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CBI-TIME-CHANGED LÉVY PROCESSES

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Abstract. We introduce and study the class of *CBI-time-changed Lévy processes* (CBITCL), obtained by time-changing a Lévy process with respect to an integrated continuous-state branching process with immigration (CBI). We characterize CBITCL processes as solutions to a certain stochastic integral equation and relate them to affine stochastic volatility processes. We provide a complete analysis of the time of explosion of exponential moments of CBITCL processes and study their asymptotic behavior. In addition, we show that CBITCL processes are stable with respect to a suitable class of equivalent changes of measure. As illustrated by some examples, CBITCL processes are flexible and tractable processes with a significant potential for applications in finance.

1. Introduction

Since their introduction in [KW71], continuous-state branching processes with immigration (CBI processes) have represented a major topic of research in the theory of stochastic processes (we refer to [Li20] for a recent overview of some of the main developments in the field). CBI processes belong to the class of affine processes (see [DFS03]) and, due to their analytical tractability, have found important applications in mathematical finance, especially in interest rate modelling (see [Fil01]). In recent years, CBI processes have attracted a renewed interest in financial modelling, due to their capability of reproducing empirical features of financial time series such as volatility clustering and self-exciting jumps. In particular, self-exciting CBI processes have been exploited for the construction of single-curve and multi-curve interest rate models in [JMS17, FGS21b], for the modelling of energy prices in [JMSS19, CMS22] and for stochastic volatility modelling in [JMSZ21].

In this paper, we introduce and study the class of *CBI-time-changed Lévy processes* (CBITCL), obtained by time-changing a Lévy process with respect to the time integral of a CBI process. This construction combines the distributional flexibility of Lévy processes with the self-exciting behavior of CBI processes, while retaining full analytical tractability. As shown in the companion paper [FGS21a], CBITCL processes have a significant potential for use in finance, notably in markets with stochastic volatility. In this paper, we set the theoretical foundations of CBITCL processes and derive some results that are especially motivated by financial applications.

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The main contributions of the present paper can be outlined as follows:

- We characterize CBITCL processes as solutions to a system of stochastic integral equations which generalizes the well-known Dawson-Li representation of a CBI process (see [DL06]) and we provide an additional characterization in terms of their semimartingale characteristics. Moreover, we study the relation with affine stochastic volatility processes, adopting an extension of the original definition of [KR11], and derive necessary and sufficient conditions for a general affine stochastic volatility process to be a CBITCL process.
- By combining general techniques of affine processes and specific properties of CBITCL processes, we provide a complete analysis of the time of explosion of exponential moments of CBITCL processes. This result is of fundamental importance in financial models, where a CBITCL process can be used to represent the log-price of an asset. We also study the asymptotic behavior of CBITCL processes. While the existence of a stationary distribution of a CBI process is well understood (see, e.g., [Li20, Section 10]), we provide an asymptotic result analogous to [KR11, Theorem 3.4] for the distribution of a CBITCL process, making use of our characterization of the lifetime of exponential moments.
- In view of financial applications, we derive a class of equivalent changes of probability that leave invariant the class of CBITCL processes, up to a change in their parameters. Moreover, we show how our results can be applied to two different specifications of CBITCL processes that have been recently considered in mathematical finance.

We emphasize that, while some of our results can be derived from the general theory of affine processes, we obtain more refined statements under minimal technical assumptions by exploiting the specific structure of CBITCL processes.

Our work is naturally related to the use of stochastic changes of time in finance. Starting from the seminal work [Cla73], time-changed processes have been widely adopted as models for asset prices and we refer to [BNS15, Swi16] for detailed accounts on the topic. In particular, our work builds on the contributions of [CGMY03, CW04], where stochastic volatility models have been constructed by relying on time-changed Lévy processes. An empirical analysis of several specifications of such models has been conducted in [HW04]. While their analysis only considers time changes driven by square-root diffusions, [HW04] point out that a promising direction of research is to investigate models where the activity rate of the time change process exhibits high-frequency jumps. This has been confirmed by the recent empirical analysis conducted in [FHM21]. This paper contributes to this line of research by developing a general theoretical framework for the use of CBI processes as time changes for Lévy processes, also allowing for the possible presence of self-exciting jumps in the activity rate of the time change.

The paper is structured as follows. In Section 2, we state the definition of CBITCL processes, characterize them as solutions to certain stochastic integral equations and study their relation to affine stochastic volatility processes. Section 3 contains the analysis of exponential moments of CBITCL processes and a study of their asymptotic behavior. In Section 4, we present a class of equivalent changes of probability that leave invariant the class of CBITCL processes. Finally, Section 5 contains two examples of CBITCL processes that are relevant for financial applications.

2. Definition and characterization

In this section, we state the definition of a CBITCL process and prove some foundational results for this class of processes. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us first recall the definition of a continuous-state branching process with immigration (see, e.g., [Li11, Chapter 3] and [Li20, Section 5]). To this effect, we introduce the following two functions:

- let $\Psi : \mathbb{R}_- \to \mathbb{R}$ be defined by

$$\Psi(u) := \beta u + \int_0^{+\infty} (e^{uz} - 1)\nu(\mathrm{d}z), \qquad \forall u \in \mathbb{R}_-, \tag{2.1}$$

  where $\beta \geqslant 0$ and ν is a Lévy measure on $(0, +\infty)$ such that $\int_0^1 z\,\nu(\mathrm{d}z) < +\infty$;
- let $\Phi : \mathbb{R}_- \to \mathbb{R}$ be defined by

$$\Phi(u) := -bu + \frac{1}{2}\sigma^2 u^2 + \int_0^{+\infty} (e^{uz} - 1 - uz)\pi(\mathrm{d}z), \qquad \forall u \in \mathbb{R}_-, \tag{2.2}$$

  where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and π is a Lévy measure on $(0, +\infty)$ such that $\int_1^{+\infty} z\,\pi(\mathrm{d}z) < +\infty$.

Definition 2.1. A Markov process $X = (X_t)_{t\geqslant 0}$ taking values in the state space $\mathbb{R}_+$ and with transition kernels $p_t : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \to [0, 1]$, for $t \geqslant 0$, is said to be a *continuous-state branching process with immigration* (CBI) with immigration mechanism Ψ and branching mechanism Φ if

$$\int_{\mathbb{R}_+} e^{uy}\, p_t(x, \mathrm{d}y) = \exp\left(\int_0^t \Psi\big(v(s, u)\big)\mathrm{d}s + v(t, u)x\right), \tag{2.3}$$

for all $u \in \mathbb{R}_-$, $x \in \mathbb{R}_+$ and $t \geqslant 0$, where the function $v(\cdot, u) : \mathbb{R}_+ \to \mathbb{R}_-$ is the unique solution to

$$\frac{\partial v}{\partial t}(t, u) = \Phi\big(v(t, u)\big), \qquad v(0, u) = u. \tag{2.4}$$

In the following, we denote by $\mathrm{CBI}(X_0, \Psi, \Phi)$ a CBI process with initial value X_0, immigration mechanism Ψ and branching mechanism Φ. Definition 2.1 corresponds to a conservative stochastically continuous CBI process in the sense of [KW71]. CBI processes are non-negative strong Markov (Feller) processes with càdlàg trajectories. As a consequence, the path integral $Y := \int_0^{\cdot} X_s\,\mathrm{d}s$ of a CBI process X is always well defined as a non-decreasing process and can therefore be used as a change of time. This motivates the following definition of CBI-time-changed Lévy processes.

Definition 2.2. A process (X, Z) is said to be a *CBI-time-changed Lévy process* (CBITCL) if

(i) $X = (X_t)_{t\geqslant 0}$ is a CBI process, and
(ii) $Z = L_Y = (L_{Y_t})_{t\geqslant 0}$, where $L = (L_t)_{t\geqslant 0}$ is a Lévy process independent of X and $Y = (Y_t)_{t\geqslant 0}$ denotes the process defined by $Y_t := \int_0^t X_s\,\mathrm{d}s$, for all $t \geqslant 0$.

The Lévy exponent Ξ of L admits the Lévy-Khintchine representation

$$\Xi(u) = b_Z u + \frac{1}{2}\sigma_Z^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{zu} - 1 - zu\mathbf{1}_{\{|z|<1\}}\big)\gamma_Z(\mathrm{d}z), \qquad \forall u \in \mathrm{i}\mathbb{R}, \tag{2.5}$$

where $(b_Z, \sigma_Z, \gamma_Z)$ is the Lévy triplet of L, with $b_Z \in \mathbb{R}$, $\sigma_Z \in \mathbb{R}$ and γ_Z a Lévy measure on $\mathbb{R}$. In the following, we write that a process (X, Z) is a $\mathrm{CBITCL}(X_0, \Psi, \Phi, \Xi)$ as a shorthand notation to denote the fact that (X, Z) is a CBI-time-changed Lévy process in the sense of Definition 2.2, where Ψ and Φ are the immigration and branching mechanisms of the CBI process X, respectively, and Ξ is the Lévy exponent of L.

2.1. **CBITCL processes as solutions to stochastic integral equations.** It is well-known that a CBI process can be characterized in two equivalent ways: as the solution to the stochastic integral equation of Dawson and Li (see [DL06]) and as the solution to a stochastic time change equation of Lamperti-type (see [CPGUB13]). It can be shown that these two representations of a CBI process are equivalent in a weak sense (see [Szu21, Theorem 2.12] for details). In the present context of CBITCL processes, Definition 2.2 is closer in spirit to the Lamperti-type representation. We now show that the Dawson-Li representation can be extended to CBITCL processes. To this effect, let us introduce the following objects, assumed to be mutually independent:

- two standard Brownian motions $B^1 = (B^1_t)_{t\geqslant 0}$ and $B^2 = (B^2_t)_{t\geqslant 0}$;
- a Poisson random measure $N_0(\mathrm{d}t, \mathrm{d}x)$ on $(0, +\infty)^2$ with compensator $\nu(\mathrm{d}x)\mathrm{d}t$;
- a Poisson random measure $N_1(\mathrm{d}t, \mathrm{d}u, \mathrm{d}x)$ on $(0, +\infty)^3$ with compensator $\pi(\mathrm{d}x)\,\mathrm{d}u\,\mathrm{d}t$;
- a Poisson random measure $N_2(\mathrm{d}t, \mathrm{d}u, \mathrm{d}x)$ on $(0, +\infty)^2 \times \mathbb{R}$ with compensator $\gamma_Z(\mathrm{d}x)\,\mathrm{d}u\,\mathrm{d}t$.

In the following, we shall always use the tilde notation to denote compensated random measures. For $X_0 \in \mathbb{R}_+$, let us consider the following two-dimensional stochastic integral equation:

$$
\begin{aligned}
X_t &= X_0 + \int_0^t (\beta - b\,X_s)\,\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\,\mathrm{d}B^1_s \\
&\quad + \int_0^t \int_0^{+\infty} x\,N_0(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_0^{X_{s-}} \int_0^{+\infty} x\,\widetilde{N}_1(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
Z_t &= b_Z \int_0^t X_s\,\mathrm{d}s + \sigma_Z \int_0^t \sqrt{X_s}\,\mathrm{d}B^2_s + \int_0^t \int_0^{X_{s-}} \int_{|x|\geqslant 1} x\,N_2(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) \\
&\quad + \int_0^t \int_0^{X_{s-}} \int_{|x|<1} x\,\widetilde{N}_2(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x).
\end{aligned} \tag{2.7}
$$

On a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and supporting the processes introduced above, there exists a unique strong solution (X, Z) to (2.6)-(2.7). Indeed, by [DL06, Theorems 5.1 and 5.2], there exists a unique strong solution X to (2.6), which corresponds to the Dawson-Li representation of a CBI process. Since the right-hand side of (2.7) does depend only on the process X, this implies the existence of a unique strong solution Z to (2.7) as well.

The next theorem asserts that defining CBITCL processes as in Definition 2.2 is equivalent to defining them as solutions to the stochastic integral equations (2.6)-(2.7).

Theorem 2.3. *A process (X, Z) is a* $\mathrm{CBITCL}(X_0, \Psi, \Phi, \Xi)$ *if and only if it is a weak solution to the stochastic integral equations* (2.6)-(2.7).

Proof. By [Li20, Theorem 8.1], a non-negative càdlàg process $X = (X_t)_{t\geqslant 0}$ with initial value X_0 is a $\mathrm{CBI}(X_0, \Psi, \Phi)$ if and only if it is a weak solution to (2.6). Therefore, to prove the theorem, we can restrict our attention to the process Z. Suppose first that $Z = L_Y$, where L is a Lévy process independent of X and $Y = \int_0^{\cdot} X_s\,\mathrm{d}s$. By the Lévy-Itô decomposition of L, it holds that

$$Z_t = b_Z\,Y_t + \sigma_Z\,W_{Y_t} + \int_0^{Y_t}\int_{|x|\geqslant 1} x\,N(\mathrm{d}s,\mathrm{d}x) + \int_0^{Y_t}\int_{|x|<1} x\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x), \qquad \forall t\geqslant 0, \tag{2.8}$$

where $W = (W_t)_{t\geqslant 0}$ is a Brownian motion and $N(\mathrm{d}t,\mathrm{d}x)$ an independent Poisson random measure on $\mathbb{R}_+\times(0,+\infty)$ with compensator $\gamma_Z(\mathrm{d}x)\,\mathrm{d}t$. By the independence of X and L, there is no loss of generality in assuming that Y_t is a stopping time with respect to the filtration generated by W and N, for every $t\geqslant 0$. The process Y is therefore a change of time in the sense of [Jac79, Definition 10.1]. By [Jac79, Theorem 10.27], we can perform a change of time in the random measure $N(\mathrm{d}t,\mathrm{d}x)$, thus obtaining

$$\left.\begin{aligned}
\int_0^{Y_t}\int_{|x|\geqslant 1} x\,N(\mathrm{d}s,\mathrm{d}x) &= \int_0^{t}\int_{|x|\geqslant 1} x\,N(X_s\,\mathrm{d}s,\mathrm{d}x),\\
\int_0^{Y_t}\int_{|x|<1} x\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x) &= \int_0^{t}\int_{|x|<1} x\,\widetilde{N}(X_s\,\mathrm{d}s,\mathrm{d}x),
\end{aligned}\right\}\qquad \forall t\geqslant 0.$$

By [IW89, Theorem II.7.4], on a suitable extension of the probability space there exists a Poisson random measure $N_2(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x)$ with compensator $\gamma_Z(\mathrm{d}x)\,\mathrm{d}u\,\mathrm{d}t$ such that

$$\left.\begin{aligned}
\int_0^{t}\int_{|x|\geqslant 1} xN(X_s\,\mathrm{d}s,\mathrm{d}x) &= \int_0^{t}\int_0^{X_{s-}}\int_{|x|\geqslant 1} x\,N_2(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x)\\
\int_0^{t}\int_{|x|<1} x\,\widetilde{N}(X_s\,\mathrm{d}s,\mathrm{d}x) &= \int_0^{t}\int_0^{X_{s-}}\int_{|x|<1} x\,\widetilde{N}_2(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x),
\end{aligned}\right\}\qquad \forall t\geqslant 0.$$

Similarly, by [IW89, Theorem II.7.1'], on a suitable extension of the probability space there exists an independent Brownian motion $B^2 = (B^2_t)_{t\geqslant 0}$ such that $W_{Y_t} = \int_0^t \sqrt{X_s}\,\mathrm{d}B^2_s$, for all $t\geqslant 0$. We have thus shown that Z is a weak solution to (2.7).

Conversely, suppose that Z is a weak solution to (2.7). In order to show that $Z = L_Y$, where L is a Lévy process independent of X with exponent Ξ, we follow the proof of [Kal06, Theorem 3.2]. Observe first that the process Z is constant on each interval $[r,s]\subseteq\mathbb{R}_+$ such that $Y_r = Y_s$ a.s. Indeed, for any such interval, it necessarily holds that $X_u = 0$ a.s. for Lebesgue-a.e. $u\in[r,s]$, and, therefore, (2.7) immediately implies that Z is a.s. constant on $[r,s]$. Let $Y_\infty := \lim_{t\to+\infty} Y_t$, which is well-defined since the process Y is non-decreasing, and define the inverse time change $\tau_z := \inf\{t\geqslant 0 : Y_t > z\}$, for all $z\geqslant 0$. Define the time-changed process $L^1 = (L^1_z)_{z<Y_\infty}$ by $L^1_z := Z_{\tau_z}$, for all $z < Y_\infty$ (note that it may happen that $Y_\infty < +\infty$, since zero is an absorbing state for X when $\Psi\equiv 0$). By [Jac79, Lemma 10.14], Z is adapted to $(\tau_z)_{z\geqslant 0}$ and it holds that $Z_t = L^1_{Y_t}$, for all $t\geqslant 0$. Without loss of generality, we can assume that the stochastic basis already supports an independent Lévy process $L^2 = (L^2_z)_{z\geqslant 0}$ with Lévy triplet (b_Z,σ_Z,γ_Z). Define then the process $L = (L_z)_{z\geqslant 0}$ by

$$L_z := L^1_z\mathbf{1}_{\{z<Y_\infty\}} + (L^1_{Y_\infty} + L^2_{z-Y_\infty})\mathbf{1}_{\{z\geqslant Y_\infty\}}, \qquad \forall z\geqslant 0,$$

where $L^1_{Y_\infty} := \lim_{t\to+\infty} Z_t =: Z_\infty$ on $\{Y_\infty < +\infty\}$, which is well-defined since equation (2.7) implies that Z is a semimartingale up to infinity on $\{Y_\infty < +\infty\}$, see [CS05]. Clearly, it holds that $Z_t = L_{Y_t}$, for all $t\geqslant 0$. To complete the proof, it remains to show that L is a Lévy process independent of

X with Lévy triplet $(b_Z, \sigma_Z, \gamma_Z)$. To this effect, let $\mathcal{F}_\infty^X := \sigma(X_t; t \geqslant 0)$ and consider the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geqslant 0}$ defined by $\mathcal{G}_t := \bigcap_{u>t}(\mathcal{F}_u \vee \mathcal{F}_\infty^X)$, for all $t \geqslant 0$. Equation (2.7) together with the independence between X and B^2 and N_2 implies that Z has semimartingale characteristics $(b_Z Y, \sigma_Z^2 Y, \gamma_Z(\mathrm{d}z)Y)$ in the filtration $\mathbb{G}$ (i.e., Z is a semimartingale with $\mathcal{F}_\infty^X$-conditionally independent increments, according to [JS03, Definition II.6.2]). Moreover, by [Jac79, Theorem 10.16], the process L^1 is a semimartingale in the time-changed filtration $(\mathcal{G}_{\tau_z})_{z\geqslant 0}$ on the stochastic interval $[\![0, Y_\infty[\![$. Similarly as in the proof of [Kal06, Theorem 3.2], [Jac79, Theorems 10.17 and 10.27] together with the fact that $Y_{\tau_z} = z$, for all $z < Y_\infty$, imply that the semimartingale characteristics (B, A, C) of L^1 in $(\mathcal{G}_z)_{z\geqslant 0}$ are given by

$$B_z = b_Z\, Y_{\tau_z} = b_Z\, z, \qquad A_z = \sigma_Z^2\, Y_{\tau_z} = \sigma_Z^2\, z, \qquad C_z(\mathrm{d}x) = \gamma_Z(\mathrm{d}x)\, Y_{\tau_z} = \gamma_Z(\mathrm{d}x)\, z,$$

for all $z < Y_\infty$. Therefore, for every $z \geqslant 0$ and $u \in \mathbb{R}$, making use of the definition of L together with the dominated convergence theorem and [JS03, Theorem II.6.6], it holds that

$$\begin{aligned}
\mathbb{E}[e^{\mathrm{i}uL_z}|\mathcal{F}_\infty^X] &= \mathbf{1}_{\{z<Y_\infty\}}\mathbb{E}[e^{\mathrm{i}uL_z^1}|\mathcal{F}_\infty^X] + \mathbf{1}_{\{z\geqslant Y_\infty\}}\mathbb{E}[e^{\mathrm{i}uZ_\infty}|\mathcal{F}_\infty^X]\,\mathbb{E}[e^{\mathrm{i}uL^2_{z-Y_\infty}}|\mathcal{F}_\infty^X] \\
&= \mathbf{1}_{\{z<Y_\infty\}}e^{\Xi(\mathrm{i}u)z} + \mathbf{1}_{\{z\geqslant Y_\infty\}}\lim_{t\to+\infty}\mathbb{E}[e^{\mathrm{i}uZ_t}|\mathcal{F}_\infty^X]\,\mathbb{E}[e^{\mathrm{i}uL_t^2}]\big|_{t=z-Y_\infty} \\
&= \mathbf{1}_{\{z<Y_\infty\}}e^{\Xi(\mathrm{i}u)z} + \mathbf{1}_{\{z\geqslant Y_\infty\}}\lim_{t\to+\infty}e^{\Xi(\mathrm{i}u)Y_t}e^{\Xi(\mathrm{i}u)(z-Y_\infty)} \\
&= e^{\Xi(\mathrm{i}u)z},
\end{aligned}$$

where the Lévy exponent Ξ is associated to the triplet $(b_Z, \sigma_Z, \gamma_Z)$ as in (2.5). Since the right-hand side of the last identity is deterministic, this shows that L is Lévy process with exponent Ξ as well as its independence of X, thus completing the proof. □

In the rest of the paper, if a CBITCL process (X, Z) is directly defined as the unique strong solution to (2.6)-(2.7), we will say that (X, Z) is given through its *extended Dawson-Li representation*. We point out that the representation (2.6)-(2.7) is especially useful for the numerical simulation of CBITCL processes (compare with [FGS21b, Appendix B] in the case CBI processes).

Remark 2.4. One of the characteristic features of CBI processes is their self-exciting behavior. As can be deduced from equations (2.6)-(2.7), this behavior is inherited by the class of CBITCL processes. In particular, we can observe the following:

- The local martingale terms of the process X depend on the current level of the process itself. This generates self-excitation since large values of the process increase its volatility. In particular, a large jump of X increases the likelihood of further jumps of X.
- The volatility of Z is determined by X. Therefore, large values of X are associated to an increased volatility of both processes, thus generating volatility clustering effects in (X, Y).

As mentioned in the introduction, self-exciting and volatility clustering phenomena are often present in financial time series. Together with their analytical tractability, this makes CBITCL processes especially appropriate for financial modeling, as illustrated for instance in [FGS21a] in the context of foreign currency markets with stochastic volatility.

The next proposition characterizes CBITCL processes in terms of their semimartingale differential characteristics (see [Kal06] for additional information on this notion of characteristics).

Proposition 2.5. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$. *Then, (X,Z) is a semimartingale (in its own natural filtration) with differential characteristics $(\mathcal{B},\mathcal{A},\mathcal{C})$ relative to the truncation function $h(x)=x\mathbf{1}_{\{|x|<1\}}$ given by*

$$
\begin{aligned}
\mathcal{B}_t &= \begin{pmatrix} \beta+\int_0^1 z\,\nu(\mathrm{d}z) \\ 0 \end{pmatrix} + X_{t-}\begin{pmatrix} -b-\int_1^{+\infty} z\,\pi(\mathrm{d}z) \\ b_Z \end{pmatrix}, \qquad \mathcal{A}_t = X_{t-}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma_Z^2 \end{pmatrix}, \\
\mathcal{C}_t(\mathrm{d}x,\mathrm{d}z) &= \nu(\mathrm{d}x)\delta_0(\mathrm{d}z) + X_{t-}\big(\pi(\mathrm{d}x)\delta_0(\mathrm{d}z)+\delta_0(\mathrm{d}x)\gamma_Z(\mathrm{d}z)\big),
\end{aligned} \tag{2.9}
$$

for all $t\geqslant 0$. Conversely, if (X,Z) is a semimartingale with differential characteristics given as in (2.9)*, with $\beta\in\mathbb{R}_+$, $b\in\mathbb{R}$, $\sigma\in\mathbb{R}$ and ν and π Lévy measures on $(0,+\infty)$ such that $\int_0^1 z\,\nu(\mathrm{d}z)<+\infty$ and $\int_1^{+\infty} z\,\pi(\mathrm{d}z)<+\infty$, then it is a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$*, where Ψ, Φ, Ξ are determined by $(\beta,b,\sigma,\nu,\pi,b_Z,\sigma_Z,\gamma_Z)$ as in* (2.1), (2.2), (2.5)*, respectively.*

Proof. Let (X,Z) be a $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$. In view of Theorem 2.3, the process (X,Z) satisfies equations (2.6)-(2.7) on an extension $(\Omega',\mathcal{F}',\mathbb{F}',\mathbb{P}')$ (see [IW89, Definition II.7.1]). This implies that (X,Z) is a quasi-left-continuous semimartingale on $(\Omega',\mathcal{F}',\mathbb{F}',\mathbb{P}')$ and its differential characteristics as stated in (2.9) can be directly deduced from (2.6)-(2.7). In view of [Jac79, Remark 10.40], it follows that (X,Z) is also a semimartingale on $(\Omega,\mathcal{F},\mathbb{F}^{(X,Z)},\mathbb{P})$, where $\mathbb{F}^{(X,Z)}=(\mathcal{F}_t^{(X,Z)})_{t\geqslant 0}$ denotes the natural filtration of (X,Z), with the same differential characteristics. Conversely, if (X,Z) is a semimartingale, then its canonical representation is determined by its differential characteristics (see [JS03, Theorem II.2.34]). Suppose that the differential characteristics of (X,Z) are given as in (2.9). By considering the canonical representation of (X,Z) and arguing similarly as in the first part of the proof of Theorem 2.3, it is easy to see that (X,Z) satisfies (2.6)-(2.7) on a suitable extension of the original probability space. Therefore, the semimartingale (X,Z) is a weak solution to (2.6)-(2.7). By Theorem 2.3, it follows that (X,Z) is a $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$. □

2.2. **CBITCL processes as affine processes.** In this section, we regard CBITCL processes as affine processes in the sense of [DFS03]. More specifically, we connect CBITCL processes to affine stochastic volatility processes as considered in [KR11]. We first state the following proposition, which provides the Fourier-Laplace transform of the joint process (X,Y,Z), where $Y:=\int_0^{\cdot} X_s\,\mathrm{d}s$. Without loss of generality, we suppose that (X,Z) is given by its extended Dawson-Li representation on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ supporting the processes introduced in Section 2.1.

Proposition 2.6. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ *and consider the process (X,Y,Z), where $Y:=\int_0^{\cdot} X_s\,\mathrm{d}s$. Then, (X,Z) and (X,Y,Z) are affine processess on the state spaces $\mathbb{R}_+\times\mathbb{R}$ and $\mathbb{R}_+^2\times\mathbb{R}$, respectively, and it holds that*

$$
\mathbb{E}\big[e^{u_1X_T+u_2Y_T+u_3Z_T}|\mathcal{F}_t\big] = \exp\big(\mathcal{U}(T-t,u_1,u_2,u_3)+\mathcal{V}(T-t,u_1,u_2,u_3)X_t+u_2Y_t+u_3Z_t\big), \tag{2.10}
$$

for all $(u_1, u_2, u_3) \in \mathbb{C}_-^2 \times \mathrm{i}\mathbb{R}$ *and* $0 \leqslant t \leqslant T < +\infty$*, where the functions* $\mathcal{U}(\cdot, u_1, u_2, u_3) : \mathbb{R}_+ \to \mathbb{C}$ *and* $\mathcal{V}(\cdot, u_1, u_2, u_3) : \mathbb{R}_+ \to \mathbb{C}_-$ *are solutions to*

$$\mathcal{U}(t, u_1, u_2, u_3) = \int_0^t \Psi\big(\mathcal{V}(s, u_1, u_2, u_3)\big)\,\mathrm{d}s, \tag{2.11}$$

$$\frac{\partial \mathcal{V}}{\partial t}(t, u_1, u_2, u_3) = \Phi\big(\mathcal{V}(t, u_1, u_2, u_3)\big) + u_2 + \Xi(u_3), \qquad \mathcal{V}(0, u_1, u_2, u_3) = u_1, \tag{2.12}$$

where $\Psi : \mathbb{C}_- \to \mathbb{C}$ *and* $\Phi : \mathbb{C}_- \to \mathbb{C}$ *denote the analytic extensions to* $\mathbb{C}_-$ *of the corresponding functions given by* (2.1) *and* (2.2)*, respectively.*

Proof. By [DFS03, Corollary 2.10], the process X is an affine process, since it is a $\mathrm{CBI}(X_0, \Psi, \Phi)$ by definition. In view of [KR09, Theorem 4.10], the process (X, Y) is also an affine process, with functional characteristics $(\widehat{\Psi}, \widehat{\Phi})$ given by

$$\widehat{\Psi}(u_1, u_2) = \Psi(u_1), \qquad \widehat{\Phi}(u_1, u_2) = \Phi(u_1) + u_2, \qquad \forall (u_1, u_2) \in \mathbb{C}_-^2.$$

Since L is by definition a Lévy process independent of X, [KR09, Theorem 4.16] implies that (X, Y, Z) is an affine process on $\mathbb{R}_+^2 \times \mathbb{R}$, with functional characteristics $(\widetilde{\Psi}, \widetilde{\Phi})$ given by

$$\begin{aligned} \widetilde{\Psi}(u_1, u_2, u_3) &= \widehat{\Psi}\big(u_1, u_2 + \Xi(u_3)\big) = \Psi(u_1), \\ \widetilde{\Phi}(u_1, u_2, u_3) &= \widehat{\Phi}\big(u_1, u_2 + \Xi(u_3)\big) = \Phi(u_1) + u_2 + \Xi(u_3), \end{aligned}$$

for all $(u_1, u_2, u_3) \in \mathbb{C}_-^2 \times \mathrm{i}\mathbb{R}$. By [DFS03, Theorem 2.7], the Fourier-Laplace transform of (X, Y, Z) is given by (2.10) in terms of the solutions to the Riccati equations (2.11)-(2.12). Finally, the fact that (X, Z) is an affine process on $\mathbb{R}_+ \times \mathbb{R}$ follows from [KR09, Proposition 4.8]. □

The availability of the explicit characterization of the conditional Fourier-Laplace transform stated in Proposition 2.6 is of great usefulness for the application of CBITCL processes in finance. More specifically, many pricing applications require the computation of conditional expectations of the form (2.10), where Y typically plays the role of a discount factor.

In line with [KR11], we say that a process (X, Z) taking values in $\mathbb{R}_+ \times \mathbb{R}$ is an *affine stochastic volatility process* if it is an affine process and its Fourier-Laplace transform has the structure (2.10) (with $u_2 = 0$), for some functions $\mathcal{U}$ and $\mathcal{V}$. This terminology is explained by the fact that, in financial applications, the process Z usually plays the role of the log-price process of a risky asset, while X represents its instantaneous variance.[1] Proposition 2.6 directly implies that CBITCL processes are affine stochastic volatility processes. In the next result, we study the converse implication. We recall that, if (X, Z) is an affine process on $\mathbb{R}_+ \times \mathbb{R}$, then the compensator of its jump measure is of the form $\nu^{(X,Z)}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) = (m_0(\mathrm{d}x, \mathrm{d}z) + X_{t-} m_1(\mathrm{d}x, \mathrm{d}z))\mathrm{d}t$, where m_0 and m_1 are Lévy measures on $\mathbb{R}_+ \times \mathbb{R}$ satisfying $\int_{(\mathbb{R}_+ \times \mathbb{R}) \setminus \{0\}} (1 \wedge (x + z^2)) m_0(\mathrm{d}x, \mathrm{d}z) < +\infty$ (see, e.g., [KR11, Section 2.1]).

[1]We point out that the notion of affine stochastic volatility process that we adopt in this paper is more general than [KR11, Definition 2.8]. Indeed, we do not require that $\exp(Z)$ is a martingale nor impose a non-degeneracy condition on Ξ (corresponding to conditions (A3)-(A4) in [KR11]). An explicit characterization of the martingale property of $\exp(Z)$ will be given below in Corollary 4.3.

Proposition 2.7. *Let (X,Z) be an affine stochastic volatility process. Then, (X,Z) is a CBITCL process if and only if the following three conditions hold:*

(i) $[X,Z]=0$ *(up to an evanescent set);*
(ii) $\int_{(1,+\infty)\times\mathbb{R}} x\, m_1(\mathrm{d}x,\mathrm{d}z)<+\infty$;
(iii) *Z is a.s. constant on every interval $[r,s]\subseteq\mathbb{R}_+$ such that $\int_r^s X_{u-}\mathrm{d}u=0$ a.s.*

Proof. If (X,Z) is an affine stochastic volatility process, then by [DFS03, Theorem 2.12] it is a semimartingale with differential characteristics $(\mathcal{B}^{(X,Z)},\mathcal{A}^{(X,Z)},\mathcal{C}^{(X,Z)})$ given by

$$\mathcal{B}^{(X,Z)}_t=\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}+X_{t-}\begin{pmatrix}b_1\\ b_2\end{pmatrix},\qquad \mathcal{A}^{(X,Z)}_t=\begin{pmatrix}0&0\\ 0&\alpha_2\end{pmatrix}+X_{t-}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix},$$

$$\mathcal{C}^{(X,Z)}_t(\mathrm{d}x,\mathrm{d}z)=m_0(\mathrm{d}x,\mathrm{d}z)+X_{t-}m_1(\mathrm{d}x,\mathrm{d}z),$$

where $(\beta_1,\beta_2)\in\mathbb{R}^+\times\mathbb{R}$, $(b_1,b_2)\in\mathbb{R}^2$, $\alpha_2\in\mathbb{R}_+$ and the matrix $\left(\begin{smallmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{smallmatrix}\right)$ is symmetric positive semi-definite. If condition (i) holds, then $\langle X^c,Z^c\rangle=0$, which implies that $a_{12}=a_{21}=0$. Furthermore, always as a consequence of condition (i), the processes X and Z do not have common jumps, which implies that the Lévy measures $m_i(\mathrm{d}x,\mathrm{d}z)$ have the following structure:

$$m_i(\mathrm{d}x,\mathrm{d}z)=m_i^X(\mathrm{d}x)\delta_0(\mathrm{d}z)+\delta_0(\mathrm{d}x)m_i^Z(\mathrm{d}z),\qquad \text{for } i=0,1.$$

Condition (ii) therefore implies that $\int_1^{+\infty}x\,m_1^X(\mathrm{d}x)<+\infty$. Moreover, if condition (iii) holds, then the process Z is constant on all intervals $[r,s]\subseteq\mathbb{R}_+$ such that $X_{u-}=0$ a.s. for a.e. $u\in[r,s]$, which in turn implies that $\beta_2=0$, $\alpha_2=0$ and $m_0^Z=0$. Recalling that $\int_{(0,1)}x\,m_0^X(\mathrm{d}x)<+\infty$, since (X,Z) is assumed to be an affine stochastic volatility process, we have thus shown that the differential characteristics $(\mathcal{B}^{(X,Z)},\mathcal{A}^{(X,Z)},\mathcal{C}^{(X,Z)})$ can be written in the form (2.9). By Proposition 2.5, it follows that (X,Z) is a CBITCL process. The converse implication is straightforward. □

3. Finiteness of exponential moments and asymptotic behavior

In this section, we study the existence of (discounted) exponential moments of CBITCL processes and their asymptotic behavior. These properties are intimately connected to the maximal lifetime of the solutions to the Riccati equations, explicitly characterized in Theorem 3.4 below.

3.1. Finiteness of exponential moments.

In financial applications of CBITCL processes, the finiteness of (discounted) exponential moments typically represents an indispensable requirement (see, e.g., [FGS21a, FGS21b]). In order to make use of some results of [KRM15] for general affine processes, let us define the convex set $\mathcal{D}_X$ as follows:

$$\mathcal{D}_X:=\left\{u\in\mathbb{R}:\int_1^{+\infty}e^{uz}\,(\nu+\pi)(\mathrm{d}z)<+\infty\right\}. \tag{3.1}$$

The set $\mathcal{D}_X$ represents the effective domain of the functions Ψ and Φ, which can be extended as finite-valued convex functions on $\mathcal{D}_X$. Similarly, let us define

$$\mathcal{D}_Z:=\left\{u\in\mathbb{R}:\int_{|z|\geqslant 1}e^{uz}\,\gamma_Z(\mathrm{d}z)<+\infty\right\}, \tag{3.2}$$

which represents the effective domain of the Lévy exponent Ξ when restricted to real arguments. By standard results on exponential moments of Lévy measures (see, e.g., [Sat99, Theorem 25.17]), the Lévy exponent Ξ can be extended as a finite-valued convex function on $\mathcal{D}_Z$.

Adapting [KRM15, Definition 2.10] to the present setup, we introduce the following definition.

Definition 3.1. For $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$, we say that $(\mathcal{U}(\cdot,u_1,u_2,u_3),\mathcal{V}(\cdot,u_1,u_2,u_3))$ is a solution to the *extended Riccati system* if it solves the following system:

$$\mathcal{U}(t,u_1,u_2,u_3)=\int_0^t\Psi\big(\mathcal{V}(s,u_1,u_2,u_3)\big)\mathrm{d}s, \tag{3.3}$$

$$\frac{\partial\mathcal{V}}{\partial t}(t,u_1,u_2,u_3)=\Phi\big(\mathcal{V}(t,u_1,u_2,u_3)\big)+u_2+\Xi(u_3),\qquad \mathcal{V}(0,u_1,u_2,u_3)=u_1, \tag{3.4}$$

up to a time $\mathrm{T}^{(u_1,u_2,u_3)}\in[0,+\infty]$, with $\mathrm{T}^{(u_1,u_2,u_3)}$ denoting the joint lifetime of the functions $\mathcal{U}(\cdot,u_1,u_2,u_3):[0,\mathrm{T}^{(u_1,u_2,u_3)})\to\mathbb{R}$ and $\mathcal{V}(\cdot,u_1,u_2,u_3):[0,\mathrm{T}^{(u_1,u_2,u_3)})\to\mathcal{D}_X$.

Definition 3.1 extends the Riccati system (2.11)-(2.12) by allowing for the possibility of explosion in finite time. In some situations, which will be precisely characterized in Theorem 3.4 below, the lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$ turns out to be infinite, in which case (3.3)-(3.4) admit a global solution.

It is well known that the branching mechanism function Φ is locally Lipschitz continuous on the interior $\mathcal{D}_X^\circ$ of the set $\mathcal{D}_X$, but it may fail to be so at the boundary $\partial\mathcal{D}_X$. Therefore, a solution $\mathcal{V}(\cdot,u_1,u_2,u_3)$ to the ODE (3.4) may not be unique when it starts at $\partial\mathcal{D}_X$ or reaches it at a later time. This observation motivates the introduction of the concept of minimal solution in [KRM15]. In our setup, for the sake of tractability, we prefer to impose an additional mild technical assumption which guarantees uniqueness of the (local) solution to the ODE (3.4) for every $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$. To this effect, we define

$$\psi:=\sup\{x\geqslant 0:\Psi(x)<+\infty\}\qquad\text{and}\qquad \phi:=\sup\{x\geqslant 0:\Phi(x)<+\infty\}. \tag{3.5}$$

Since $\mathcal{D}_X$ is a convex set containing $\mathbb{R}_-$, it can be written as $\mathcal{D}_X=(-\infty,\psi\wedge\phi)$, or $(-\infty,\psi\wedge\phi]$ when $\Psi(\psi\wedge\phi)\vee\Phi(\psi\wedge\phi)<+\infty$ (which is equivalent to $\int_1^{+\infty}e^{(\psi\wedge\phi)z}(\nu+\pi)(\mathrm{d}z)<+\infty$). The function Φ is convex and, hence, differentiable almost everywhere on $\mathcal{D}_X^\circ$, with derivative given by

$$\Phi'(x)=-b+\sigma^2x+\int_0^{+\infty}z(e^{xz}-1)\pi(\mathrm{d}z),\qquad \forall x\in\mathcal{D}_X^\circ. \tag{3.6}$$

If $\psi\wedge\phi=+\infty$, then $\mathcal{D}_X=\mathbb{R}$, in which case $\Phi\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$ obviously holds. If $\psi\wedge\phi<+\infty$, the following assumption ensures that $\Phi'(\psi\wedge\phi)<+\infty$, which in turn implies that $\Phi\in\mathcal{C}^1(\mathcal{D}_X,\mathbb{R})$.

Assumption 3.2. If $\psi\wedge\phi<+\infty$, then $\int_1^{+\infty}ze^{(\psi\wedge\phi)z}\pi(\mathrm{d}z)<+\infty$.

The validity of Assumption 3.2 can be easily checked for each specific type of CBITCL process. Moreover, as illustrated in Section 5, it is satisfied by a large class of models. Under Assumption 3.2, there exists a unique solution $(\mathcal{U}(\cdot,u_1,u_2,u_3),\mathcal{V}(\cdot,u_1,u_2,u_3))$ to the extended Riccati system (3.3)-(3.4) up to time $\mathrm{T}^{(u_1,u_2,u_3)}$, for all $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$. This enables us to state the following result, which extends Proposition 2.6 and follows directly from [KRM15, Theorem 2.14].

Lemma 3.3. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ *and* $Y := \int_0^{\cdot} X_s\,\mathrm{d}s$. *Suppose that Assumption 3.2 holds. Then, for all $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$ and $0\leqslant t\leqslant T<\mathrm{T}^{(u_1,u_2,u_3)}$, it holds that*

$$\mathbb{E}\big[e^{u_1X_T+u_2Y_T+u_3Z_T}\big|\mathcal{F}_t\big]=\exp\big(\mathcal{U}(T-t,u_1,u_2,u_3)+\mathcal{V}(T-t,u_1,u_2,u_3)X_t+u_2Y_t+u_3Z_t\big), \tag{3.7}$$

where $(\mathcal{U}(\cdot,u_1,u_2,u_3),\mathcal{V}(\cdot,u_1,u_2,u_3))$ is the unique solution to the extended Riccati system (3.3)-(3.4) *defined up to time* $\mathrm{T}^{(u_1,u_2,u_3)}$.

The lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$ is closely connected to the finiteness of exponential moments of the process (X,Y,Z). Indeed, in view of [KRM15, Proposition 3.3], it holds that

$$\mathrm{T}^{(u_1,u_2,u_3)}=\sup\big\{t\geqslant 0:\mathbb{E}[e^{u_1X_t+u_2Y_t+u_3Z_t}]<+\infty\big\}. \tag{3.8}$$

The next theorem is the main result of this section and provides an explicit formula for $\mathrm{T}^{(u_1,u_2,u_3)}$, for every $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$. The proof is based on techniques similar to [KR11, Theorem 4.1], which covers the case of affine stochastic volatility processes. However, our theorem is specific to CBITCL processes and avoids the additional assumptions of [KR11, Theorem 4.1]. In particular, it allows for CBI processes X with an arbitrary (not necessarily strictly subcritical, i.e. $b>0$) branching mechanism Φ. For $(u_2,u_3)\in\mathbb{R}\times\mathcal{D}_Z$, we introduce the following notation:

$$\mathcal{S}^{(u_2,u_3)}:=\big\{x\in\mathcal{D}_X:\Phi(x)+u_2+\Xi(u_3)\leqslant 0\big\}\qquad\text{and}\qquad\chi^{(u_2,u_3)}:=\sup\mathcal{S}^{(u_2,u_3)}\in[-\infty,\psi\wedge\phi],$$

with the convention $\chi^{(u_2,u_3)}=-\infty$ if the set $\mathcal{S}^{(u_2,u_3)}$ is empty.

Theorem 3.4. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ *and suppose that Assumption 3.2 holds. Then, for all $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$, the lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$ is given as follows:*

(i) *if $u_1\leqslant\chi^{(u_2,u_3)}$, then $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$;*

(ii) *if $u_1>\chi^{(u_2,u_3)}$, then*

$$\mathrm{T}^{(u_1,u_2,u_3)}=\int_{u_1}^{\psi\wedge\phi}\frac{\mathrm{d}x}{\Phi(x)+u_2+\Xi(u_3)}. \tag{3.9}$$

Proof. For simplicity of notation, for fixed $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$, we denote by $\mathrm{T}^{\mathcal{U}}$ and $\mathrm{T}^{\mathcal{V}}$ the lifetimes of the functions $\mathcal{U}(\cdot,u_1,u_2,u_3)$ and $\mathcal{V}(\cdot,u_1,u_2,u_3)$ solutions to (3.3)-(3.4), respectively. Making use of this notation, the lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$ can be decomposed as $\mathrm{T}^{(u_1,u_2,u_3)}=\mathrm{T}^{\mathcal{U}}\wedge\mathrm{T}^{\mathcal{V}}$. Always for simplicity of notation, we omit to write the superscript (u_2,u_3) in $\chi^{(u_2,u_3)}$ and $\mathcal{S}^{(u_2,u_3)}$. Let us first consider the case $u_1\leqslant\chi$. If $\Phi(u_1)+u_2+\Xi(u_3)=0$, then the constant function $\mathcal{V}(\cdot,u_1,u_2,u_3)\equiv u_1$ is the unique solution to (3.3), so that $\mathrm{T}^{\mathcal{V}}=+\infty$. Since $u_1\in\mathcal{D}_X$, we also have $\mathrm{T}^{\mathcal{U}}=+\infty$, which implies that $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$. Suppose now that $\Phi(u_1)+u_2+\Xi(u_3)<0$. Let us define $\xi:=\inf\mathcal{S}$, with $\xi=+\infty$ if the set $\mathcal{S}$ is empty. Note that, in the present case, $\xi<u_1$ and, if $\xi>-\infty$, then $\Phi(\xi)+u_2+\Xi(u_3)=0$. By convexity of Φ, it holds that $\Phi(x)+u_2+\Xi(u_3)<0$, for all $x\in(\xi,u_1]$. Therefore, equation (3.3) implies that the function $\mathcal{V}(\cdot,u_1,u_2,u_3)$ is strictly decreasing and we can write

$$t=-\int_{\mathcal{V}(t,u_1,u_2,u_3)}^{u_1}\frac{\mathrm{d}x}{\Phi(x)+u_2+\Xi(u_3)},\qquad\text{for all } t\geqslant 0.$$

Letting $t \to +\infty$ on both sides of this identity, we obtain that $\mathcal{V}(t, u_1, u_2, u_3) \to \xi$ as $t \to +\infty$, while $\xi < \mathcal{V}(t, u_1, u_2, u_3) \leqslant u_1$ for all $t \geqslant 0$. This shows that $\mathrm{T}^{\mathcal{V}} = +\infty$. Moreover, making use of the structure of Ψ, we obtain $-\infty < \mathcal{U}(t, u_1, u_2, u_3) \leqslant t\Psi(u_1)$ for all $t \geqslant 0$, implying that $\mathrm{T}^{\mathcal{U}} = +\infty$. We have thus shown that $\mathrm{T}^{(u_1,u_2,u_3)} = +\infty$. If $u_1 \leqslant \chi$ and $\Phi(u_1) + u_2 + \Xi(u_3) > 0$, then we necessarily have $u_1 < \xi \in \mathcal{D}_X$ and $\Phi(x) + u_2 + \Xi(u_3) > 0$, for all $x \in [u_1, \xi)$. Arguing similarly as above, this implies that $u_1 \leqslant \mathcal{V}(t, u_1, u_2, u_3) < \xi$ for all $t \geqslant 0$, which in turn leads to $\mathrm{T}^{(u_1,u_2,u_3)} = +\infty$.

Let us now consider the case $u_1 > \chi$ (which includes the case $\chi = -\infty$). By using the convexity of Φ, we have $\Phi(u_1) + u_2 + \Xi(u_3) > 0$, implying that the function $\mathcal{V}(\cdot, u_1, u_2, u_3)$ is strictly increasing with values in $[u_1, \phi]$. The function $\mathcal{V}(\cdot, u_1, u_2, u_3)$ can be extended to a maximal interval of existence $[0, \mathrm{T}^*)$ such that one of the following two cases occurs:

(i) $\mathrm{T}^* = +\infty$;
(ii) $\mathrm{T}^* < +\infty$ and $\lim_{t \to \mathrm{T}^*} \mathcal{V}(t, u_1, u_2, u_3) = \phi$.

In case (i), since $\mathcal{V}(\cdot, u_1, u_2, u_3)$ is strictly increasing, the limit $l := \lim_{t \to +\infty} \mathcal{V}(t, u_1, u_2, u_3)$ is well-defined with values in $(u_1, \phi] \cup \{+\infty\}$. Suppose that $l < +\infty$, i.e., the line $y = l$ is a horizontal asymptote for $\mathcal{V}(\cdot, u_1, u_2, u_3)$ as $t \to +\infty$. This implies that $\frac{\partial \mathcal{V}}{\partial t}(t, u_1, u_2, u_3) \to 0$ as $t \to +\infty$. Letting $t \to +\infty$ on both sides of (3.3), this yields $\Phi(l) + u_2 + \Xi(u_3) = 0$, contradicting the fact that $\Phi(x) + u_2 + \Xi(u_3) > 0$ for all $x > \chi$. Therefore, the limit l must necessarily be infinite, which can only happen if $\phi = +\infty$ and, in this case, $\lim_{t \to +\infty} \mathcal{V}(t, u_1, u_2, u_3) = \phi$, analogously to case (ii). In case (ii), let $(\mathrm{T}_n)_{n \in \mathbb{N}}$ be an increasing sequence such that $\mathrm{T}_n \to \mathrm{T}^*$ as $n \to +\infty$. Similarly as above, making use of equation (3.3), we can write

$$\mathrm{T}_n = \int_{u_1}^{\mathcal{V}(\mathrm{T}_n, u_1, u_2, u_3)} \frac{\mathrm{d}x}{\Phi(x) + u_2 + \Xi(u_3)}, \qquad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Letting $n \to +\infty$ on both sides of (3.10) yields

$$\mathrm{T}^* = \int_{u_1}^{\phi} \frac{\mathrm{d}x}{\Phi(x) + u_2 + \Xi(u_3)},$$

which represents the lifetime $\mathrm{T}^{\mathcal{V}}$ of the function $\mathcal{V}(\cdot, u_1, u_2, u_3)$. To complete the proof, it suffices to observe that, if $\phi \leqslant \psi$, then $\int_0^t \Psi(\mathcal{V}(s, u_1, u_2, u_3))\mathrm{d}s$ is always finite whenever $\mathcal{V}(t, u_1, u_2, u_3)$ is finite, so that $\mathrm{T}^{(u_1,u_2,u_3)} = \mathrm{T}^{\mathcal{V}}$. If $\phi > \psi$, then $\mathrm{T}^{(u_1,u_2,u_3)} = \inf\{t \in \mathbb{R}_+ : \mathcal{V}(t, u_1, u_2, u_3) = \psi\}$. Without loss of generality, we can assume that there exists $n \in \mathbb{N}$ such that $\mathrm{T}_n = \mathrm{T}^{(u_1,u_2,u_3)}$ and $\mathcal{V}(\mathrm{T}_n, u_1, u_2, u_3) = \psi$. Inserting this into equation (3.10) and combining the two cases $\phi \leqslant \psi$ and $\psi < \phi$ gives formula (3.9). □

Remark 3.5. (1) The result of Theorem 3.4 is of great importance in financial applications. Indeed, many derivatives can be efficiently priced by resorting to Fourier representations of their payoffs and exploiting the knowledge of the conditional characteristic function of the joint process (X, Y, Z) (see, e.g., [Fil09, Section 10.3]). This requires an extension of (3.7) to the complex domain. As shown in [KRM15], the feasibility of this extension crucially depends on the fact that the lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$ is greater than the maturity of the derivative to be priced, for suitable $(u_1, u_2, u_3) \in \mathcal{D}_X \times \mathbb{R} \times \mathcal{D}_Z$.

(2) In asset pricing models, the finiteness of the lifetime $\mathrm{T}^{(u_1,u_2,u_3)}$, for $(u_1,u_2,u_3)\in\mathcal{D}_X\times\mathbb{R}\times\mathcal{D}_Z$, is intimately related to the shape of the implied volatility smile at extreme strikes, see [Lee04] and [KR11, Section 5.1]. Therefore, the availability of an explicit description of $\mathrm{T}^{(u_1,u_2,u_3)}$ in Theorem 3.4 permits to characterize the tail behavior of the implied volatility smile in financial models driven by CTBICL processes, as will be illustrated in the examples considered in Section 5.

(3) In the case of classical CBI processes, a characterization of the time of explosion of exponential moments has been obtained in [FGS21b, Theorem 2.7]. The latter result can be recovered as a special case of Theorem 3.4 by taking $u_2\in\mathbb{R}_-$ and $u_3=0$.

The following corollary provides a necessary and sufficient condition for the finiteness of exponential moments for all $u_1\in\mathcal{D}_X$, for fixed but arbitrary $(u_2,u_3)\in\mathbb{R}\times\mathcal{D}_Z$. Whenever $\psi\wedge\phi=+\infty$, we denote $\Phi(\psi\wedge\phi):=\lim_{u\to+\infty}\Phi(u)$, which is well-defined with values in $\{-\infty,+\infty\}$.

Corollary 3.6. *Let (X,Z) be a* CBITCL(X_0,Ψ,Φ,Ξ) *and suppose that Assumption 3.2 holds. Assume furthermore that, if $\psi<+\infty$ and $\psi\leqslant\phi$, then $\int_1^{+\infty}e^{\psi z}\nu(\mathrm{d}z)<+\infty$. Let $(u_2,u_3)\in\mathbb{R}\times\mathcal{D}_Z$. Then, $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$ holds for all $u_1\in\mathcal{D}_X$ if and only if $\Phi(\psi\wedge\phi)+u_2+\Xi(u_3)\leqslant 0$.*

Proof. Note first that, under the present assumptions, $\mathcal{D}_X=(-\infty,\psi\wedge\phi]$ whenever $\psi\wedge\phi<+\infty$. Suppose first that $\Phi(\psi\wedge\phi)+u_2+\Xi(u_3)\leqslant 0$. In this case, $\chi^{(u_2,u_3)}=\psi\wedge\phi$ (in both cases $\psi\wedge\phi<+\infty$ and $\psi\wedge\phi=+\infty$). By Theorem 3.4, it follows that $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$ for all $u_1\in\mathcal{D}_X$. Conversely, suppose that $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$ for all $u_1\in\mathcal{D}_X$. If $\psi\wedge\phi<+\infty$, then $\psi\wedge\phi\in\mathcal{D}_X$ and, therefore, $\mathrm{T}^{(\psi\wedge\phi,u_2,u_3)}=+\infty$. Arguing by contradiction, suppose that $\Phi(\psi\wedge\phi)+u_2+\Xi(u_3)>0$. In that case, by the properties of the function Φ, we would have $\chi^{(u_2,u_3)}<\psi\wedge\phi$. But then formula (3.9) would imply that $\mathrm{T}^{(\psi\wedge\phi,u_2,u_3)}=0$, thus leading to a contradiction. On the other hand, if $\psi\wedge\phi=+\infty$, then $\mathrm{T}^{(u_1,u_2,u_3)}=+\infty$ for all $u_1\in\mathcal{D}_X=\mathbb{R}$. Arguing by contradiction, suppose that $\Phi(\psi\wedge\phi)+u_2+\Xi(u_3)>0$. In this case, there exists $M>0$ such that $\Phi(x)+u_2+\Xi(u_3)>0$ for all $x\geqslant M$. In turn, this yields $\chi^{(u_2,u_3)}<M$, which by Theorem 3.4 would imply that $\mathrm{T}^{(M,u_2,u_3)}<+\infty$, thus leading to a contradiction. ☐

3.2. **Asymptotic behavior of CBITCL processes.** In this section, we study the long-term behavior of a CBITCL process (X,Z). If the CBI process X is strictly subcritical (i.e., $b>0$), then it converges in law to a unique stationary distribution η, with Laplace transform L_η given by

$$L_\eta(\lambda)=\exp\left(\int_\lambda^0\frac{\Psi(x)}{\Phi(x)}\mathrm{d}x\right),\qquad\text{for all }\lambda\leqslant 0,$$

see [Li20, Theorem 10.4]. In general, the time-changed Lévy process Z does not admit an ergodic distribution. However, similarly as in [KR11, Section 3.2] but under weaker technical assumptions, we can prove that the rescaled cumulant generating function $\frac{1}{t}\log\mathbb{E}[e^{uZ_t}]$ converges to a limit that corresponds to the cumulant generating function of an infinitely divisible random variable.

Let (X,Z) be a CBITCL(X_0,Ψ,Φ,Ξ) and suppose that Assumption 3.2 is satisfied. We recall from Lemma 3.3 that

$$\mathbb{E}\big[e^{uZ_t}\big]=\exp\big(\mathcal{U}(t,0,u)+\mathcal{V}(t,0,u)X_0\big),\qquad\text{for all }u\in\mathcal{D}_Z,$$

where $(\mathcal{U}(\cdot,0,u),\mathcal{V}(\cdot,0,u))$ is the unique solution to the following extended Riccati system:

$$\mathcal{U}(t,0,u) = \int_0^t \Psi\big(\mathcal{V}(s,0,u)\big)\mathrm{d}s, \tag{3.11}$$

$$\frac{\partial \mathcal{V}}{\partial t}(t,0,u) = \Phi\big(\mathcal{V}(t,0,u)\big) + \Xi(u), \qquad \mathcal{V}(0,0,u) = 0, \tag{3.12}$$

for all $0 \leqslant t < \mathrm{T}(u)$, where $\mathrm{T}(u) := \mathrm{T}^{(0,0,u)}$ denotes the maximal lifetime of $(\mathcal{U}(\cdot,0,u),\mathcal{V}(\cdot,0,u))$. The next corollary follows directly from Theorem 3.4 and provides an explicit description of $\mathrm{T}(u)$. In the following, for simplicity of notation, we denote $\chi(u) := \chi^{(0,u)}$.

Corollary 3.7. *Let (X,Z) be a $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ and suppose that Assumption 3.2 holds. Then, for all $u \in \mathcal{D}_Z$, the lifetime $\mathrm{T}(u)$ is given as follows:*

(i) *if $\chi(u) \geqslant 0$, then $\mathrm{T}(u) = +\infty$;*
(ii) *if $\chi(u) < 0$, then*

$$\mathrm{T}(u) = \int_0^{\psi\wedge\phi} \frac{\mathrm{d}x}{\Phi(x)+\Xi(u)}. \tag{3.13}$$

By Corollary 3.7, $\mathrm{T}(u) = +\infty$ for all $u \in \mathcal{D}_Z$ such that $\chi(u) \geqslant 0$, meaning that the functions $\mathcal{U}(t,0,u)$ and $\mathcal{V}(t,0,u)$ are finite for all $t \geqslant 0$. The study of the asymptotic behavior of $\mathbb{E}[e^{uZ_t}]$ therefore requires analysing the asymptotic properties of the functions $\mathcal{U}(\cdot,0,u)$ and $\mathcal{V}(\cdot,0,u)$ for all $u \in \mathcal{X} := \{u \in \mathcal{D}_Z : \chi(u) \geqslant 0\}$. This is the content of the next proposition, which specializes [KR11, Theorem 3.4] to the case of CBITCL processes. More precisely, by relying on Corollary 3.7 and exploiting the specific structure of a CBITCL process, we obtain an asymptotic result which only requires the CBI process X to be strictly subcritical, besides the technical requirement of Assumption 3.2, thereby weakening the assumptions of [KR11].

Proposition 3.8. *Let (X,Z) be a $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ with $b>0$ and suppose that Assumption 3.2 holds. For all $u \in \mathcal{X}$, define $\xi(u) := \inf\{x \in \mathcal{D}_X : \Phi(x)+\Xi(u) \leqslant 0\}$. Then, it holds that*

$$\lim_{t\to+\infty} \mathcal{V}(t,0,u) = \xi(u) \qquad \textit{and} \qquad \lim_{t\to+\infty} \frac{1}{t}\mathcal{U}(t,0,u) = \Psi\big(\xi(u)\big), \qquad \textit{for every } u \in \mathcal{X}.$$

Proof. The branching mechanism Φ satisfies $\Phi(0)=0$ and is continuous and convex. Moreover, if $b>0$, then $\lim_{x\to-\infty}\Phi(x) = +\infty$. Making use of these properties, for each $u \in \mathcal{X}$, the fact that $\{x \in \mathcal{D}_X : \Phi(x)+\Xi(u) \leqslant 0\} \neq \varnothing$ implies that the quantity $\xi(u)$ is finite-valued and belongs to $\mathcal{D}_X$. In addition, by continuity of Φ, it holds that

$$\Phi\big(\xi(u)\big) + \Xi(u) = 0, \qquad \text{for all } u \in \mathcal{X}. \tag{3.14}$$

For $u \in \mathcal{X}$, let us consider separately the three cases $\xi(u)=0$, $\xi(u)<0$, and $\xi(u)>0$. If $\xi(u)=0$, then $\Xi(u)=0$ by (3.14) and the function $\mathcal{V}(\cdot,0,u)\equiv 0$ is the unique solution to (3.12), so that $\mathcal{V}(t,0,u)\to\xi(u)$ as $t\to+\infty$ trivially holds. If $\xi(u)<0$, then we necessarily have that $\Xi(u)<0$. By convexity of Φ, it holds that $\Phi(x)+\Xi(u)<0$ for all $x \in (\xi(u),0]$. Equation (3.12) therefore implies that the function $\mathcal{V}(\cdot,0,u)$ is strictly decreasing and satisfies $\xi(u) < \mathcal{V}(t,0,u) \leqslant 0$ and

$$t = \int_{\mathcal{V}(t,0,u)}^0 \frac{-\mathrm{d}x}{\Phi(x)+\Xi(u)}, \qquad \text{for all } t \geqslant 0.$$

Letting $t \to +\infty$ on both sides of this identity and recalling (3.14), we obtain that $\mathcal{V}(t,0,u) \to \xi(u)$ as $t \to +\infty$. If $\xi(u) > 0$, then we necessarily have that $\Phi(x) + \Xi(u) > 0$ for all $x \in [0, \xi(u))$, which implies that the function $\mathcal{V}(\cdot, 0, u)$ is strictly increasing and satisfies $0 \leqslant \mathcal{V}(t,0,u) < \xi(u)$ and

$$t = \int_{\mathcal{V}(t,0,u)}^{0} \frac{-\mathrm{d}x}{\Phi(x) + \Xi(u)}, \qquad \text{for all } t \geqslant 0.$$

Analogously to the preceding case, letting $t \to +\infty$ on both sides of the latter identity and making use of (3.14), we obtain that $\mathcal{V}(t,0,u) \to \xi(u)$ as $t \to +\infty$. Finally, for all $u \in \mathcal{X}$, the convergence of the function $t \mapsto (1/t)\,\mathcal{U}(t,0,u)$ directly follows from equation (3.11):

$$\frac{1}{t}\mathcal{U}(t,0,u) = \frac{1}{t}\int_0^t \Psi\big(\mathcal{V}(s,0,u)\big)\,\mathrm{d}s \underset{t\to+\infty}{\longrightarrow} \Psi\big(\xi(u)\big).$$

□

Proposition 3.8 yields the following long-term behavior of the time-changed Lévy process Z:

$$\frac{1}{t}\log \mathbb{E}\big[e^{uZ_t}\big] = \frac{1}{t}\mathcal{U}(t,0,u) + \frac{1}{t}\mathcal{V}(t,0,u)X_0 \underset{t\to+\infty}{\longrightarrow} \Psi\big(\xi(u)\big), \qquad \text{for all } u \in \mathcal{X}.$$

Similarly as in [KR11, Theorem 3.4], it can be shown that $\xi(\cdot)$ and $\Psi(\xi(\cdot))$ are cumulant generating functions of infinitely divisible random variables. We can therefore conclude that the marginal distributions of Z are asymptotically equivalent to those of a Lévy process with characteristic exponent $\Psi(\xi(\cdot))$. Notice that $\Psi(\xi(\cdot))$ corresponds to the exponent obtained by subordinating a Lévy process with exponent ξ by an independent Lévy process with exponent Ψ, see [Sat99, Theorem 30.1]. In particular, this subordinator is equivalent to the one appearing in the Lamperti-type representation of the CBI process X (see [CPGUB13] and also [Szu21, Section 2.5]).

4. CBITCL-preserving changes of probability

In this section, we describe a class of equivalent changes of probability that leave invariant the class of CBITCL processes. More precisely, we consider Esscher-type changes of measure under which a CBITCL process remains a CBITCL process, with modified branching and immigration mechanisms and Lévy exponent. The results of this section are motivated by financial applications, where one typically wants to ensure that a model preserves its structural characteristics under both the statistical and the risk-neutral probability, as well as under risk-neutral probabilities associated to different numéraires (see [FGS21a] for an application to a multi-currency market).

Let us fix two constants $\zeta \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ and consider the process $\mathcal{W} = (\mathcal{W}_t)_{t\geqslant 0}$ defined by

$$\mathcal{W}_t := \zeta(X_t - X_0) + \lambda Z_t, \qquad \forall t \geqslant 0. \tag{4.1}$$

In view of [JS03, Proposition II.8.26], the process $\mathcal{W}$ is an exponentially special semimartingale if and only if $\zeta \in \mathcal{D}_X$ and $\lambda \in \mathcal{D}_Z$. In this case, $\mathcal{W}$ admits a unique exponential compensator, i.e., a predictable finite variation process $\mathcal{K} = (\mathcal{K}_t)_{t\geqslant 0}$ such that $\exp(\mathcal{W} - \mathcal{K})$ is a local martingale. The following lemma provides the explicit representation of the exponential compensator $\mathcal{K}$.

Lemma 4.1. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$. *Consider the process $\mathcal{W}$ defined by* (4.1), *with $\zeta\in\mathcal{D}_X$ and $\lambda\in\mathcal{D}_Z$. Then, the exponential compensator $\mathcal{K}$ of $\mathcal{W}$ is given by*

$$\mathcal{K}_t = t\Psi(\zeta)+Y_t\big(\Phi(\zeta)+\Xi(\lambda)\big), \qquad \forall t\geqslant 0. \tag{4.2}$$

Proof. In view of [KS02, Theorems 2.18 and 2.19], taking into account that CBITCL processes are quasi-left-continuous, the exponential compensator $\mathcal{K}$ coincides with the modified Laplace cumulant process of (X,Z) computed at $\theta=(\zeta,\lambda)$. The latter can be explicitly expressed in terms of the semimartingale differential characteristics of (X,Z), given in Proposition 2.5. Hence, for all $t\geqslant 0$,

$$\begin{aligned}
\mathcal{K}_t &= \int_0^t\left(\theta^\top\mathcal{B}_s+\frac{1}{2}\theta^\top\mathcal{A}_s\,\theta+\int_{\mathbb{R}^2}(e^{\theta^\top x}-1-\theta^\top x\mathbf{1}_{\{|x|<1\}})\,\mathcal{C}_s(\mathrm{d}x)\right)\mathrm{d}s\\
&= t\left(\beta\zeta+\int_0^{+\infty}(e^{\zeta x}-1)\nu(\mathrm{d}x)\right)+Y_t\left(-b\zeta+\frac{1}{2}\sigma^2\zeta^2+\int_0^{+\infty}(e^{\zeta x}-1-\zeta x)\pi(\mathrm{d}x)\right)\\
&\quad+Y_t\left(b_Z\lambda+\frac{1}{2}\sigma_Z^2\lambda^2+\int_{\mathbb{R}}(e^{\lambda x}-1-\lambda x\mathbf{1}_{\{|x|<1\}})\gamma_Z(\mathrm{d}x)\right)\\
&= t\Psi(\zeta)+Y_t\big(\Phi(\zeta)+\Xi(\lambda)\big).
\end{aligned}$$

□

The process $\mathcal{W}$ introduced in (4.1) can be used to define an equivalent change of probability that leaves invariant the class of CBITCL processes. We consider a time horizon $\mathcal{T}<+\infty$ and assume that (X,Z) is directly given through its extended Dawson-Li representation (2.6)-(2.7) on a filtered stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. In view of Theorem 2.3, this entails no loss of generality.

Theorem 4.2. *Let (X,Z) be a* $\mathrm{CBITCL}(X_0,\Psi,\Phi,\Xi)$ *and suppose that Assumption 3.2 holds. Consider the process $\mathcal{W}$ defined in* (4.1), *with $\zeta\in\mathcal{D}_X$ and $\lambda\in\mathcal{D}_Z$, and its exponential compensator $\mathcal{K}$ given by* (4.2). *Then, the process $(\exp(\mathcal{W}_t-\mathcal{K}_t))_{t\in[0,\mathcal{T}]}$ is a martingale. Moreover, under the probability measure $\mathbb{P}'\sim\mathbb{P}$ defined on $(\Omega,\mathcal{F})$ by*

$$\frac{\mathrm{d}\mathbb{P}'}{\mathrm{d}\mathbb{P}}:=e^{\mathcal{W}_\mathcal{T}-\mathcal{K}_\mathcal{T}}, \tag{4.3}$$

the process (X,Z) remains a CBITCL process up to time $\mathcal{T}$, with parameters β', ν', b', σ', π', b'_Z, σ'_Z, and γ'_Z reported in Table 1.

Proof. By Lemma 4.1, the process $\exp(\mathcal{W}-\mathcal{K})$ is a local martingale and, by Fatou's lemma, also a supermartingale. Therefore, to prove the martingale property of $(\exp(\mathcal{W}_t-\mathcal{K}_t))_{t\in[0,\mathcal{T}]}$, it suffices to show that $\mathbb{E}[\exp(\mathcal{W}_\mathcal{T}-\mathcal{K}_\mathcal{T})]=1$. More specifically, making use of equations (4.1) and (4.2), we will prove that

$$e^{-\zeta X_0-\mathcal{T}\Psi(\zeta)}\mathbb{E}\big[e^{\zeta X_\mathcal{T}-(\Phi(\zeta)+\Xi(\lambda))Y_\mathcal{T}+\lambda Z_\mathcal{T}}\big]=1. \tag{4.4}$$

Recalling the notation introduced in Section 3.1, we have that $\zeta\leqslant\chi^{(-\Phi(\zeta)-\Xi(\lambda),\lambda)}$. Theorem 3.4 therefore implies that $\mathrm{T}^{(\zeta,-\Phi(\zeta)-\Xi(\lambda),\lambda)}=+\infty$, thus showing that the expectation in (4.4) is finite. Moreover, under Assumption 3.2, there exists a unique solution to the extended Riccati system (3.3)-(3.4) with $(u_1,u_2,u_3)=(\zeta,-\Phi(\zeta)-\Xi(\lambda),\lambda)$. The solution to (3.4) is given by the constant

CBITCL parameters under $\mathbb{P}'$
$\beta' := \beta$
$\nu'(\mathrm{d}z) := e^{\zeta z}\nu(\mathrm{d}z)$
$b' := b - \zeta\sigma^2 - \int_0^{+\infty} z(e^{\zeta z}-1)\pi(\mathrm{d}z)$
$\sigma' := \sigma$
$\pi'(\mathrm{d}z) := e^{\zeta z}\pi(\mathrm{d}z)$
$b'_Z := b_Z + \lambda\sigma_Z^2 + \int_{\lvert z\rvert<1} z\,(e^{\lambda z}-1)\gamma_Z(\mathrm{d}z)$
$\sigma'_Z := \sigma_Z$
$\gamma'_Z(\mathrm{d}z) := e^{\lambda z}\,\gamma_Z(\mathrm{d}z)$

TABLE 1. Parameter transformations from $\mathbb{P}$ to $\mathbb{P}'$ for the CBITCL process (X,Z).

function $\mathcal{V}(\cdot,\zeta,-\Phi(\zeta)-\Xi(\lambda),\lambda)=\zeta$, which in turn implies that $\mathcal{U}(t,\zeta,-\Phi(\zeta)-\Xi(\lambda),\lambda)=t\Psi(\zeta)$, for all $t\geqslant 0$. The validity of (4.4) then follows directly from Lemma 3.3. We have thus shown that (4.3) defines a probability measure $\mathbb{P}'\sim\mathbb{P}$ with density process $(\exp(\mathcal{W}_t-\mathcal{K}_t))_{t\in[0,\mathcal{T}]}$.

In order to show that (X,Z) is a CBITCL process under $\mathbb{P}'$, we first express $(\exp(\mathcal{W}_t-\mathcal{K}_t))_{t\in[0,\mathcal{T}]}$ as a stochastic exponential, making use of [JS03, Theorem II.8.10] together with the extended Dawson-Li representation (2.6)-(2.7) of (X,Z):

$$
\begin{aligned}
e^{\mathcal{W}-\mathcal{K}} &= \mathcal{E}\bigg(\zeta\sigma\int_0^{\cdot}\sqrt{X_s}\,\mathrm{d}B_s^1+\lambda\sigma_Z\int_0^{\cdot}\sqrt{X_s}\,\mathrm{d}B_s^2+\int_0^{\cdot}\int_0^{+\infty}(e^{\zeta x}-1)\widetilde{N}_0(\mathrm{d}s,\mathrm{d}x)\bigg)\\
&\quad\times\mathcal{E}\bigg(\int_0^{\cdot}\int_0^{X_{s-}}\int_0^{+\infty}(e^{\zeta x}-1)\widetilde{N}_1(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x)+\int_0^{\cdot}\int_0^{X_{s-}}\int_{\mathbb{R}}(e^{\lambda x}-1)\widetilde{N}_2(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x)\bigg).
\end{aligned}
$$

By Girsanov's theorem, the processes $(B_t'^{,1})_{t\in[0,\mathcal{T}]}$ and $(B_t'^{,2})_{t\in[0,\mathcal{T}]}$ defined by

$$
B_t'^{,1} := B_t^1-\zeta\sigma\int_0^t\sqrt{X_s}\,\mathrm{d}s \qquad\text{and}\qquad B_t'^{,2} := B_t^2-\lambda\sigma_Z\int_0^t\sqrt{X_s}\,\mathrm{d}s, \qquad \forall t\in[0,\mathcal{T}],
$$

are independent Brownian motions under $\mathbb{P}'$. Again by Girsanov's theorem, under $\mathbb{P}'$ the compensated Poisson random measures associated to $N_0(\mathrm{d}t,\mathrm{d}x)$, $N_1(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x)$, and $N_2(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x)$ are respectively given by

$$
\begin{aligned}
\widetilde{N}_0'(\mathrm{d}t,\mathrm{d}x) &:= N_0(\mathrm{d}t,\mathrm{d}x)-e^{\zeta x}\nu(\mathrm{d}x)\,\mathrm{d}t,\\
\widetilde{N}_1'(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x) &:= N_1(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x)-e^{\zeta x}\pi(\mathrm{d}x)\,\mathrm{d}u\,\mathrm{d}t,\\
\widetilde{N}_2'(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x) &:= N_2(\mathrm{d}t,\mathrm{d}u,\mathrm{d}x)-e^{\lambda x}\gamma_Z(\mathrm{d}x)\,\mathrm{d}u\,\mathrm{d}t.
\end{aligned}
$$

Therefore, under the probability $\mathbb{P}'$, the extended Dawson-Li representation (2.6)-(2.7) of (X,Z) can be rewritten as follows:

$$
X_t = X_0+\int_0^t(\beta'-b'X_s)\mathrm{d}s+\sigma'\int_0^t\sqrt{X_s}\,\mathrm{d}B_s'^{,1}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^{+\infty} x\,N_0(\mathrm{d}s,\mathrm{d}x)+\int_0^t\int_0^{X_{s-}}\int_0^{+\infty} x\widetilde{N}_1'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x),\\
Z_t &= b_Z'\int_0^t X_s\,\mathrm{d}s+\sigma_Z'\int_0^t\sqrt{X_s}\,\mathrm{d}B_s'^{,2}+\int_0^t\int_0^{X_{s-}}\int_{|x|\geqslant 1} xN_2(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x)\\
&+\int_0^t\int_0^{X_{s-}}\int_{|x|<1} x\widetilde{N}_2'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x),
\end{aligned}
$$

where the parameters $\beta', b', b_Z', \sigma', \sigma_Z'$ are given as in Table 1. By Theorem 2.3, it follows that (X,Z) is a CBITCL process under $\mathbb{P}'$, thus completing the proof. $\square$

As pointed out at the end of Section 2.2, in financial applications the component Z of a CBITCL process (X,Z) is typically related to the log-price process of an asset. In order to ensure absence of arbitrage, it is useful to have conditions characterizing the martingale property of $\exp(Z)$. To this end, by exploiting the previous results, we can state the following corollary.

Corollary 4.3. *Let (X,Z) be a* CBITCL(X_0,Ψ,Φ,Ξ) *and suppose that Assumption 3.2 holds. Then, the process $(e^{Z_t})_{t\in[0,\mathcal{T}]}$ is a martingale if and only if $1\in\mathcal{D}_Z$ and $\Xi(1)=0$.*

Proof. If $1\in\mathcal{D}_Z$ and $\Xi(1)=0$, by making use of (4.1) with $(\zeta,\lambda)=(0,1)$ together with Lemma 4.1 and Theorem 4.2, we directly obtain that $(e^{Z_t})_{t\in[0,\mathcal{T}]}$ is a martingale. Conversely, if $(e^{Z_t})_{t\in[0,\mathcal{T}]}$ is a martingale, then $\mathbb{E}[e^{Z_{\mathcal{T}}}]=1$. By [KRM15, Theorem 2.14-(a)], it follows that $1\in\mathcal{D}_Z$. Moreover, in view of (3.8), we have that $\mathrm{T}^{(0,0,1)}>\mathcal{T}$. By Lemma 3.3, the martingale property of $(e^{Z_t})_{t\in[0,\mathcal{T}]}$ necessarily implies that $\mathcal{V}(t,0,0,1)=0$, for all $t\in[0,\mathcal{T}]$. Since the ODE (3.4) admits a unique solution under Assumption 3.2, it follows that $\Xi(1)=0$. $\square$

5. Examples and applications

In this section, we present some examples of CBITCL processes that are particularly appropriate for financial applications and that possess a self-exciting behavior. In Section 5.1, we analyze the alpha-CIR process recently studied in [JMS17, JMSZ21] from the viewpoint of CBITCL processes. In Section 5.2, we discuss the CBITCL process adopted in [FGS21a] for the modelling of multi-currency markets with stochastic volatility.

5.1. Alpha-CIR process and geometric Brownian motion.

Let us consider a process (X,Z) such that

(i) X is an α-CIR process;
(ii) $Z=L_Y$, where the Lévy process L is given by the drifted Brownian motion $L_t:=B_t-t/2$, for all $t\geqslant 0$, and the change of time process is given by $Y=\int_0^{\cdot} X_s\,\mathrm{d}s$.

We recall from [JMS17] that an *α-CIR process* is defined as the unique strong solution to the following SDE (see [FL10, Corollary 6.3]):

$$
X_t = X_0+\int_0^t(\beta-b\,X_s)\mathrm{d}s+\sigma\int_0^t\sqrt{X_s}\,\mathrm{d}\bar{B}_s+\eta\int_0^t\sqrt[\alpha]{X_{s-}}\,\mathrm{d}L_s^{\alpha}, \tag{5.1}
$$

with $b > 0$, $\eta > 0$, $\beta, \sigma \in \mathbb{R}_+$ and where $\bar{B}$ is a Brownian motion independent of B and L^α is a spectrally positive compensated α-stable Lévy process independent of $\bar{B}$ and B, with stability parameter $\alpha \in (1,2)$ and Lévy measure $C_\alpha\, z^{-1-\alpha}\mathbf{1}_{\{z>0\}}\mathrm{d}z$, where C_α is a suitable normalization constant (see [JMS17] for additional details). It can be easily verified that X is a CBI process with $\nu = 0$ and $\pi(\mathrm{d}z) = \eta^\alpha C_\alpha z^{-1-\alpha}\mathbf{1}_{\{z>0\}}\mathrm{d}z$. The immigration mechanism is simply given by $\Phi(u) = \beta u$, while the branching mechanism is

$$\Phi(u) = -bu + \frac{1}{2}(\sigma u)^2 + C_\alpha\, \Gamma(-\alpha)(-\eta u)^\alpha, \tag{5.2}$$

where Γ denotes the Gamma function extended to $\mathbb{R}\backslash\mathbb{Z}_-$ (see [Leb72]). With this specification, we have $\phi = 0$ and $\psi = +\infty$, implying that $\mathcal{D}_X = (-\infty, 0]$. Since $\int_1^{+\infty} z\pi(\mathrm{d}z) < +\infty$, Assumption 3.2 is satisfied. As a consequence of Theorem 3.4, the process X does not admit exponential moments of any order, i.e., $\mathbb{E}[e^{uX_t}] = +\infty$ for all $u > 0$ and $t > 0$. This fact will motivate the study of *tempered* α-stable processes in Section 5.2.

In this example, the process Z is defined as a time-changed Brownian motion with drift. This specification ensures that $\exp(Z) = \mathcal{E}(B_Y)$ is a martingale (see Corollary 4.3). In view of financial applications, Z can therefore represent the discounted log-price process of a risky asset under a risk-neutral probability measure. The Lévy exponent of L is given by $\Xi(u) = u(u-1)/2$ and, using the notation introduced in Section 3.2, we have that

$$\begin{cases} \chi(u) = 0, & \text{for } u \in [0,1], \\ \chi(u) = -\infty, & \text{otherwise.} \end{cases}$$

Corollary 3.7 therefore implies that $\mathrm{T}(u) = +\infty$, for every $u \in [0,1]$, while $\mathrm{T}(u) = 0$, for every $u \notin [0,1]$. In other words, for an α-CIR-time-changed geometric Brownian motion it holds that $\mathbb{E}[e^{uZ_t}] < +\infty$ for all $u \in [0,1]$ and $t > 0$, while $\mathbb{E}[e^{uZ_t}] = +\infty$ for all $u \notin [0,1]$ and $t > 0$.

As explained in part (2) of Remark 3.5, these results on the finiteness of exponential moments of Z can be used to study the behavior of the implied volatility smile. Let us denote by $\sigma(T,k)$ the implied volatility of a European Call option written on an asset with price process $\exp(Z)$, with maturity T and strike $\exp(k)$. By applying [Lee04, Theorems 3.2 and 3.4], we can deduce that the asymptotic behavior of $\sigma(T,k)$ at extreme strikes is explicitly described as follows:

$$\limsup_{k\to\pm\infty} \frac{\sigma^2(T,k)}{|k|} = \frac{2}{T}, \qquad \text{for all } T > 0.$$

Concerning the long-term behavior of the process Z, by applying Proposition 3.8 we obtain that

$$\frac{1}{t}\log \mathbb{E}\big[e^{uZ_t}\big] \underset{t\to+\infty}{\longrightarrow} \beta\,\xi(u), \qquad \text{for } u \in [0,1].$$

Moreover, making use of equation (3.14), the quantity $\xi(u)$ is explicitly given by

$$\xi(u) = \Phi^{-1}\left(\frac{u(1-u)}{2}\right),$$

where Φ^{-1} denotes the inverse function of the branching mechanism Φ given by (5.2), which can be easily seen to be a bijection from $\mathbb{R}_-$ to $\mathbb{R}_+$.

Remark 5.1. The α-Heston model recently introduced in [JMSZ21] extends the above model by allowing for negative correlation between the Brownian motions B and $\bar{B}$, in order to capture the leverage effect between the price process of a risky asset and its volatility. The results stated above continue to hold in the same form even in the presence of correlation, as shown in [JMSZ21, Proposition 4.1 and Corollary 4.2].

5.2. **Tempered α-stable CBI process and CGMY process.** The CBITCL process considered in the previous subsection has the drawback that its CBI component does not possess exponential moments. In finance applications, the existence of exponential moments often represents an essential modelling requirement. For this reason, we now present a CBITCL process that enjoys good integrability properties. The example considered in this subsection relies on tempered α-stable CBI processes, as introduced in [FGS21b] in interest rate modelling (see also [Szu21, Section 2.7]).

We recall from [FGS21b] that a CBI process $X = (X_t)_{t \geqslant 0}$ is said to be *tempered α-stable* if the Lévy measures appearing in (2.1)-(2.2) are respectively given by

$$\nu = 0 \qquad \text{and} \qquad \pi(\mathrm{d}z) = C_\alpha \, z^{-1-\alpha} \, e^{-\theta z} \mathbf{1}_{\{z>0\}} \mathrm{d}z,$$

where $\theta > 0$, $\alpha \in (1,2)$ and $C_\alpha > 0$ is a suitable normalization constant. Under this specification, the immigration mechanism reduces to $\Psi(u) = \beta u$, while the branching mechanism can be explicitly computed as

$$\Phi(u) = -bu + \frac{1}{2}(\sigma u)^2 + C_\alpha \, \Gamma(-\alpha)\big((\theta - u)^\alpha - \theta^\alpha + \alpha\theta^{\alpha-1}u\big), \qquad \forall u \leqslant \theta. \tag{5.3}$$

It can be easily verified that $\mathcal{D}_X = (-\infty, \theta]$ and Assumption 3.2 is satisfied (see also [Szu21, Lemma 2.19]). The existence of exponential moments of X can be characterized by relying on Corollary 3.6 (noting that, in the present case, $\phi = \theta$ and $\psi = +\infty$). Indeed, taking $u_2 = u_3 = 0$ in Corollary 3.6, it follows that $\mathbb{E}[e^{uX_T}] < +\infty$ holds for all $u \leqslant \theta$ and $T > 0$ if and only if $\Phi(\theta) \leqslant 0$, namely, if and only if

$$b \geqslant \frac{\sigma^2}{2}\theta + C_\alpha \, \Gamma(-\alpha) \, \theta^{\alpha-1}(\alpha - 1). \tag{5.4}$$

We have therefore shown that, in the case of tempered α-stable CBI processes, the existence of exponential moments amounts to a simple condition on the parameters characterizing the process.

Remark 5.2. Tempered α-stable CBI processes can be constructed from non-tempered α-stable CBI processes by means of an equivalent change of probability. More specifically, if X is a non-tempered α-stable CBI process and $\mathcal{W}$ is defined as in (4.1) with $\zeta = -\theta$ and $\lambda = 0$ (so that $\mathcal{W} = \theta(X_0 - X)$), then Theorem 4.2 implies that the probability $\mathbb{P}'$ defined by (4.3) is well-defined and X is a tempered α-stable CBI process under $\mathbb{P}'$, with tempering parameter θ (and a different parameter b). This technique has been also employed in [JMS17, Proposition 4.1]. Alternatively, tempered α-stable CBI processes can be directly defined as solutions to a certain stochastic time change equation (see [Szu21, Section 2.7]).

To construct a CBITCL process, let us then consider a process (X, Z) such that

(i) X is a tempered α-stable CBI process;

(ii) $Z = L_Y$, where the Lévy process L is a CGMY process (see [CGMY03]) and the change of time process is given by $Y = \int_0^{\cdot} X_s \, \mathrm{d}s$.

We recall that L is a CGMY process if its Lévy measure γ is of the form

$$\gamma(\mathrm{d}z) = C_Y\big(z^{-1-Y}e^{-Mz}\mathbf{1}_{\{z>0\}} + |z|^{-1-Y}e^{-G|z|}\mathbf{1}_{\{z<0\}}\big)\mathrm{d}z, \tag{5.5}$$

where the normalization constant can be chosen as $C_Y = 1/\Gamma(-Y)$. The parameters $G > 0$ and $M > 0$ temper the downward and the upward jumps, respectively, while the parameter $Y \in (1,2)$ determine the local behavior of the process L, similarly to α above. The Lévy exponent Ξ associated to a CGMY process L with Lévy triplet $(0,0,\gamma)$ is given by

$$\Xi(u) = \int_{\mathbb{R}} (e^{zu} - 1 - zu)\gamma_Z(\mathrm{d}z), \qquad \forall u \in \mathrm{i}\mathbb{R}.$$

It can be easily checked that $\mathcal{D}_Z = [-G, M]$ and the Lévy exponent Ξ takes the explicit form

$$\Xi(u) = (M-u)^Y - M^Y + (G+u)^Y - G^Y + u\,Y\,(M^{Y-1} - G^{Y-1}), \qquad \forall u \in [-G, M].$$

For simplicity of presentation, let us assume that $G = M$. In this case, $\Xi : [-M, M] \to \mathbb{R}_+$ is a convex function with minimum $\Xi(0) = 0$ and maximum $\Xi(-M) = \Xi(M) = 2M^Y(2^{Y-1}-1)$. If $2M^Y(1-2^{Y-1}) \geqslant \Phi(\theta)$, which can be rewritten in the form

$$M \leqslant \left(\frac{\Phi(\theta)}{2\big(1-2^{Y-1}\big)}\right)^{1/Y}, \tag{5.6}$$

we have that $\chi(u) = \theta \geqslant 0$ for every $u \in [-M, M]$, using the notation introduced in Section 3.2. Corollary 3.7 then implies that $\mathrm{T}(u) = +\infty$ for every $u \in [-M, M]$, while $\mathrm{T}(u) = 0$ for every $u \notin [-M, M]$. Similarly as in Section 5.1, these results on the finiteness of exponential moments of Z can be used to characterize the tail behavior of the implied volatility smile. Indeed, under condition (5.6) and assuming $M > 1$, an application of [Lee04, Theorems 3.2 and 3.4] yields that

$$\left.\begin{aligned} \limsup_{k\to-\infty} \frac{\sigma^2(T,k)}{|k|} &= \frac{2}{T}\big(1 - 2(\sqrt{M^2+M} - M)\big), \\ \limsup_{k\to+\infty} \frac{\sigma^2(T,k)}{k} &= -\frac{2}{T}\big(1 + 2(\sqrt{M^2-M} - M)\big), \end{aligned}\right\} \qquad \text{for all } T > 0.$$

The long-term behavior of the process Z can be determined by relying on Proposition 3.8. Under conditions (5.4) and (5.6), we have that

$$\frac{1}{t}\log\mathbb{E}\big[e^{uZ_t}\big] \underset{t\to+\infty}{\longrightarrow} \beta\,\xi(u), \qquad \text{for all } u \in [-M, M],$$

where $\xi(u)$ is defined in the statement of Proposition 3.8.

The quantity $\xi(u)$ can be explicitly determined under the following additional condition:

$$b \geqslant \sigma^2\theta + C_\alpha\Gamma(-\alpha)\,\theta^{\alpha-1}\alpha. \tag{5.7}$$

Condition (5.7) is stronger than condition (5.4) and, together with the convexity and continuity of Φ, it implies that Φ is decreasing on $(-\infty, \theta]$. In this case, making use of (3.14), we have that

$$\xi(u) = \Phi^{-1}\big(2M^Y - (M+u)^Y - (M-u)^Y\big),$$

where Φ^{-1} denotes the inverse function of the branching mechanism Φ given by (5.3), which under condition (5.7) is a bijection from $(-\infty, \theta]$ to $[\Phi(\theta), +\infty)$.

Remark 5.3. (1) Note that the present specification does not necessarily guarantee the martingale property of the process $\exp(Z)$ (compare with Corollary 4.3). Therefore, in view of financial applications and similarly as in [FGS21a], the CBITCL process (X, Z) may be used to model the discounted price process $S = (S_t)_{t\geqslant 0}$ of a risky asset as follows:

$$\log S_t := \lambda Z_t + \zeta(X_t - X_0) - \mathcal{K}_t, \qquad \forall t \geqslant 0,$$

where $\mathcal{K}$ denotes the exponential compensator (see Lemma 4.1), with $\zeta \leqslant \theta$ and $\lambda \in [-G, M]$. Under this specification, the CBI process X plays the role of stochastic volatility, while the parameter ζ determines the correlation between the log-price process and its volatility. A direct application of Theorem 4.2 yields that S is a martingale, thereby ensuring absence of arbitrage. Moreover, since Assumption 3.2 is satisfied, the existence of moments $\mathbb{E}[S_t^u]$ can be characterized by Theorem 3.4.

(2) As a direct consequence of Theorem 4.2, the class of CBITCL processes considered in this subsection is stable with respect to equivalent changes of probability of the form considered in Section 4. This property has been used to construct risk-neutral measures that preserve the structure of the model in [FGS21a], where CBITCL processes have been applied to the modelling of multi-currency markets in the presence of stochastic volatility and self-exciting jumps.

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