{k_0}_{CCS}, C^{k_3}) - \mathcal{X}_t^{k_0,k} S_t^k(A^k_{CCS}, C^{k_3})\right),$$

with $A^{k_0}_{CCS}$ and A^k_{CCS} representing the cash flow of the domestic and foreign legs associated to two generic market indices I^{k_0} and I^k. In particular, for each $t \le \tau^e$, the leg $\ell \in \{k_0, k\}$ is evaluated via

$$\begin{aligned} S_t^\ell(A^\ell_{CCS}, C^{k_3}) = \mathcal{N}^\ell \Bigg(& -B^{\ell,k_3}(t,\tau^s)\mathbb{I}_{\{t\le\tau^s\}} + B^{\ell,k_3}(t,\tau^e) \\ & + \sum_{n=1}^{N^\ell} \delta_n^\ell \mathbb{E}^{\mathbb{Q}^\ell}\left[\frac{B_t^{c,\ell,k_3}}{B_{t_n^\ell}^{c,\ell,k_3}}\left(I^\ell_{t^\ell_{n-1}+\delta_n^{\ell,f}}(t^\ell_{n-1}, t^\ell_{n-1}+\delta_n^{\ell,f}, t_n^\ell) + \mathcal{S}_0^\ell(\tau^e)\right)\middle|\mathcal{G}_t\right]\mathbb{I}_{\{t\le t_n^\ell\}}\Bigg), \end{aligned} \tag{7.1}$$

where the indicator function $\mathbb{I}$ means that each term in the summations exists until its payment date, namely the n-th term in the domestic leg disappears for $t > t_n^{k_0}$, and the n-th term in the foreign leg disappears for $t > t_n^k$. Notice that the same formulas allow also to treat the case of fixed-versus-fixed and fixed-versus-floating cross-currency swaps by suitably setting the desired indices to zero and interpreting the spreads $\mathcal{S}_0^{k_0}(\tau^e)$ and $\mathcal{S}_0^k(\tau^e)$ as fixed rates.

We now illustrate how the modeling quantities analyzed in the present work are crucial in the evaluation of the formula (7.1). By Definition 6.1, we express the leg $\ell \in \{k_0, k\}$ in terms of the forward contract I^{ℓ,k_3} written on the index I^ℓ with collateralization in currency k_3, namely

$$\begin{aligned} S_t^\ell(A^\ell_{CCS}, C^{k_3}) = \mathcal{N}^\ell \Bigg(& -B^{\ell,k_3}(t,\tau^s)\mathbb{I}_{\{t\le\tau^s\}} + B^{\ell,k_3}(t,\tau^e) \\ & + \sum_{n=1}^{N^\ell} \delta_n^\ell \left(I_t^{\ell,k_3}(t^\ell_{n-1}, t^\ell_{n-1}+\delta_n^{\ell,f}, t_n^\ell) + \mathcal{S}_0^\ell(\tau^e)\right) B^{\ell,k_3}(t, t_n^\ell)\,\mathbb{I}_{\{t\le t_n^\ell\}}\Bigg). \end{aligned} \tag{7.2}$$

In particular, for every $t \ge 0$, the forward contract I^{ℓ,k_3} can be written in compact form starting from equation (6.20) as

$$I_t^{\ell,k_3}(t^\ell_{n-1}, t^\ell_{n-1}+\delta_n^{\ell,f}, t_n^\ell) = \frac{1}{\delta_n^\ell}\left(\frac{B_t^{c,\ell,k_3}}{B_{t\wedge t^\ell_{n-1}}^{c,\ell,k_3}}\frac{B^{\ell,k_3}(t\wedge t^\ell_{n-1}, t^\ell_{n-1})}{B^{\ell,k_3}(t,t_n^\ell)}\mathcal{S}_t^{\ell,k_3}(t^\ell_{n-1}, t^\ell_{n-1}+\delta_n^{\ell,f}, t_n^\ell) - 1\right). \tag{7.3}$$

By substituting (7.3) into (7.2), we get

$$S_t^\ell(A_{CCS}^\ell, C^{k_3}) = \mathcal{N}^\ell\Bigg(-B^{\ell,k_3}(t,\tau^s)\mathbb{I}_{\{t\le\tau^s\}} + B^{\ell,k_3}(t,\tau^e)$$

$$+\sum_{n=1}^{N^\ell}\left(\frac{B_t^{c,\ell,k_3}}{B_{t\wedge t_{n-1}^\ell}^{c,\ell,k_3}}\frac{B^{\ell,k_3}(t\wedge t_{n-1}^\ell, t_{n-1}^\ell)}{B^{\ell,k_3}(t,t_n^\ell)}\mathcal{S}_t^{\ell,k_3}(t_{n-1}^\ell, t_{n-1}^\ell+\delta_n^{\ell,f}, t_n^\ell) - 1 + \delta_n^\ell \mathcal{S}_0^\ell(\tau^e)\right)B^{\ell,k_3}(t,t_n^\ell)\,\mathbb{I}_{\{t\le t_n^\ell\}}\Bigg).$$

Finally, we use the definition of the foreign-collateral discount curves as product between the domestic bond with domestic collateral and the ℓ-k_3 cross-currency bond spread in equation (5.20) and obtain

$$S_t^\ell(A_{CCS}^\ell, C^{k_3}) =\mathcal{N}^\ell\Bigg(-B^{\ell,\ell}(t,\tau^s)Q^{\ell,k_3}(t,\tau^s)\mathbb{I}_{\{t\le\tau^s\}} + B^{\ell,\ell}(t,\tau^e)Q^{\ell,k_3}(t,\tau^e)$$

$$+\sum_{n=1}^{N^\ell}\Big(\frac{B_t^{c,\ell}Q_t^{\ell,k_3}}{B_{t\wedge t_{n-1}^\ell}^{c,\ell}\,Q_{t\wedge t_{n-1}^\ell}^{\ell,k_3}}\frac{B^{\ell,\ell}(t\wedge t_{n-1}^\ell, t_{n-1}^\ell)Q^{k_0,k_3}(t\wedge t_{n-1}^\ell, t_{n-1}^\ell)}{B^{\ell,\ell}(t,t_n^\ell)Q^{\ell,k_3}(t,t_n^{k_0})}\cdot$$

$$\cdot\,\mathcal{S}_t^{\ell,k_3}(t_{n-1}^\ell, t_{n-1}^\ell+\delta_n^{\ell,f}, t_n^\ell) - 1 + \delta_n^\ell\mathcal{S}_0^\ell(\tau^e)\Big)B^{\ell,\ell}(t,t_n^\ell)Q^{\ell,k_3}(t,t_n^\ell)\,\mathbb{I}_{\{t\le t_n^\ell\}}\Bigg).$$

This shows how all the quantities modelled in the previous sections enter into play in the evaluation of cross-currency swap contracts. In particular, the formulas obtained also highights that constant-notional cross-currency swaps are linear with respect to all the quantities introduced in our cross-currency HJM framework.

7.2. **Resetting cross-currency swaps.** A resetting (or marked-to-market) cross-currency swap (MtMCCS) is constructed via a sequence of one-period cross-currency swaps. Every single cross-currency swap in the sequence involves an exchange of notional, so that the contract can be interpreted as a rolling strategy on loans with varying notional. The rolling mechanism implies exchanges of notional at every payment date, and it reduces the outstanding counterparty risk exposure. This version of the instrument is cheaper if one takes into account the possibility of default of the agents which are involved in the transaction.

Typically, quoted instruments feature notional resets on the leg indexed to the *stronger* currency (e.g., USD for the EURUSD pair), and a constant notional for the *weaker* currency (i.e., EUR in the EURUSD pair example). In this case, the spread is introduced only on the weaker leg and it is fixed in such a way that the value of the contract at initiation is zero. This means in particular that the spread is the market quote. For our example, we shall analyze the case when the notional resets affect either the k_0- or the k-denominated leg. The collateralization is in currency k_3.

We then denote by $MtMCCS^{k_0,k_3}$ the value in domestic currency of the resetting cross-currency swap collateralized in currency k_3. With the notional resets affecting the k_0-denominated leg, for $t\le\tau^e$ we have that

$$MtMCCS^{k_0,k_3} = \phi\left(S_t^{k_0}(A_{MtMCCS}^{k_0}, C^{k_3}) - \mathcal{X}_t^{k_0,k}S_t^k(A_{CCS}^k, C^{k_3})\right),$$

where $A_{MtMCCS}^{k_0}$ represents the cash flow of the domestic leg with resetting. Similarly, if the notional resets are affecting the foreign leg instead, the valuation formula is

$$MtMCCS^{k_0,k_3} = \phi\left(S_t^{k_0}(A_{CCS}^{k_0}, C^{k_3}) - \mathcal{X}_t^{k_0,k}S_t^k(A_{MtMCCS}^k, C^{k_3})\right),$$

with A^k_{MtMCCS} being the cash flow of the foreign leg with resetting. In particular, then the value of the leg with resetting is given for $\ell \in \{k_0, k\}$ by

$$\begin{aligned}
&S^\ell_t(A^\ell_{MtMCCS}, C^{k_3}) = \\
&\mathcal{N}^\kappa \left(\sum_{n=1}^{N^\ell} \mathbb{E}^{\mathbb{Q}^\ell} \left[\frac{B^{c,\ell,k_3}_t}{B^{c,\ell,k_3}_{t^\ell_n}} \mathcal{X}^{\ell,\kappa}_{t^\ell_{n-1}} \left(1 + \delta^\ell_n \left(I^\ell_{t^\ell_{n-1}+\delta^{\ell,f}_n}(t^\ell_{n-1}, t^\ell_{n-1} + \delta^{\ell,f}_n, t^\ell_n) + \mathcal{S}^\ell_0(\tau^e) \right) \right) \middle| \mathcal{G}_t \right] \mathbb{I}_{\{t \le t^\ell_n\}} \right. \\
&\left. - \sum_{n=1}^{N^\ell} \mathbb{E}^{\mathbb{Q}^\ell} \left[\frac{B^{c,\ell,k_3}_t}{B^{c,\ell,k_3}_{t^\ell_{n-1}}} \mathcal{X}^{\ell,\kappa}_{t^\ell_{n-1}} \middle| \mathcal{G}_t \right] \mathbb{I}_{\{t \le t^\ell_n\}} \right),
\end{aligned} \tag{7.4}$$

where we notice that the notional for the leg $\ell \in \{k_0, k\}$ is now denominated in the other currency, namely in currency $\kappa \in \{k_0, k\}$ with $\kappa \neq \ell$. The leg with no resetting, $S^\kappa_t(A^\kappa_{CCS}, C^{k_3})$, is evaluated as in (7.1) for $\kappa \in \{k_0, k\}$. Starting from (7.4), with similar steps as in Section 7.1, one recovers all the modelling quantities studied in the paper, including the dynamics for the FX rate in Section 5.4.

7.3. **The impact of the LIBOR transition.** For concreteness, we take the perspective of a USD agent who entered at time $t = 0$ before any benchmark reform into an OTC EURUSD cross-currency swap with notional resets and fixed spread applied on the EUR-denominated leg. In this case, we then have $k_0 = \$$ and $k = €$. As a legacy product, we assume that the swap was entered when the USD 3M LIBOR and the 3M EURIBOR were the market standard floating rates applied to the two legs in this kind of contracts. Both the floating rates are fixed at the beginning of the period and paid at the end, so in this case $\delta^{€,f}_n = \delta^{\$,f}_n = 0$ for all n, which allows us to use the more familiar notation

$$I^\ell_{t^\ell_{n-1}+\delta^{\ell,f}_n}(t^\ell_{n-1}, t^\ell_{n-1} + \delta^{\ell,f}_n, t^\ell_n) = I^\ell_{t^\ell_{n-1}}(t^\ell_{n-1}, t^\ell_{n-1}, t^\ell_n) = \mathcal{I}^\ell_{t^\ell_{n-1}}(t^\ell_{n-1}, t^\ell_n), \text{ for } \ell \in \{€, \$\},$$

to denote the IBOR rate of the two currencies which is fixed in t^ℓ_{n-1} for the interval $(t^\ell_{n-1}, t^\ell_n]$, and is paid in t^ℓ_n. In line with the prevailing market standard, we assume that the collateral is exchanged in USD, namely $r^{c,k_3} = r^{c,k_0} = r^{c,\$}$, with $r^{c,\$}$ being initially the Fed Fund rate which is an unsecured overnight rate. The value of the resetting cross-currency swap is then

$$MtMCCS^{\$,\$} = \phi \left(S^\$_t(A^\$_{CCS}, C^\$) - \mathcal{X}^{\$,€}_t S^€_t(A^€_{MtMCCS}, C^\$) \right),$$

where, before the LIBOR discontinuation, the USD leg is evaluated as in (7.1), namely

$$\begin{aligned}
S^\$_t(A^\$_{CCS}, C^\$) &= \mathcal{N}^\$ \Bigg(-B^{\$,\$}(t, \tau^s)\mathbb{I}_{\{t \le \tau^s\}} + B^{\$,\$}(t, \tau^e) \\
&\quad + \sum_{n=1}^{N^\$} \delta^\$_n \mathbb{E}^{\mathbb{Q}^\$} \left[\frac{B^{c,\$}_t}{B^{c,\$}_{t^\$_n}} \mathcal{I}^\$_{t^\$_{n-1}}(t^\$_{n-1}, t^\$_n) \middle| \mathcal{G}_t \right] \mathbb{I}_{\{t \le t^\$_n\}} \Bigg) \\
&= \mathcal{N}^\$ \Bigg(-B^{\$,\$}(t, \tau^s)\mathbb{I}_{\{t \le \tau^s\}} + B^{\$,\$}(t, \tau^e) \\
&\quad + \sum_{n=1}^{N^\$} \delta^\$_n B^{\$,\$}(t, t^\$_n) \mathbb{E}^{\mathbb{Q}^{t^\$_n,\$,\$}} \left[\mathcal{I}^\$_{t^\$_{n-1}}(t^\$_{n-1}, t^\$_n) \middle| \mathcal{G}_t \right] \mathbb{I}_{\{t \le t^\$_n\}} \Bigg).
\end{aligned} \tag{7.5}$$

Here the spread $\mathcal{S}_0^{\$}(\tau^e)=0$ because the the USD currency is the strong one in the USD-EUR pair. The EUR leg is instead evaluated as in (7.4) by

$$S_t^{€}(A_{MtMCCS}^{€},C^{\$})=\mathcal{N}^{\$}\left(\sum_{n=1}^{N^{€}}\mathbb{E}^{\mathbb{Q}^{€}}\left[\frac{B_t^{c,€,\$}}{B_{t_n^{€}}^{c,€,\$}}\mathcal{X}_{t_{n-1}^{€}}^{€,\$}\left(1+\delta_n^{€}\left(\mathcal{I}_{t_{n-1}^{€}}^{€}(t_{n-1}^{€},t_n^{€})+\mathcal{S}_0^{€}(\tau^e)\right)\right)\middle|\mathcal{G}_t\right]\mathbb{I}_{\{t\le t_n^{€}\}}\right.$$
$$\left.-\sum_{n=1}^{N^{€}}\mathbb{E}^{\mathbb{Q}^{€}}\left[\frac{B_t^{c,€,\$}}{B_{t_{n-1}^{€}}^{c,€,\$}}\mathcal{X}_{t_{n-1}^{€}}^{€,\$}\middle|\mathcal{G}_t\right]\mathbb{I}_{\{t\le t_n^{€}\}}\right),$$

where the rate $r^{c,€}$ was initially represented by the EONIA rate. We remark that the expression above depends also on the infinitesimal cross-currency basis $q^{\$,€}$.

The benchmark reform had multiple impacts on the valuation of this type of products. The first one is a switch of the collateral rate by central counterparties for both the currency areas, which also became the market standard for OTC markets. On 27 July 2020, the central counterparties switched $r^{c,€}$ from EONIA to ESTR. Subsequently, on 16 October 2020, the rate $r^{c,\$}$ has been switched from the Fed Fund rate to SOFR. Both the switches of discount rates triggered exchanges of cash flows among agents. Even more relevant is the discontinuation of the USD 3M LIBOR rate, which is impacting directly the USD leg. In this case, the agents can agree on different alternatives, all of which are captured by our framework.

A first choice is to replace the USD 3M LIBOR by means of AMERIBOR T90. In this case, the new benchmark is still an unsecured forward-looking rate that embeds a dynamic credit component, hence the valuation formula (7.5) is virtually unchanged. A second choice is to adopt the ISDA fallback protocol[6], where USD 3M LIBOR is replaced by the sum of the backward-looking rate and a fixed credit spread. This means that in (7.5) we replace the spot index $\mathcal{I}_{t_{n-1}^{\$}}^{\$}(t_{n-1}^{\$},t_n^{\$})$ by[7]

$$I_{t_{n-1}^{\$}}^{\$}(t_{n-1}^{\$},t_n^{\$},t_n^{\$}):=\frac{1}{t_n^{\$}-t_{n-1}^{\$}}\left(e^{\int_{t_{n-1}^{\$}}^{t_n^{\$}}r_u^{c,\$}du}-1\right)+\mathcal{CS}=\frac{1}{t_n^{\$}-t_{n-1}^{\$}}\left(\frac{B_{t_n^{\$}}^{c,\$}}{B_{t_{n-1}^{\$}}^{c,\$}}-1\right)+\mathcal{CS},$$

where $\mathcal{CS}$ denotes a credit spread between the USD 3M LIBOR and the backward-looking rate. This is usually estimated as a mean or a median on the basis of historical data. For an in-depth analysis of LIBOR fallbacks from a quantitative perspective, we refer to Henrard [2019]. We limit ourselves to notice that a constant credit spread is a questionable choice, since it is clearly unable to capture the dynamic nature of inter-bank risk Filipović and Trolle [2013].

In the aftermath of the benchmark reform, cross-currency swaps still fall within our HJM framework: the market practice has moved to the use of the backward-looking rate on both legs.

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[6] https://assets.isda.org/media/3062e7b4/23aa1658-pdf/

[7] Concretely, the formula involves a daily compounding of rates over the whole interval $(t_{n-1}^{\$},t_n^{\$}]$ which we do not present here to avoid the introduction of further notation.

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(Alessandro Gnoatto) University of Verona, Department of Economics,
Via Cantarane 24, 37129 Verona, Italy
Email address: `alessandro.gnoatto@univr.it`

(Silvia Lavagnini) BI Norwegian Business School, Department of Data Science and Analytics,
Nydalsveien 37, 0484 Oslo, Norway
Email address: `silvia.lavagnini@bi.no`