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Abstract

We consider the test statistic devised by Christensen, Oomen and Renò in 2020 to obtain insight into the causes of *flash crashes* occurring at particular moments in time in the price of a financial asset. Under an Ito semimartingale model containing a drift component, a Brownian component and finite variation jumps, it is possible to identify when the cause is a drift burst (the statistic explodes) or otherwise (the statistic is asymptotically Gaussian). We complete the investigation showing how infinite variation jumps contribute asymptotically. The result is that the jumps never cause the explosion of the statistic. Specifically, when there are no bursts, the statistic diverges only if the Brownian component is absent, the jumps have finite variation and the drift is non-zero. In this case the triggering is precisely the drift. We also find that the statistic could be adopted for a variety of tests useful for investigating the nature of the data generating process, given discrete observations.

Keywords: test statistic, Ito semimartingale, infinite variation jumps, jump activity index, asymptotic behavior

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1. Introduction

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and on a fixed time horizon $T > 0$, we consider a càdlàg Ito semimartingale (SM)

$$dY_t = b_t dt + \sigma_t dW_t + dX_t,\ \ t \in [0, T], \tag{1}$$

Y_0 being $\mathcal{F}_0$-measurable, modeling the evolution in time of the price of a financial asset. The drift process $\{b_t\}_{t\geq 0}$ is predictable, $\{\sigma_t\}_{t\geq 0}$ is an adapted, càdlàg positive volatility process; $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and X is a pure-jump process represented as the sum of its compensated small jumps plus the sum of the not-compensated big jumps,

$$X_t = \int_0^t \int_{|x|\leq 1} x\tilde{\mu}(dx, ds) + \int_0^t \int_{|x|>1} x\mu(dx, ds),\ \ t \in [0, T], \tag{2}$$

where $\mu(dx, ds)$ is a random jump measure defined on $(\Omega \times \mathbb{R} \times [0, T])$ and endowed with a compensator of type $\nu(dx, dt) = \lambda(x, s)dxds$, where $\lambda(x, s)$ is random, and $\tilde{\mu} = \mu - \nu$ is the compensated measure. Formal conditions on the components of Y are given in Section 2.

For fixed $\bar{t} \in (0, T)$, we focus on the asymptotic behavior of

$$T^n_{\bar{t}} \doteq \frac{\sum_{i=1}^n K_i \Delta_i Y}{\sqrt{\sum_{i=1}^n K_i (\Delta_i Y)^2}}, \tag{3}$$

where: for any integer $n > 0$, $\{t_i = t_i^{(n)}, i = 1, \ldots, n\}$ gives a non-random partition of $[0, T]$; $\Delta_i Y \doteq Y_{t_i} - Y_{t_{i-1}}; K_i = K\left(\frac{\bar{t}-t_{i-1}}{h}\right)$; $K : \mathbb{R} \to \mathbb{R}_+$ is a kernel continuous function and h is a bandwidth parameter. We are interested in the framework where

$$n \to +\infty \text{ while } h \to 0 \text{ in such a way that } nh \to +\infty, \tag{4}$$

and we assume that the partition does not differ asymptotically significantly from the equally spaced partition, as explained below.

The statistic $T^n_{\bar{t}}$ is devised in [2], where an Ito SM is considered to model the price evolution of a financial asset. Christensen, Oomen and Renò wished to test whether a sudden large movement of the asset price at a particular time

$\bar{t}$ (*flash crash*) is due to a *drift burst*, i.e. a local explosion of the drift coefficient around $\bar{t}$. They were particularly interested in understanding whether a flash crash occurring at $\bar{t}$ is more compatible with an *explosion* (burst) at $\bar{t}$ of the Brownian coefficient (the volatility) or with an explosion at $\bar{t}$ of the drift coefficient. Thus their test statistic was intended to compare the magnitudes of $\sigma_{\bar{t}}$ and $b_{\bar{t}}$, and $T^n_{\bar{t}}$ is given by $\sqrt{h}$ times the ratio of the two kernel-based estimators $\hat{b}_{\bar{t}} \doteq \frac{1}{h}\sum_{i=1}^n K_i \Delta_i Y$ of the drift and $\hat{\sigma}_{\bar{t}} \doteq \left(\frac{1}{h}\sum_{i=1}^n K_i (\Delta_i Y)^2\right)^{\frac{1}{2}}$ of the volatility at $\bar{t}$. Under the null hypothesis of either no drift bursts, or the occurrence of bursts with the one in drift smaller than the one in volatility, the statistic is asymptotically normal, while, under a given class of alternative models including bursts, $T^n_{\bar{t}}$ is shown to explode when there is a burst in the drift larger than a burst in the volatility.

However their framework only considers finite variation (FV) jumps, and it is natural to wonder what role infinite variation (IV) jumps would play within $T^n_{\bar{t}}$, for instance whether the explosion observed in the empirical implementation of the statistic on finite samples may be due to a jump component of IV, possibly present in the data generating process (DGP). Or, how the statistic would behave if the DGP did not contain any Brownian components. Therefore we expand the analysis on the asymptotic behavior of $T^n_{\bar{t}}$, under the hypothesis of no bursts, when Y also contains IV jumps and/or does not contain the Brownian motion, and we complete the picture given in [2].

Three elements are crucial for this analysis. First, separately measuring the contribution of the jump component X of the model is necessary because we need to know the exact speed of convergence of each term involving the increments $\Delta_i X$ so as to be able to decide which terms in $T^n_{\bar{t}}$ are leading when considering the complete model. For this reason our first step is to illustrate the behavior of $T^n_{\bar{t}}$ for a pure jump model. After that the behavior in the complete framework will be an immediate consequence. Note however that, as pure jump models exist and are currently used for financial asset prices, the analysis of the statistic within the first step framework is also important in itself.

Second, in the pure jump framework it turns out that the behaviour of $T^n_{\bar{t}}$

is different in the two cases where $\bar{t}$ is or is not a jump time. In fact, denoting by $\Delta X_{\bar{t}}$ the size of the jump that possibly occurred at $\bar{t}$, the numerator tends (ω-wise if the jumps have FA, in probability if they have IA) to $K(0)\Delta X_{\bar{t}}$, and the denominator to $\sqrt{K(0)} \cdot |\Delta X_{\bar{t}}|$. Thus if $\Delta X_{\bar{t}} \neq 0$ the statistic has a well defined finite limit, otherwise both numerator and denominator tend to 0 and, as soon as $T^n_{\bar{t}}$ is defined, the limit is determined by the dominant terms.

Third, the contribution of the jump terms is essentially determined by the freneticism of the jumps, and for this reason we deal with processes X with constant *jump activity index* α on $\Omega \times [0,T]$. When $\sigma \equiv 0$ the asymptotic distribution of the statistic is substantially different depending on whether the jumps have finite or infinite variation. In the former case ($\alpha < 1$), as well as when $\alpha = 1$, we obtain the explosion of $T^n_{\bar{t}}$, while if $\alpha > 1$ then $\left|T^n_{\bar{t}}\right|$ does not explode, as numerator and denominator tend to 0 at the same speed, which depends on the magnitude of α.

To get an insight into how things are going, let us consider the simple case when $\Delta_i Y = a\Delta + \sigma\Delta_i W + \Delta_i J$, with constant drift and volatility coefficients, a symmetric $\alpha-$stable Lévy jump process J and evenly spaced observations. If $\sigma \neq 0$ the three components have the following different magnitude orders: $a\Delta = O(\Delta), \sigma\Delta_i W = O_P(\sqrt{\Delta}), \Delta_i J \overset{d}{=} \Delta^{\frac{1}{\alpha}} J_1$: while for any $\alpha \in (0,2)$ we obtain $\Delta^{\frac{1}{\alpha}} << \sqrt{\Delta}$, the term $a\Delta$ is dominated or dominates $\Delta^{\frac{1}{\alpha}}$ depending on whether $\alpha > 1$ or $\alpha < 1$, respectively. It is thus easy to convince ourselves that if $\sigma \neq 0$ then the Brownian component gives the leading term both of $\Delta_i Y$ and of $(\Delta_i Y)^2$, and since under our assumptions we have $\sum_{i=1}^n K_i\Delta/h \to 1$, we obtain $\sum_{i=1}^n K_i\sigma(\Delta_i W)^2/h \overset{P}{\to} \sigma^2_{\bar{t}}$ (see also Lemma 3 in [2]), and that $\sum_{i=1}^n K_i\sigma\Delta_i W/\sqrt{h}$ is asymptotically Gaussian (as in the proof of Thm 1 there). Thus under the null of no drift or volatility bursts, even in the presence of IV jumps $T^n_{\bar{t}}$ is asymptotically Gaussian.

Let us now deal with the case of $\sigma = 0$, when the model is of the pure jump type and with drift: we have $\sum_{i=1}^n K_i\Delta_i Y = \sum_{i=1}^n K_i a\Delta + \sum_{i=1}^n K_i\Delta_i J$ and $\sum_{i=1}^n K_i(\Delta_i Y)^2 = \sum_{i=1}^n K_i a^2\Delta^2 + 2a\Delta\sum_{i=1}^n K_i\Delta_i J + \sum_{i=1}^n K_i(\Delta_i J)^2$. It turns out that when $\alpha < 1$ the sum $\sum_{i=1}^n K_i a\Delta$ dominates all the other sums at both the numerator and the denominator of $T^n_{\bar t}$, while for $\alpha > 1$ the jumps always dominate. More in detail, when $\alpha < 1$ and $\sum_{i=1}^n K_i a\Delta$ dominates, denoting by $\simeq$ that two expressions have the same limit and by $\overset{d}{\simeq}$ that they have the same limit in distribution, we obtain $\sum_{i=1}^n K_i(a\Delta) \simeq ah$, while $\sum_{i=1}^n K_i(a\Delta)^2 \simeq a^2 h\Delta$. Since $\frac{h}{\Delta} \to \infty$, then $|T^n_{\bar t}| \to +\infty$.

For the case $\alpha > 1$ when the jump component dominates, for sake of simplicity we illustrate the case where the kernel function is given by a continuous approximation of the indicator $I_{\{|x|\le\frac{1}{2}\}}$. The jump contribution is as follows:

$$\sum_{i=1}^n K_i\Delta_i J \simeq \sum_{t_{i-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]} \Delta_i J \simeq J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}} \overset{d}{\simeq} h^{\frac{1}{\alpha}} J_1,$$

and

$$\sum_{i=1}^n K_i(\Delta_i J)^2 \simeq \sum_{t_{i-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]}(\Delta_i J)^2 \simeq$$
$$\left(J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}}\right)^2 - \sum_{i\neq k:\ t_{i-1},t_{k-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]} \Delta_i J\Delta_k J \overset{d}{\simeq} \left(J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}}\right)^2 \overset{d}{\simeq} h^{\frac{2}{\alpha}} J_1^2,$$

thus the numerator and the denominator of $|T^n_{\bar t}|$ tend to 0 at the same speed $h^{\frac{1}{\alpha}}$, and the statistic converges in distribution.

More generally, our results are that, when $\Delta X_{\bar t} = 0$, if a non-zero Brownian term is present in the model Y then, under the no-burst hypothesis, $T^n_{\bar t}$ never explodes: it is asymptotically normal, whatever the jump activity index, because the Brownian terms dominate all the others at numerator and denominator. The conclusion is that a flash crash cannot be explained by infinite variation jumps, i.e.: in the presence of a Brownian component in the model, a drift b which is exploding in relation to the volatility is the only case in which $T^n_{\bar t}$ explodes. This happens precisely because the numerator asymptotically behaves as $\sqrt{h} b_{\bar t}$ while the denominator approaches $\sigma_{\bar t}$.

By contrast, the IV jumps happen to dominate any other term only when $\sigma \equiv 0$ and $\alpha > 1$, but then they contribute by the same amount both to the numerator and the denominator, and the statistic cannot explode.

Note that in the absence of the Brownian component and when the jumps have finite variation then the drift of Y bursts in relation to the zero volatility, and consistently $|T^n_{\bar t}|$ explodes.

The finite activity jump case (the simplest case of FV jumps) is dealt with under more general conditions for the choice of partitions and for the jump sizes. For the infinite activity case, on the other hand, we assume evenly spaced observations and that the small jumps have constant jump activity index α. In the latter framework we first analyze the case where the compensated small jumps are the ones in a (not necessarily symmetric) α-stable Lévy process, we denote them $\tilde{J}$. In this way we can study the asymptotic behavior for the characteristic functions of the statistic numerator and squared denominator separately, and, when $\alpha > 1$ and the jump sizes have symmetric law, also for the characteristic function of the joint law of squared numerator and squared denominator, and we provide closed form expressions for the limit characteristic functions. Subsequently the results are extended to more general jump processes X with jump index α. In fact, under our assumptions we can split the compensated small jumps $\tilde{X}$ into the sum $\tilde{J} + \tilde{X}'$ of the ones in an α-stable model plus those in a residual process $\tilde{X}'$ with a lower jump activity index, and we show that the contribution of $\tilde{X}'$ does not substantially change the results which hold for the α-stable case.

Actually, $T^n_{\bar{t}}$ could be exploited for many different tests. Assuming model (1) we firstly check whether $T^n_{\bar{t}}$ is asymptotically Gaussian. In case, the DGP contains a BM, otherwise it is an SM only containing jumps, compensator of the small jumps, and possibly a further drift component: if $\left|T^n_{\bar{t}}\right|$ does not explode the DGP has IV jumps, if $\left|T^n_{\bar{t}}\right| \to \infty$ then the DGP has FV jumps, but no jumps occurred at $\bar{t}$; if $|T^n_{\bar{t}}| \to \sqrt{K(0)}$ then a jump occurred at $\bar{t}$. Assessment of whether through $T^n_{\bar{t}}$ we can further distinguish FA from IA jumps is ongoing.

The rest of the paper is organized as follows: Section 2 sets out details of the model considered and provides some notation; Section 3 analyzes the behavior of $T^n_{\bar{t}}$ for the pure jump SM X. In particular, Section 3.1 deals with the case of finite activity jumps: the necessary assumptions are established and the first main theorem is stated; Section 3.2 deals with the case of infinite activity jumps: further assumptions are made and the second main result of the paper is stated. Section 4 shows the behavior of $T^n_{\bar{t}}$ for the complete SM model (1), possibly including infinite variation jumps. Section 5 briefly illustrates the theoretical results from simulated data, and Section 6 discusses a possible extension of the results to a multivariate framework. Section 7 includes the proof of our first Lemma, the statements of other five necessary Lemmas and the proofs of the Theorems. The statements of two further Lemmas, the proofs of the second to eighth Lemmas and of the Corollary to Theorem 2 are shown in the Appendix.

2. Setting

We start by introducing our setting and some notation. We assume that model (1) further satisfies the following conditions: $\{b_t\}_{t\geq 0}$ is locally bounded; $\lambda(\omega, x, s)$, from $\Omega \times \mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}$, is a predictable function, i.e. it is measurable in relation to $\mathcal{P} \times \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the predictable σ-algebra of $\Omega \times [0, T]$ and $\mathcal{B}(\mathbb{R})$ is the Borelian σ-algebra of $\mathbb{R}$; if $\mu(\omega, \mathbb{R}, \{s\}) \neq 0$ then $\int_{\mathbb{R}} x\mu(dx, \{s\}) \neq 0$.

The local boundedness of b ensures that no drift bursts occur; the predictability condition for λ is required to make the two processes $\int_0^t \int_{|x|\leq 1} x\tilde{\mu}(dx, ds)$ and $\int_0^t \int_{|x|>1} x\lambda(x, s)dxds$ well-defined. The last requirement simply means that if a jump occurs at s then its size is non-zero.

Notation 1. $K_+ \doteq \int_0^{+\infty} K(u)du$, $K_- \doteq \int_{-\infty}^0 K(u)du$. For any random process b,

$$b^\star_{\bar{t}} \doteq b_{\bar{t}-} \cdot K_+ + b_{\bar{t}+} \cdot K_-. \tag{5}$$

When X has FV jumps, we define $a_s \doteq \int_{|x|\leq 1} x\lambda(x, s)dx$, so that the compensator of the small jumps is given by $\int_0^t a_s ds$.

After defining $\Delta = \Delta_n = \frac{T}{n}$ and $\Delta_{max} = \Delta_{max,n} = \max_{i=1..n} |t_i - t_{i-1}|$ we assume that

$$\Delta_{max} \leq C\Delta$$

for a fixed constant C, which means that the partition should not differ too much, asymptotically, from the equally spaced partition. The framework (4), under which we look for our asymptotic results, means that $\Delta \to 0$ and $\frac{\Delta}{h} \to 0$.

As mentioned in the Introduction, in the presence of the Brownian part in the model, when $\Delta X_{\bar{t}} = 0$ the contribution of the jumps turns out always to be negligible. To illustrate this, we start by analyzing the jump contibution in the pure jump model (2), then return to the general model in Section 4.

3. Pure jump model

Within the framework in (2) note that for fixed $\bar{t} \in (0, T)$ the statistic $T^n_{\bar{t}}$ of our interest is well-defined when the denominator is non-zero. As will be clear from the proofs of Lemma 1 and Theorem 2 (part a), this is the case at least when X jumps at $\bar{t}$ or when X has IA jumps (in which case in any small interval some jumps occur). When no jumps occur at $\bar{t}$ and X has FA jumps, the statistic is well-defined at least when $a^\star_{\bar{t}} \neq 0$ (see (16)).

As mentioned in the Introduction, for a fixed ω it turns out that the behaviour of $T^n_{\bar{t}}$ is different in the two cases where $\bar{t}$ is or is not a jump time, and the statistic asymptotic distribution is substantially different depending on whether the jumps have finite or infinite variation. We tackle the finite activity jump case first, while the infinite activity case is dealt with in Section 3.2.

Notation 2. C always indicates a constant. Within the algebraic expressions we retain the constant C even where the two sides of an equality yield different constants. Given two functions f, g, then $f(h) \simeq g(h)$ indicates that $\lim_{h\to 0} f(h) = \lim_{h\to 0} g(h)$, while $f(h) \sim g(h)$ indicates that $\lim_{h\to 0} \frac{f(h)}{g(h)} = C$, $f(h) \ll g(h)$ indicates asymptotic negligibility of f w.r.t. g, i.e. $\lim_{h\to 0} \frac{f(h)}{g(h)} = 0$. Given two sequences T^n, U^n of random variables, $T^n \overset{d}{\simeq} U^n$ means that they have the same limit in distribution. Recall that ΔX_t indicates the size of the jump that possibly occurred at t (in our framework $\Delta X_t = 0$ iff $\mu(\omega, \mathbb{R}, \{t\}) = 0$). $K_s \doteq K\left(\frac{\bar{t}-s}{h}\right)$. For any $\alpha > 0$, $K_{(\alpha)} \doteq \int_{\mathbb{R}} K^\alpha(u)du$. $\mathbb{R}_+ = (0, +\infty)$, $\quad \mathbb{R}_- = (-\infty, 0)$. $\mu(dx, ds), \tilde{\mu}(dx, ds)$ can be abbreviated using $d\mu, d\tilde{\mu}$, respectively.

3.1. Finite activity jumps

We now consider the case in which $\int_0^T \int_{\mathbb{R}} 1\nu(dx, ds) = \int_0^T \int_{\mathbb{R}} \lambda(x, s)dxds < \infty$ a.s.. Hence we obtain a.s.

$$|a_s| \leq \int_0^t \int_{|x|\leq 1} |x|\lambda(x, s)dxds \leq \int_0^t \int_{\mathbb{R}} \lambda(x, s)dxds < \infty,$$

so X can be written as

$$X_t = \int_0^t \int_{\mathbb{R}} x\mu(dx, ds) - \int_0^t \int_{|x|\leq 1} x\lambda(x, s)dxds.$$

The latter term $-\int_0^t \int_{|x|\leq 1} x\lambda(x, s)dxds = -\int_0^t a_s ds$ is a random drift component. On the other hand $\int_0^t \int_{\mathbb{R}} xd\mu$ coincides with $\sum_{p=1}^{N_t} c_p$ for any $t \in [0, T]$, where N is the process counting the finitely many jumps, occurring at some random times $S_1(\omega), \ldots, S_{N_T(\omega)}(\omega)$ on $[0, T]$, and $c_p = c_p(\omega) \doteq \int_{\mathbb{R}} x\mu(dx, \{S_p\}) = c(\omega, x_p, S_p)$ is the random finite size of the jump at S_p. Thus we also can write X as

$$X_t = \sum_{p=1}^{N_t} c_p - \int_0^t a_s ds \doteq L_t - \int_0^t a_s ds.$$

Note that while, for any s, $|a_s| < \infty$, a.s., in general the drift process a could not be uniformly bounded in (ω, s).

*Assumption A*1. **Kernel function.**

A1.1 $K : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant L and satisfies

$\lim_{x\to+\infty} K(x) = 0, \lim_{x\to-\infty} K(x) = 0$ and $\int_{\mathbb{R}} K(x)dx = 1$.

A1.2 K satisfies the following:

· if $|a| < |b|$ then $K(\frac{b}{h}) << K(\frac{a}{h})$

· for any fixed $x \neq 0$, $K(\frac{x}{h}) << h\Delta$, as $h \to 0$, under (4).

Remark 1. i) The Gaussian kernel $K(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ satisfies Assumption **A1** for instance with $h = \Delta^\gamma$ with $\gamma \in (0, 1)$. This is the case if for instance $h = k_n\Delta$ with $k_n = C\Delta^{-\frac{1}{2}}$.

ii) To know how T_t^n behaves asymptotically if the kernel is an indicator function, our results can be used where the kernel is a Lipschitz continuous approximation of the indicator function.

*Assumption A*2. **Partitions of** $[0, T]$**.** After defining

$$H_t^{(n)} \doteq \frac{1}{\Delta} \sum_{t_i \le t} \Delta_i^2,$$

we assume that:

· for any $t \in (0, T]$ the $\lim_{n\to+\infty} H_t^{(n)} \doteq H_t > 0$ exists and is finite,

· H is Lebesgue differentiable in $(0, T)$ except for a finite and fixed number $m \ge 0$ of points $\tau_1, \dots, \tau_m$, and H' is bounded,

· if $I_H^{(n)} = \{i : \exists k, \tau_k \in [t_{i-1}, t_i)\}$, then $\sup_{\{i \notin I_H^{(n)}\}} \sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 0$, as $n \to \infty$.

Remark 2. The previous Assumption **A2** is similar to Assumption 2.2 in [7] but less restrictive.

When we have equally spaced observations all the Δ_i coincide with $\frac{T}{n}$ and $H' \equiv 1$. When the observations are more (less) concentrated around t, we have $H_t < 1$ ($H_t > 1$).

Note that, where it is defined, $H' \ge 0$, however if for example $n \cdot \min_i \Delta_i \to C > 0$ then $H' > 0$.

As an example, consider the sequence of partitions where the length of the first $[n/2]$ intervals $[t_{i-1}, t_i)$ is 2Φ and the length of the remaining $n - [n/2]$ is Φ. Then $\Phi = \frac{T}{n} \frac{1}{1+[\frac{n}{2}]\frac{1}{n}}$ and, for any $t \in (0, T]$, $H_t = \frac{4t}{3} I_{t\le\tau_1} + (\frac{4T}{9} + \frac{2t}{3}) I_{t>\tau_1}$ where $\tau_1 = 2T/3$. This function H is not differentiable at τ_1, so $m = 1$ and for any n, $I_H^{(n)}$ is the only i for which $[t_{i-1}, t_i)$ contains τ_1. Further, the interval $[t_{i-1}, t_i)$ for which $i \in I_H^{(n)}$ is the first interval with length Φ. As for the third condition in Assumption **A2**, for any n if $t_{i-1} \le \tau_1 < t_i$ then $\sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 2/3$, but if both t_{i-1}, t_i are on the same side of τ_1 (thus $i \notin I_H^{(n)}$) then $\sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 0$. Further, $\sup_{\{i \notin I_H^{(n)}\}} \sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| = |\frac{4}{3} - \frac{2}{1+[\frac{n}{2}]\frac{1}{n}}| \to 0$, and Assumption **A2** is satisfied.

*Assumption A*3. **Jump intensity and sizes.** For the process $a_s = \int_{|x|\le1} x\lambda(x, s)dx$ one of the following conditions holds true:

(i) a.s. $\sup_{i=1,\dots,n} \sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}| \to 0$;

(ii) $\sup_{i=1,\dots,n} \sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}| \xrightarrow{P} 0$;

(iii) there exists $\rho > 0 :\ \forall s, u$ such that $|s-u| \le \Delta$ then $E[|a_s - a_u|] \le C\Delta^\rho$.

Remark 3. i) The above requires regularity of the paths of the drift coefficient a. For instance, if a path $a_s(\omega)$ is continuous, then on $[0,T]$ it is uniformly continuous in s, and (i) is satisfied.

ii) If a.s. $\lambda(x,s)$ is bounded in x and, for any x, $\lambda(x,s)$ is continuous in s, then, in this framework of finite activity jumps, **A3** (i) is satisfied.

iii) If λ does not depend on s then a_t collapses on the finite r.v. $a \equiv \int_{|x|\le 1} x\lambda(x)dx$ for any t, and trivially all the three conditions (i) - (iii) are satisfied. For instance **A3** is satisfied if X has jumps with identically distributed Gaussian sizes.

iv) Condition (ii) of **A3** amounts to saying that the sequence of processes $G_s^{(n)} \doteq \sum_{i=1}^n (a_s - a_{t_{i-1}}) I_{s\in[t_{i-1},t_i)}$ tends to 0 ucp.

v) Condition (iii) is similar to a requirement given in Assumption 2.1 in [7].

The following definition helps to focus on the asymptotic behavior of $T_{\bar t}^n$: given a deterministic function $f(x)$ we set

$$F^n(X) \doteq \sum_{i=1}^n K_i f(\Delta_i X). \tag{6}$$

With $f(x) = x$ we obtain the numerator of $T_{\bar t}^n$, with $f(x) = x^2$ the squared denominator. Note that here we are only interested in the r.v. $F^n(X)$ (rather than in a process), which is computed using *all* the increments $\Delta_i X$ with t_i from t_1 to t_n. The next Lemma describes the asymptotic behavior of $F^n(X)$, and is used in the proofs of both Theorems 1 and 2. It is proved in Sec. 7.

Lemma 1. *If a.s.* $\lambda(\mathbb{R}\times[0,T]) < \infty$ *and* $L \doteq \big(\int_0^t \int_{\mathbb{R}} x\mu(dx,ds)\big)_{t\ge 0}$, *then under (4), if K is continuous at 0 and* $\lim_{x\to\pm\infty} K(x) = 0$, *then for any real function $f(x)$ continuous on $\mathbb{R}$ we have*

$$F^n(L) \xrightarrow{a.s.} F(L) \doteq K(0) f(\Delta L_{\bar t}).$$

From the Lemma, the limit of $T^n_{\bar t}$ is almost immediately obtained if $\Delta X_{\bar t} \neq 0$. On the other hand, if $\Delta X_{\bar t} = 0$ both the numerator and the denominator of $T^n_{\bar t}$ tend to 0, and we need some work to catch the leading terms. Note that in the case where $\bar t$ is not a jump time, if the drift in X is absent $T^n_{\bar t}(X)$ may not be defined. This is the case for instance when $N_T = 0$; or when $N_T \geq 1$ but the support of K is bounded. If e.g. $K(x)$ is a Lipschitz continuous approximation of $I_{\{|x|\leq \frac{1}{2}\}}$, for sufficiently small h then both $\sum_{i=1}^n K_i \Delta_i X = 0$ and $\sum_{i=1}^n K_i (\Delta_i X)^2 = 0$, thus $T^n_{\bar t}(X)$ is not defined. Note that it is always true that if $\sum_{i=1}^n K_i (\Delta_i X)^2 = 0$ then also $\sum_{i=1}^n K_i \Delta_i X = 0$.
The behavior of $T^n_{\bar t}$ in this framework is as follows:

Theorem 1. *Under model (2), conditions (4) and $\frac{\Delta}{h^2} \to 0$,*

*a) If K satisfies Assumption **A1.1** and a.s. $\sup_{s\in[0,T]} \int_{|x|\leq 1} |x|\lambda(x,s)dx < +\infty$, then the following holds true a.s.: if $\bar t$ is a jump time then*

$$T^n_{\bar t} \to \sqrt{K(0)} \cdot sgn(\Delta X_{\bar t}).$$

*b) Under Assumptions **A1**, **A2** and **A3**(i) and if $(a_s)_{s\geq 0}$ is làdlàg then the following holds true a.s.: if $\Delta X_{\bar t} = 0$ but $N_T > 0$, $a^\star_{\bar t} \neq 0$ and $H'_{\bar t \pm} > 0$, then*

$$T^n_{\bar t} \to sgn(-a^\star_{\bar t}) \cdot \infty,$$

where $a^\star$ is defined as in (5).
*If, within b), Assumption **A3**(ii) is replaced by either Assumption **A3**(ii) or Assumption **A3**(iii) then the result is in probability.*

Remark 4. i) If, on ω, a is continuous at $\bar t$ then $a^\star_{\bar t} = a_{\bar t}$.

ii) Note that, since our process X is an Ito semimartingale, it has "no fixed times of discontinuities," namely $P\{\Delta X_{\bar t} \neq 0\} = 0$. Despite this, point a) of the theorem is relevant from the practical point of view, because we only have at hand one specific path $\{X_s(\omega), s \in [0,T]\}$, on which at $\bar t$ a jump could well have occurred.

Corollary 1. Contribution of the drift to $T^n_{\bar t}$. *Let $D_t = \int_0^t b_s ds$. Under **A1**, **A2**, **A3**(i) and $\frac{\Delta}{h^2} \to 0$, if $(b_s)_{s\geq 0}$ is làdlàg; $b^\star_{\bar t} \neq 0$; and $H'_{\bar t\pm} > 0$, then*

$$T^n_{\bar t}(D) \to sgn(b^\star_{\bar t}) \cdot \infty.$$

If instead Assumption **A3***(i) is replaced by either Assumption* **A3***(ii) or Assumption* **A3***(iii) then the result is in probability.*

In fact from the Proof of the Theorem it follows that $\sum_{i=1}^n K_i \Delta_i D \simeq b_{\bar t}^\star h$, while $\sum_{i=1}^n K_i (\Delta_i D)^2 \simeq (H' b^2)_{\bar t}^\star h \Delta$. $\square$

If the jump process is represented in the form

$$L_t = \sum_{p=1}^{N_t} c_p,$$

without compensation, then the drift coefficient $a_s \equiv 0$, and part b) of the theorem above does not apply. However, the limit behavior of $T_{\bar t}^n(L)$ does not change if $\bar t$ is a jump time, while $T_{\bar t}^n(L) \to 0$ if $\Delta L_{\bar t} = 0$. This is summarized below.

Corollary 2. Contribution of the sum of the jumps to $T_{\bar t}^n$. *Let $L_t = \sum_{p=1}^{N_t} c_p$. We have*

a) under **A1.1** *and $\frac{\Delta}{h^2} \to 0$, if $\bar t$ is a jump time then*

$$T_{\bar t}^n(L) \to \sqrt{K(0)} \cdot sgn(c_{\bar t});$$

b) under **A1**, **A2**, **A3***(i) and $\frac{\Delta}{h^2} \to 0$, if $\Delta L_{\bar t} = 0$ but $N_T > 0$ and $spt(K) = \mathbb{R}$, then*

$$T_{\bar t}^n(L) \to 0.$$

In fact, from the proof of the Theorem and using the same notation, we obtain the following: if $\bar t$ is a jump time, then for small Δ we obtain

$$T_{\bar t}^n(L) = \frac{\sum_{p=1}^{N_T} K_{i_p} c_p}{\sqrt{\sum_{p=1}^{N_T} K_{i_p} c_p^2}} \simeq \frac{K(0) c_{\bar t}}{\sqrt{K(0) c_{\bar t}^2}} = \sqrt{K(0)} \cdot sgn(c_{\bar t}).$$

If $\bar t$ is not a jump time, since $N_T \geq 1$ and $spt(K) = \mathbb{R}$, with $[t_{i_p - 1}, t_{i_p}[$ being the unique interval of the partition containing the time of the p-th jump and $\underline{p}$ being the number such that $|\bar t - S_{\underline{p}}| \doteq \min_p |\bar t - S_p| > 0$, then, for small Δ, we have

$$T_{\bar t}^n(L) = \frac{\sum_{p=1}^{N_T} K_{i_p} c_p}{\sqrt{\sum_{p=1}^{N_T} K_{i_p} c_p^2}} \simeq \frac{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right) c_{\underline{p}}}{\sqrt{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right) c_{\underline{p}}^2}} = \sqrt{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right)} \cdot sgn(c_{\underline{p}}) \to 0. \quad \square$$

Note that, in this framework of FA jumps, $T^n_{\bar{t}}$ could provide a test for the presence of a drift component in the DGP: if a drift $\int a_s ds$ is present in X then either $|T^n_{\bar{t}}| \to \sqrt{K(0)}$ or $|T^n_{\bar{t}}| \to \infty$; if not then $T^n_{\bar{t}} \to 0$. We comment on the potential use of $T^n_{\bar{t}}$ as a test for a jump at $\bar{t}$ in the next Section.

Remark 5. The above result is consistent with Thm 4 in ([2]). The Authors consider a process of type $Y + \bar{J}$, where Y as in (1) and $\bar{J}_t = UI_{0<\tau\leq t}$ is a single jump occurring at time τ. They analyze T^n precisely at the jump time, with the result that $T^n_\tau \xrightarrow{P} \sqrt{\frac{K(0)}{K_2}} \cdot sgn(U)$. Their constant K_2 is derived from their definition $\sqrt{\frac{h}{K_2}} \frac{\hat{b}_{\bar{t}}}{\hat{\sigma}_{\bar{t}}}$ of the test statistic, while in this paper we consider $\sqrt{h} \frac{\hat{b}_{\bar{t}}}{\hat{\sigma}_{\bar{t}}}$. Within their framework the model contains a non-vanishing Brownian component. When no jumps occur at $\bar{t}$, the Brownian motion dominates all the other components, thus the specific contribution of a non-exploding drift and of the jumps are not explicit. It follows that it is not possible to deduce the asymptotics for $T^n_{\bar{t}}$ from their framework in the limit case when the Brownian term is absent.

3.2. Infinite activity jumps

When the jumps have infinite activity, it turns out that if $\Delta X_{\bar{t}} \neq 0$ (again an event of zero probability), then $T^n_{\bar{t}}$ has the same limit as in the FA jump case. When $\Delta X_{\bar{t}} = 0$, as above, both the numerator and the denominator tend to 0 in probability, and the freneticism of the activity of the small jumps is crucial in determining the convergence speeds. Therefore we assume that the jump activity index is a constant α, and we consider a generalized α-stable process (assumption **IA3**), for which the jump activity is wilder when α is higher. The large jumps are always of FA, their jump activity index is 0 and they do not contribute to determining the convergence speeds we are interested in. We show that the limit of $T^n_{\bar{t}}$ is different when $\alpha < 1$ (finite variation jumps) or $\alpha > 1$ (infinite variation jumps). For sake of simplicity we concentrate on the case of equally spaced observations (assumption **IA2**); further, we add the technical requirement **IA1** on the Kernel function, which is satisfied at least in the Gaussian kernel case.

*Assumption IA*1. **Kernel.** Given a deterministic function φ defined on $\mathbb{R}_+$, we say that K satisfies **IA1** for φ if **A1** is satisfied and: K is monotonically non-decreasing on $\mathbb{R}_-$ and non-increasing on $\mathbb{R}_+$ and there exists a deterministic function ε_h such that as $h \to 0$

$$\varepsilon_h \to 0, \quad \frac{\varepsilon_h}{h} \to +\infty \quad \text{and} \quad \frac{K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)} \to +\infty. \tag{7}$$

Remark 6. For instance, with φ equal to any of the speed functions $\varphi_\alpha(h)$ or $\psi_\alpha(h)$ at (10) below, with the Gaussian kernel, and with the function

$$\varepsilon_h \doteq h\sqrt{\log\log\frac{1}{h}} \tag{8}$$

the above conditions (7) are satisfied for any $\alpha \in (0, 2)$.

*Assumption IA*2. **Partitions.** We take $\Delta_i = \Delta$ for all n, for all $i = 1, \ldots, n$.

*Assumption IA*3. **Small jumps.** The jump process has the form $X = \tilde{X} + X^1$, where

$$\tilde{X}_t = \int_0^t \int_{|x|\le 1} x\tilde{\mu}(dx, ds), \quad X_t^1 = \int_0^t \int_{|x|>1} x\mu(dx, ds),$$

the compensating measure of the jumps smaller than 1 has the form $\nu(dx, ds) = \lambda(x, s)dxds$, where the, possibly random, intensity $\lambda(x, s)$is given by

$$\lambda(x, s) = \frac{A_+ g(x, s)}{x^{1+\alpha}} I_{\{0<x\le 1\}} + \frac{A_- g(x, s)}{|x|^{1+\alpha}} I_{\{-1\le x<0\}},$$

where $A_+, A_- > 0$, $\alpha \in (0, 2)$, and $0 \le g(x, s) \le 1$ is a random predictable function defined on $\Omega \times \mathbb{R} \times [0, T]$. Further, the random function g is such that: if $\alpha \le 1$: there exists $r < \alpha < 1$ such that $\int_{|x|\le 1} |x|^r \frac{1-g(x,s)}{|x|^{1+\alpha}} dx \le C$ for any $(\omega, s) \in \Omega \times [0, T]$; and $a_s = \int_{|x|\le 1} x\lambda(x, s)dx$ satisfies **A3** of Sec. 3.1;
if $\alpha > 1$ we have $\int_{|x|\le 1} |x| \frac{1-g(x,s)}{|x|^{1+\alpha}} dx \le C$ for any (ω, s).

Remark 7. i) About process a. When X is an α-stable Lévy process, or a CGMY process with $\alpha \in (0, 1)$ then assumption **A3** is satisfied, because λ does not depend on (ω, s) and a_s is a constant.
ii) Examples of processes satisfying **IA3**. The small jumps of an α-stable process

satisfy **IA3** with the constant $g(x,s) \equiv 1$. We recall that α-stable processes necessarily have $\alpha \in (0,2]$ and the only 2-stable process is the Brownian motion. In particular our framework includes cases where X is a subordinated process. For instance, a γ stable subordinator S without drift, has infinite activity and $\gamma < 1$; if we subordinate a symmetric β stable process Z with $\beta \in (0,2]$, then the subordinated process $X = Z_S$ is stable with index $\alpha = \gamma\beta \in (0,2)$ (see [4], p.110). The case where Z is a Brownian motion is included.
The small jumps of a CGMY process satisfy **IA3** with $g(x,s) \equiv e^{-Gx} I_{\{x<0\}} + e^{-Mx} I_{\{x>0\}}$. More generally, if $1 - g(x,s) \leq C|x|^\eta$, for all (ω, s) and some $\eta > 0$, then the assumption is satisfied for instance in the following cases: if $\alpha < 1$ and $\eta \in (0, 2\alpha)$, with $r = \alpha - \eta/2$; if $\alpha \geq 1$ and $\eta > \alpha - 1$ (for instance $\eta = \alpha/2$).
iii) Assumption **IA3** aims to have a constant *jump activity index* α for X (as defined in [1] p.2). Such an index is identified by the component $\frac{1}{|x|^{1+\alpha}}$ of the Lévy measure of X, as the latter conditions prevent g from increasing the jump activity.
Assumption 2 in [1] is similar to **IA3**, and requires a constant jump activity index as well. The α-stable process is the prototypical example within both frameworks. Showing some results for such a process is a crucial first step, because then, with specific technical tools, it is often possible to extend their validity under the more general Ito SM framework.
iv) We obtain the same results if Assumption **A3** is made for the compensated measure of the jumps smaller than any boundary $c > 0$ in place of 1.

Notation 3. $E_{i-1}[Z] = E[Z|\mathcal{F}_{t_{i-1}}]$. For each $\alpha \in (0,2)$ let $Z_{i,\alpha}$, $i = 1,2$, be random variables characterized by $\beta = \frac{A_+ - A_-}{A_+ + A_-}$,

$$E[e^{isZ_{1,\alpha}}] = e^{-|s|^\alpha K_{(\alpha)} \left|\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)\right| \cdot (A_+ + A_-)\left(1 - i\beta\tan\left(\frac{\alpha\pi}{2}\right) sgn(s)\right)}; \tag{9}$$

$Z_{2,\alpha} \geq 0$,

$$E[e^{-sZ_{2,\alpha}}] = \begin{cases} e^{-s^{\frac{\alpha}{2}} \cdot \frac{2^\alpha}{\sqrt{\pi}} K_{(\alpha/2)} (A_+ + A\_) \Gamma\left(\frac{\alpha+1}{2}\right) \left|\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right|}, & \alpha \in (0,1) \cup (1,2) \\ e^{-s^{\frac{\alpha}{2}} \cdot 2^{\alpha-1} \sqrt{\pi} K_{(\alpha/2)} (A_+ + A\_) \Gamma\left(\frac{\alpha+1}{2}\right)}, & \alpha = 1. \end{cases}$$

For each $\alpha \in (0,2)$ let us define on $\mathbb{R}_+$ the speed functions of our interest

$$\varphi_\alpha(h) \doteq \begin{cases} h & \text{if } \alpha \in (0,1), \\ h \log \frac{1}{h} & \text{if } \alpha = 1, \\ h^{\frac{1}{\alpha}} & \text{if } \alpha \in (1,2); \end{cases} \qquad \psi_\alpha(h) \doteq h^{\frac{2}{\alpha}}, \tag{10}$$

where φ_α is shown to be the speed (of convergence to 0 when $\Delta X_{\bar{t}} = 0$) of the numerator of $T^n_{\bar{t}}$ and ψ_α the speed of the squared denominator.

Remark 8. The random variable $Z_{1,\alpha}$ is α-stable of type $S_\alpha(c, \beta, 0)$, with scale parameter $c = K_{(\alpha)} \, |\Gamma(-\alpha)| \cdot \left|\cos\left(\frac{\alpha\pi}{2}\right)\right| (A_+ + A_-)$, skewness parameter β and zero shift parameter (parametrization of [8], thm 14.15).

By contrast, the law of $Z_{2,\alpha}$ cannot be stable, in that $Z_{2,\alpha}$ is non-negative with positive jump sizes, so it would have to be $\beta = 1$ but then the characteristic function of an $S_{\alpha/2}(c, 1, 0)$ would be not compatible with the above Laplace transform. $Z_{2,\alpha}$ comes from the leading term of a squared α-stable random variable in Lemma 5, but does not have the law of a squared α-stable r.v..

Note that $\Gamma(-\alpha) < 0$ and $\cos\left(\frac{\pi\alpha}{2}\right) > 0$ for $\alpha \in (0,1)$, while $\Gamma(-\alpha) > 0$ and $\cos\left(\frac{\pi\alpha}{2}\right) < 0$ for $\alpha \in (1,2)$. Thus $\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)$ is negative for all $\alpha \neq 1$.

The following Theorem provides the asymptotic behavior of the drift burst test statistic $T^n_{\bar{t}}$ within the pure jump model X.

Theorem 2. *a) Under Assumption* **A1***, (4) and* $\frac{\Delta}{h^2} \to 0$*, with either* $f(x) = x$ *or* $f(x) = x^2$ *we still obtain*

$$F^n(X) \xrightarrow{P} F(X) \doteq K(0) f(\Delta X_{\bar{t}}), \tag{11}$$

having used the notation in (6).

b) Let the kernel satisfy **A1** *and be such that* $K^{\alpha/2}$ *is Lipschitz and in* $L^1(\mathbb{R})$*. Assume that* K *satisfies* **IA1** *for both the two functions* φ_α *and* ψ_α *in (10), and assume* **IA2, IA3***, the asymptotics (4) and* $\frac{\Delta}{h^2} \to 0$*.*
In the case $\alpha < 1$ *assume also that* $a^\star_{\bar{t}} \neq 0$*.*
In the case $\alpha = 1$ *let* $A_+ \neq A_-$*,* $\sqrt{K} \log K$ *be bounded and* $\frac{\Delta}{h^2} \log^2 \frac{1}{h} \to 0$*.*

Then we have

$$\text{if } \alpha \in (0,1), \quad T_{\bar t}^n \xrightarrow{P} -sgn(a_{\bar t}^\star)\cdot\infty,$$

$$\text{if } \alpha = 1, \quad T_{\bar t}^n \xrightarrow{P} -sgn(A_+ - A_-)\cdot\infty.$$

$$\text{If } \alpha \in (1,2), \quad T_{\bar t}^n \text{ cannot diverge,}$$

because numerator and denominator have the same speed of convergence to 0.

c) Under the assumptions of part b) in the case $\alpha > 1$, if $A_+ = A_-$ then

$$\alpha \in (1,2), \quad |T_{\bar t}^n| \xrightarrow{d} Z_\alpha \doteq \frac{|Z_{1,\alpha}|}{\sqrt{Z_{2,\alpha}}}.$$

Remark 9. i) Result a) above implies that there exists a subsequence $T_{\bar t}^{n_k}$ such that $T_{\bar t}^{n_k} I_{\{\Delta X_{\bar t}\neq 0\}} \xrightarrow{a.s.} \sqrt{K(0)}\cdot sgn(\Delta X_{\bar t}) I_{\{\Delta X_{\bar t}\neq 0\}}$. In particular, if on a given ω we have $\Delta X_{\bar t} \neq 0$ then $T_{\bar t}^{n_k} \to \sqrt{K(0)}\cdot sgn(\Delta X_{\bar t})$. However $P\{\Delta X_{\bar t} \neq 0\} = 0$.

ii) At point b), in case $\alpha < 1$ we have a.s. $|a_{\bar t}^\star| < \infty$, and the above result is in continuity with Theorem 1, part b). If $\tilde X$ is given by the compensated jumps smaller than 1 of an α stable process with $\alpha < 1$, then $a_{\bar t}^\star \equiv a = \frac{A_+ - A_-}{1-\alpha}$.

iii) The requirement $A_+ \neq A_-$ is in line with the requirement $a^\star \neq 0$ of the case $\alpha < 1$ or of Theorem 1 part b), and ensures that the drift of X is the leading term at the numerator of $T_{\bar t}^n$.

Consider the case where $\alpha = 1$ and $\Delta X_{\bar t} = 0$. When $A_+ \neq A_-$, the numerator of $T_{\bar t}^n$ tends to 0 at speed $h \log \frac{1}{h}$. When instead $A_+ = A_-$ then a.s. the numerator of $T_{\bar t}^n$ tends to 0 at the faster rate h. In fact the term determining the speed of the numerator is $\sum_{i=1}^n K_i \Delta_i \tilde X$, and within the first step of the proof of Lemma 4 we see that the exponent of the characteristic function loses the term containing $\sin v - v$, and we can apply Lemma 3 with $\varphi(h) = h$, rather than with $\varphi(h) = h \log \frac{1}{h}$. It follows that $T_{\bar t}^n$ does not diverge, because by Lemma 5 numerator and denominator have the same speed.

The same happens for $\alpha < 1$ when $a^\star = 0$: if also we assume that for any fixed $x > 0$ we have $K(\frac{x}{h}) << h^{\frac{2}{\alpha}}$, then (by Lemmas 4 and 5) numerator and denominator of $T_{\bar t}^n$ have the same speed $h^{\frac{1}{\alpha}}$ and $T_{\bar t}^n$ does not diverge.

iv) As for point c), the case $\alpha \in (1,2)$ with $A_+ \neq A_-$ requires further investigation. From the proof of Lemma 6, when $A_+ = A_-$ we obtain the joint Laplace transform of $(Z^2_{1,\alpha}, Z_{2,\alpha})$, and, since it cannot be factorized when $\alpha < 2$, the two random variables $Z_{1,\alpha}, Z_{2,\alpha}$ turn out not to be independent.

v) The jumps never cause $T^n_{\bar{t}}$ to explode: when the jumps have FV ($\alpha < 1$) or $\alpha = 1$ then the explosion is due to the compensator (drift part of X); when the jumps have IV ($\alpha > 1$) then $T^n_{\bar{t}}$ does not diverge. This proves that the presence of IV jumps in an Ito SM model cannot make the statistic $T^n_{\bar{t}}$ in [2] explode. This will be even more clear in the next Section.

vi) It is not clear whether it is possible to construct confidence intervals for Z_α starting from the Laplace transform of $(Z^2_{1,\alpha}, Z_{2,\alpha})$.
In case, at least under the assumption $A_+ = A_-$, $T^n_{\bar{t}}$ provides a test for FV jumps (in which case $|T^n_{\bar{t}}| \to +\infty$) against $\alpha > 1$ (in which case $|T^n_{\bar{t}}| \to Z_\alpha$), or a test for whether a jump occurred at $\bar{t}$ (in which case $|T^n_{\bar{t}}| \to \sqrt{K(0)}$) or did not occur (either $|T^n_{\bar{t}}| \to +\infty$ or $|T^n_{\bar{t}}| \to Z_\alpha$).

4. In the presence of a Brownian component

We now come back to the behavior of $T^n_{\bar{t}}$ when Y at (1) contains both a Brownian term and infinite variation jumps. In [2] it has been proved that in the presence of a Brownian component, when the jumps have finite variation, corresponding here to the case $\alpha < 1$, and there is no drift burst, then $T^n_{\bar{t}} \xrightarrow{d} \mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ denotes the law of a standard normal r.v.. The following corollary certifies that the same result also holds when the jumps have infinite variation, because in any case the Brownian component introduces the leading terms, both at the numerator and at the denominator of $T^n_{\bar{t}}$.

Corollary 3. *Let Y evolve following $dY_t = b_t dt + \sigma_t dW_t + dX_t$, Y_0 being $\mathcal{F}_0$-measurable, where $\{b_t\}_{t\geq 0}$ is a locally bounded and predictable drift process, $\{\sigma_t\}_{t\geq 0}$ is an adapted, càdlàg, a.s. strictly positive volatility process; $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and X is a pure-jump process for which the compensated small jumps are of generalized α-stable type, as in* **IA3**, *with $\alpha \in$*

$[1, 2)$. *Let the assumptions of Theorem 2, part b), be fulfilled. Then*

$$T_t^n(Y) = \frac{\sum_{i=1}^n K_i \Delta_i Y}{\sqrt{\sum_{i=1}^n K_i (\Delta_i Y)^2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

It follows that

(a) In the presence of a volatility component not vanishing at $\bar{t}$ we have

$\cdot$ $T_{\bar{t}}^n \xrightarrow{d} \mathcal{N}(0, 1)$ when there is no drift burst (whatever the variation of the jumps)

$\cdot$ $|T_{\bar{t}}^n| \xrightarrow{P} +\infty$ when at $\bar{t}$ there is a drift burst bigger than a volatility burst

(b) In the absence of a Brownian component and of drift burst, under **IA3** then

$\cdot$ $|T_{\bar{t}}^n|$ does not diverge if $\alpha \in (1, 2)$, for instance $|T_{\bar{t}}^n| \xrightarrow{d} Z_\alpha$ if $A_+ = A_-$

$\cdot$ $|T_{\bar{t}}^n| \xrightarrow{P} +\infty$ if $\alpha \in (0, 1]$.

As mentioned in the Introduction, tests based on discrete observations are available for assessing whether in an SM model without drift bursts a Brownian component is needed for a better explanation of the data. Potentially $|T_{\bar{t}}^n|$ may provide a further test.

5. Practical illustration

In this section we briefly illustrate the different behavior of $T_{\bar{t}}^n$ when $Y = \sigma W_t + X_t$ has different features. We first consider the case where we are given $n = 252 \cdot 84$ evenly spaced discrete observations of $H = 100$ simulated paths from the same data generating process. The step between two consecutive observations is $\Delta = 1/(252 * 84)$, the time horizon is $T = n\Delta = 1$ year and the Gaussian kernel of Remark 1 is used with bandwidth $h = \Delta^{0.45}$.

The first column on the left in Figure 1 shows the histograms of the values of $T_{\bar{t}}^n$ when $X_t = \sum_{i=1}^{N_t} Z_i$ is a compound Poisson process (CPP) possibly superposed with a Brownian motion with different volatilities. The second column shows the histograms of the values of $T_{\bar{t}}^n$ when $X_t = \sum_{i=1}^{N_t} Z_i - t\lambda \int_{|x|\le 1} x f(x) dx$ is a compound Poisson process with compensation of the jumps smaller than 1 (CPPComp) and possibly superposed with a Brownian motion. For both columns the annual jump intensity is $\lambda = 10$ and the jump sizes are i.i.d. Gaussian with law $\mathcal{N}(-0.1, 0.05^2)$ and density $f(x)$.[2]

The theoretical findings are clearly visible: the statistic explodes only when Y has no Brownian component and the drift component (the compensator of the small jumps) is not null (second column, top plot).
The columns from the third to the last show the histograms of the values of $T_{\bar{t}}^n$ when X is a one-sided CGMY jump process (only positive jumps) with compensation of the jumps smaller than 1 and jump intensity $\lambda(x) = 0.003 \cdot \frac{e^{-x}}{x^{1+\alpha}} I_{x>0}$, possibly superposed with a Brownian motion.[3] In this case too we can visualize the theoretical results: for $\sigma = 0$ when $\alpha < 1$ (top row of 3rd to 5th columns) the drift given by the compensator of the small jumps leads $T_{\bar{t}}^n$ to explode towards $-\infty$, while for $\alpha > 1$ the statistic displays a different, not symmetric, law. By contrast, as soon as $\sigma \neq 0$ the leading term both at numerator and at denominator of $T_{\bar{t}}^n$ is the Brownian motion, which pushes the statistic close to a Gaussian r.v..

We remark that if we could use higher frequency observations filtered out for microstructure noises, the asymptotic results would be even more evident, as in Figure 2, where in order to highlight the results we set $n = 252 \times 840$, then for CPP and CGMY jumps $\Delta = 1/(252 \times 84000)$, while for CPPComp $\Delta = 1/(252 \times 8400)$.

6. Multivariate extension

One may wonder whether for a multivariate process it is possible to obtain results similar to those obtained in the univariate case. This is subject to further research, and we briefly illustrate the problem.

[2] In order to produce more observations of $T_{\bar{t}}^n$, for each simulated path the statistic is computed on 50 evenly spaced time instants $\bar{t}$ within [0,T], as for each $\bar{t}$ the statistic has the same law.

[3] Simulation of the CGMY model is carried out by approximation with a compound Poisson process with jumps larger than $\varepsilon = 10^{-4}$ and proper intensity, as in [4], Example 6.9.

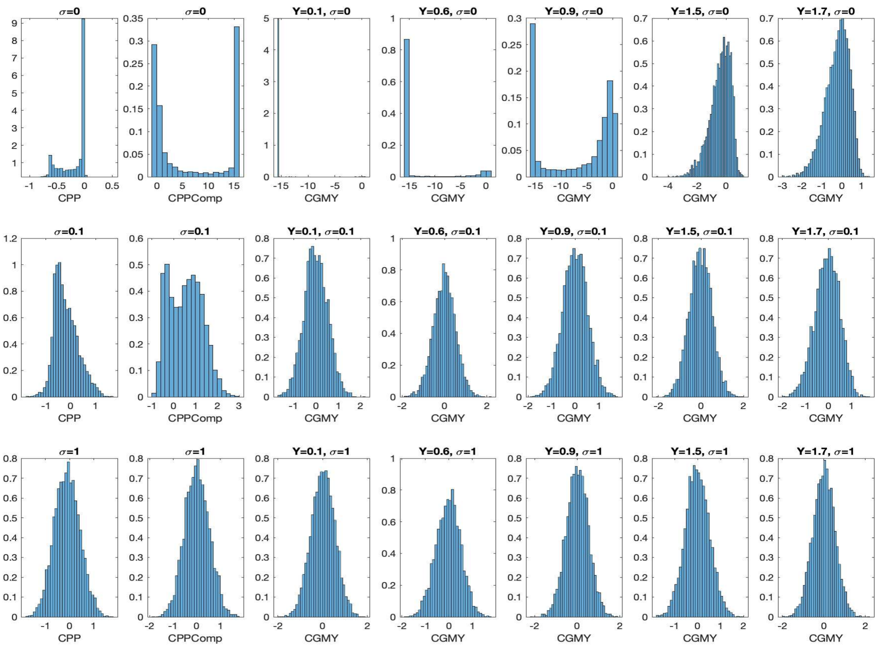

Figure 1: Histograms of $T^n_{\bar{t}}$ under different models Y. First column: first row, $Y \equiv X =$ CPP with no drift; second and third rows, $Y = X + \sigma W$ with different volatilities. Second column: first row, $Y \equiv X =$ CPPComp, with drift given by the compensator of the small jumps; second and third rows, $Y = X + \sigma W$. From the first to the last column: first row, $Y \equiv X =$ CGMY model with compensation of the small jumps; second and third rows, $Y = X + \sigma W$. $n = 252 \times 84, \Delta = 1/n$.

One could start by analyzing the pure jump model. Let us consider the bivariate jump process $X = (X^{(1)}, X^{(2)})$, where the components have constant jump indices α_1, α_2. When both $X^{(i)}$ satisfy our assumptions, we already know the behavior of the relative statistics $T^{(1),n}_{\bar{t}}, T^{(2),n}_{\bar{t}}$, and we would like to know the limit in distribution of the joint $\big(T^{(1),n}_{\bar{t}}, T^{(2),n}_{\bar{t}}\big)$. Depending on how the marginal statistics covariate, the confidence intervals of the joint law may differ, and the power of the joint test may be different.

If both the jump indices are larger than 1, along the lines of this paper, the above-mentioned limit could be obtained from the convergence in distribution

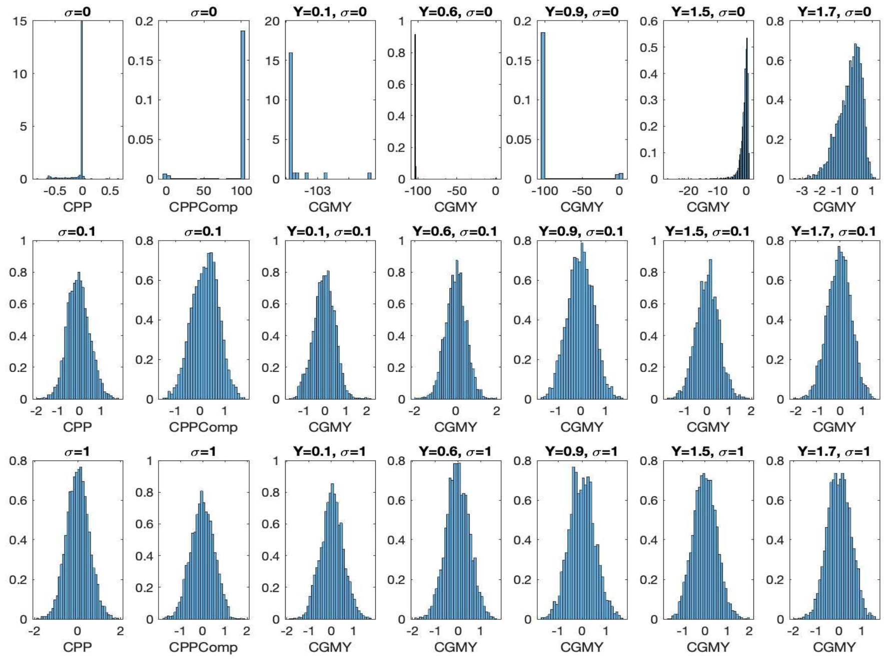

Figure 2: Histograms of $T_{\bar{t}}^n$ under the same models Y as in the previous figure. Now $n = 252 \times 840$ and for CPP and CGMY jumps $\Delta = 1/(252 \times 84000)$, while for CPPComp jumps $\Delta = 1/(252 \times 8400)$.

of

$$\left(\frac{\left(\sum_{i=1}^n K_i \Delta_i X^{(1)}\right)^2}{h^{2/\alpha_1}}, \frac{\sum_{i=1}^n K_i (\Delta_i X^{(1)})^2}{h^{2/\alpha_1}}, \frac{\left(\sum_{i=1}^n K_i \Delta_i X^{(2)}\right)^2}{h^{2/\alpha_2}}, \frac{\sum_{i=1}^n K_i (\Delta_i X^{(2)})^2}{h^{2/\alpha_2}} \right).$$

We expect that, at least when $A_+^{(i)} = A_-^{(i)}$, i=1,2 as in Lemma 6, the above to have the same convergence in distribution as

$$\left(\frac{\sum_{i=1}^n K_i^2 (\Delta_i \tilde{X}^{(1)})^2}{h^{2/\alpha_1}}, \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{X}^{(1)})^2}{h^{2/\alpha_1}}, \frac{\sum_{i=1}^n K_i^2 (\Delta_i \tilde{X}^{(2)})^2}{h^{2/\alpha_2}}, \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{X}^{(2)})^2}{h^{2/\alpha_2}} \right). \tag{12}$$

One could start by finding the result for a process X where $X^{(2)} = \rho X^{(1)} + X^{(3)}$, with $\left(X^{(1)}, X^{(3)}\right)$ a Lévy process with stable marginals and with respective jump indices α_1, α_3, so if $\rho \neq 0$ then $\alpha_2 = \max\{\alpha_1, \alpha_3\}$. In this way, X is a linear transformation of $\left(X^{(1)}, X^{(3)}\right)$, so it still has independent increments and an expression for its characteristic function is available ([4], p.107). Using the approach of Lemmas 5, we now have

$$E\left[e^{-u_1(X_1^{(1)})^2-u_2(X_1^{(2)})^2}\right]=\int_{\mathbb{R}^2}\varphi_{(B^1,B^3)}(x_1,x_3)\cdot f(x_1,x_3)dx_1dx_3$$

where (B^1, B^3) is a bivariate centered Gaussian r.v. with variance-covariance matrix Σ explicitly depending on u_1, u_2, ρ and with characteristic function $\varphi_{(B^1,B^3)}(x_1, x_3)$, while $f(x_1, x_3)$ is the joint density of $(X_1^{(1)}, X_1^{(3)})$. However the above equals

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} & e^{iy_1x_1+iy_3x_3}g(y_1,y_3)dy_1dy_3\cdot f(x_1,x_3)dx_1dx_3\\ &=\int_{\mathbb{R}^2}E\left[e^{iy_1X_1^{(1)}+iy_3X_1^{(3)}}\right]g(y_1,y_3)dy_1dy_3,\end{aligned}$$

where $g(y_1, y_3)$ is the density of the joint law of (B^1, B^3). Thus we expect that the limit in n of the Laplace transform of the joint law of (12) can be computed, and thus information on the asymptotic law of $(T_{\bar{t}}^{(1),n}, T_{\bar{t}}^{(2),n})$ can be obtained.

As for a bivariate version $Y = (Y^{(1)}, Y^{(2)})$, of the complete model, we already know that in the presence of a Brownian component, the latter dominates all the other parts, thus the joint asymptotic distribution of $\left(T_{\bar{t}}^{(1),n}, T_{\bar{t}}^{(2),n}\right)$ can be reduced to finding the joint asymptotic distribution of

$$\left(\frac{\left(\sum_{i=1}^n K_i\Delta_i\sigma^{(1)}.W^{(1)}\right)^2}{h},\ \frac{\sum_{i=1}^n K_i(\Delta_i\sigma^{(1)}.W^{(1)})^2}{h},\ \frac{\left(\sum_{i=1}^n K_i\Delta_i\sigma^{(2)}.W^{(2)}\right)^2}{h},\ \frac{\sum_{i=1}^n K_i(\Delta_i\sigma^{(2)}.W^{(2)})^2}{h}\right),$$

where $\sigma.W$ indicates the Ito integral process of σ in dW. But since $\frac{\sum_{i=1}^n K_i(\Delta_i\sigma.W)^2}{h}\xrightarrow{P}\sigma_{\bar{t}}^2$, if we show that

$$\left(\frac{\left(\sum_{i=1}^n K_i\Delta_i\sigma^{(1)}.W^{(1)}\right)^2}{h},\ \frac{\left(\sum_{i=1}^n K_i\Delta_i\sigma^{(2)}.W^{(2)}\right)^2}{h}\right)$$

converges stably to a bivariate r.v. then we can immediately conclude. Again one could consider $W^{(2)} = \rho^W W^{(1)} + W^{(3)}$ with independent Brownian motions $W^{(1)}, W^{(3)}$. Then we expect the result to be obtained using the multidimensional theorem on the stable convergence of triangular arrays ([6]).

7. Proofs of Lemma 1 and of the two Theorems

Proof of Lemma 1. For fixed ω, for any given jump time $S_p = S_p(\omega)$ of L and any integer n, let $i_p= i_p(\omega)$ be the right extreme of the unique interval $[t_{i-1}, t_i)$ containing S_p.

For the fixed ω, $\sum_{p=1}^{N_t} c_p$ is a step-wise constant function of t, so each increment $\Delta_i L$ either is 0, if $[t_{i-1}, t_i)$ does not contain jump times, or is $\sum_{p=1}^{\Delta_i N} c_p$, if $[t_{i-1}, t_i)$ contains some instants S_p. Since the time horizon T is finite and fixed, for sufficiently small Δ we have $0 \le \Delta_i N \le 1$ for all $i = 1, \dots, n$, thus $\Delta_i L$ either is 0 or reduces to a single $c_p \in \mathbb{R} - \{0\}$, and $\sum_{i=1}^n K_i f\Big(\sum_{p=1}^{\Delta_i N} c_p\Big)$ reduces to $\sum_{p=1}^{N_T} K_{i_p} f(c_p)$.

a) When $\bar{t}$ is a jump time then it coincides with one of the S_p, say $S_{\bar{p}} \doteq \bar{t}$, while, if some other jumps occurred (i.e. $N_T \ge 2$), for the other indices p we have $\Delta S \doteq \min_{p \ne \bar{p}} |S_p - \bar{t}| > 0$. For $\Delta \to 0$ we have that, for all $p = 1, \dots, N_T$, $t_{i_p-1} \to S_p$, so that $\bar{t} - t_{i_{\bar{p}}-1} \to 0$, and since $|\bar{t} - t_{i_{\bar{p}}-1}| \le \Delta_{i_p} \le \Delta$, we have $\frac{|\bar{t} - t_{i_{\bar{p}}-1}|}{h} \le \frac{\Delta}{h} \to 0$, thus $K_{i_{\bar{p}}} f(c_{\bar{p}}) \to K(0) f(c_{\bar{p}}) = K(0) f(\Delta L_{\bar{t}})$.
On the other hand, if $N_T \ge 2$, for $p \ne \bar{p}$ we have that $|\bar{t} - t_{i_p-1}| \to |\bar{t} - S_p| \ge \Delta S > 0$, thus $\frac{|\bar{t} - t_{i_p-1}|}{h} \to +\infty$, and $K_{i_p} \to 0$. So, for $p \ne \bar{p}$, $K_{i_p} f(c_p) \to 0$.
In other words, for sufficiently small Δ, $\sum_{i=1}^n K_i f\Big(\sum_{p=1}^{\Delta_i N} c_p\Big)$ only contains N_T non-zero terms, and all of them tend to 0 but one. Only the term for which $[t_{i-1}, t_i)$ contains $S_{\bar{p}} = \bar{t}$ has a non-zero limit, amounting to $K(0) f(c_{\bar{p}}) = K(0) f(\Delta L_{\bar{t}})$.

b) When $\bar{t}$ is not a jump time, we have that, for any given ω, each S_p is at positive distance from $\bar{t}$: we define $\underline{p}$ through

$$|\bar{t} - S_{\underline{p}}| \doteq \min_p |\bar{t} - S_p| > 0,$$

and again, for sufficiently small $\Delta = \Delta(\omega)$, we have $\sum_{p=1}^{N_T} K_{i_p} f(\Delta_{i_p} L) = \sum_{p=1}^{N_T} K_{i_p} f(c_p)$, which is a sum of N_T terms, where now all the terms K_{i_p} tend to 0, because, similarly as above, $t_{i_p-1} \to S_p$ but $|\bar{t} - S_p| \ge |\bar{t} - S_{\underline{p}}| > 0$, thus $\frac{|\bar{t} - t_{i_p-1}|}{h} \to +\infty$. However, since $f(\Delta L_{\bar{t}}) = 0$ we can also write $\sum_{i=1}^n K_i f(\Delta_i L) \to K(0) f(\Delta L_{\bar{t}})$. $\square$

gathers properties of the kernel function which are used numerous times

The following Lemma, which is proved in the Appendix, gathers properties of the kernel function which are used numerous times. Point 1) is similar to point 1) of Lemma A.1 in [7], but is adapted to the present framework.

Lemma 2. *Whatever $\bar{t} \in (0,T)$ is, under (4), the following hold true:*
1) [Lemma A.1 (i) in [7]]. For a sequence of processes $b^{(n)}$ bounded by the same constant C, for any Lipschitz function $K(x)$ with Lipschitz constant L and $\frac{\Delta}{h^2} \to 0$ then

$$\int_0^T \frac{1}{h} K\Big(\frac{\bar{t}-s}{h}\Big) b_s^{(n)} ds - \sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar{t}-t_{i-1}}{h}\Big) \int_{t_{i-1}}^{t_i} b_s^{(n)} ds = O_{a.s.}\left(\frac{\Delta}{h^2}\right)$$

2) If K is Lipschitz, $K \in L^1(\mathbb{R})$ and $\frac{\Delta}{h^2} \to 0$ then $\frac{\sum_{i=1}^n K_i \Delta_i}{h} \to K_{(1)} = \int_{\mathbb{R}} K(u)du$.
3) If K^2 is Lipschitz, has $K_{(2)} = \int_{\mathbb{R}} K^2(x)dx < \infty$ and $\frac{\Delta}{h^2} \to 0$ then $\frac{\sum_{i=1}^n K_i^2 \Delta_i}{h} \to K_{(2)}$.
4) For a làdlàg bounded process b and any density function $K(x)$ on $\mathbb{R}$ we have a.s.

$$\int_0^T \frac{1}{h} K\Big(\frac{\bar{t}-s}{h}\Big) b_s ds \to b_{\bar{t}}^{\star}.$$

5) If K is Lipschitz, $K \in L^1(\mathbb{R})$, $\frac{\Delta}{h^2} \to 0$ and $b^{(n)}$ are processes for which
(i) a.s. $\sup_{i=1,\dots,n} \sup_{s\in[t_{i-1},t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \to 0$,
then a.s.

$$\sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar{t}-t_{i-1}}{h}\Big) b_{t_{i-1}}^{(n)} \Delta_i \simeq \sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar{t}-t_{i-1}}{h}\Big) \int_{t_{i-1}}^{t_i} b_s^{(n)} ds.$$

If the last assumption is replaced by either
(ii) $\sup_{i=1,\dots,n} \sup_{s\in[t_{i-1},t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \stackrel{P}{\to} 0$
or
(iii) there exists $\rho > 0 :\ \forall s, u$ such that $|s-u| \le \Delta$ then $E[|b_s^{(n)} - b_u^{(n)}|] \le C\Delta^\rho$,
then the above result holds in probability rather than a.s..
6) If K^2 is Lipschitz and in $L^1(\mathbb{R})$, then under (4) and $\frac{\Delta}{h^2} \to 0$

$$\sum_{i=1}^n \sum_{j<i} K_i^2 K_j^2 \Delta_j \Delta_i \simeq \int_0^T K_u^2 \int_0^u K_s^2 ds du \simeq h^2 C_K,$$

where $C_k \doteq \int_{\mathbb{R}} K^2(v) \int_v^{+\infty} K^2(w)dwdv > 0.$

Proof of Theorem 1.

a) When $\bar{t}$ is a jump time. We show that a.s.

1a) $\sum_{i=1}^n K_i \Delta_i X \to K(0)\Delta X_{\bar{t}}$,

2a) $\sum_{i=1}^n K_i (\Delta_i X)^2 \to K(0)\Big(\Delta X_{\bar{t}}\Big)^2$,

which are sufficient to conclude.

As for 1a), using Lemma 1 for process $L \cdot \int_0^{\cdot} \int_{\mathbb{R}} x\mu(dx, ds)$, it remains to check that $\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds \overset{a.s.}{\to} 0$, which is almost immediate. In fact, we have a.s.

$$\left| \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds \right| \leq \left(\sup_s \int_{|x|\leq 1} |x|\lambda(x,s)dx \right) h \cdot \frac{\sum_{i=1}^n K_i \Delta_i}{h}.$$

Since the last factor above tends a.s. to $K_{(1)} = 1$ we are done.

In order to show 2a) we write $\sum_{i=1}^n K_i (\Delta_i X)^2$ as

$$\sum_{i=1}^n K_i \Big(\sum_{p=1}^{\Delta_i N} c_p \Big)^2 + \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} a_s ds \Big)^2 - 2 \sum_{i=1}^n K_i \Big(\sum_{p=1}^{\Delta_i N} c_p \Big) \int_{t_{i-1}}^{t_i} a_s ds. \quad (13)$$

By Lemma 1 the first term tends to $K(0)(\Delta X_{\bar{t}})^2$. The second term of (13) similarly as above tends to 0, because it is bounded from above by

$$\sum_{i=1}^n K_i \big(\sup_s \int_{|x|\leq 1} \lambda(x,s)dx \big)^2 \Delta_i^2 \leq C\Delta h \frac{\sum_{i=1}^n K_i \Delta_i}{h} \to 0.$$

The third term in (13) is a negligible mixed term. In fact, for small Δ it becomes

$$-2 \sum_{p=1}^{N_T} K_{i_p} c_p \int_{t_{i_p-1}}^{i_p} a_s ds : \quad (14)$$

since on the fixed ω only finitely many jumps occurred, and each jump has finite size, the random number $\bar{c} \doteq \max_{p=1,\ldots,N_T} |c_p|$ is finite, further under Assumption **A1.1** the kernel K is bounded, then the latter sum is dominated in absolute value by

$$C \sum_{p=1}^{N_T} \Delta_{i_p} \sup_s \int_{|x|\leq 1} \lambda(x,s)dx \leq CN_T\Delta \to 0.$$

Thus 2a) follows and a) is proved.

b) When $\bar{t}$ is not a jump time. Within

$$\sum_{i=1}^n K_i \Delta_i X = \sum_{p=1}^{N_T} K_{i_p} \Delta_{i_p} L - \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds,$$

as above, the second sum tends a.s. to 0, and now also the first one does, by Lemma 1. The same happens at the denominator of $T_{\bar{t}}^n$, thus we have a limit form $\frac{0}{0}$, and we look for the speed at which the two terms of the quotient tend to zero.

For that, note that, by virtue of the assumption that if $\mu(\omega, \mathbb{R}, \{s\}) \neq 0$ then $\int_{\mathbb{R}} x\mu(dx, \{s\}) \neq 0$, for the fixed ω we have $|\underline{c}| \doteq \min_{p=1,\dots,N_T} |c_p| > 0$, and we can write $\sum_{i=1}^n K_i \Delta_i X$ as follows

$$\sum_{i=1}^n K_i \Delta_i X = \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|x|>|\underline{c}|} x\mu(dx, ds) - \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds. \tag{15}$$

For a sufficiently small $\Delta = \Delta(\omega)$ the first sum contains the N_T vanishing terms $K_{i_p} c_p = K\Big(\frac{\bar{t} - t_{i_p - 1}}{h}\Big) c_p$, the leading of which, when $h \to 0$, by Assumption **A1.2** is the one having the smallest $\Big|\frac{\bar{t} - t_{i_p - 1}}{h}\Big|$. Since for all p we have $t_{i_p - 1} \to S_p$, the slowest term is $K\Big(\frac{\bar{t} - t_{i_{\underline{p}} - 1}}{h}\Big)|c_{\underline{p}}|$, being $|c_{\underline{p}}| > 0$. In other words, for the given ω the first sum in (15) tends to zero at speed $K\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big)$. Using Lemma 2, points 1) and 4),

$$\frac{1}{h} \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds = \int_0^T \frac{1}{h} K\Big(\frac{\bar{t} - s}{h}\Big) a_s ds + O_{a.s.}\left(\frac{\Delta}{h^2}\right) \to a_{\bar{t}}^\star(\omega),$$

thus if $a_{\bar{t}}^\star(\omega) \neq 0$ the last sum in (15) tends to 0 as $-h a_{\bar{t}}^\star$, which, by Assumption **A1.2**, dominates $K\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big)$, so the numerator of $T_{\bar{t}}^n$ tends to zero as $-h a_{\bar{t}}^\star$.

As for the denominator of $T_{\bar{t}}^n$, from (13) analogously as above we find that the leading term of the first sum is $K\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big) c_{\underline{p}}^2$; the third sum, a.s., for small Δ is as in (14), thus it is bounded in absolute value by $C \sum_{p=1}^{N_T} K_{i_p} |c_p| \Delta_{i_p}$. The latter is in turn asymptotically dominated by $CK\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big)|c_{\underline{p}}|\Delta << CK\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big)$. This shows that the third sum is negligible with respect to the first one.

The second sum $\sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2$ in (13) is now shown to tend a.s. to 0 at speed $h\Delta \cdot (H' a^2)_{\bar{t}}^\star$. For that we proceed based on the following schedule:

1b) $\frac{1}{\Delta h} \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 \simeq \frac{1}{\Delta h} \sum_{i=1}^n K_i a_{t_{i-1}}^2 \Delta_i^2$

2b) $\frac{1}{\Delta h}\sum_{i=1}^{n} K_i a_{t_{i-1}}^2 \Delta_i^2 \simeq \int_0^T \frac{1}{h} K_s H_s' a_s^2 ds$

3b) $\int_0^T \frac{1}{h} K_s H_s' a_s^2 ds \to (H'a^2)^\star_{\bar t}$,

which proves that the denominator of $T^n_{\bar t}$ tends to 0 as

$$\sqrt{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right) + h\Delta (H'a^2)^\star_{\bar t}}. \tag{16}$$

However, from Assumption **A1.2** it will follow that the latter tends to 0 as $\sqrt{h\Delta \cdot (H'a^2)^\star_{\bar t}}$. Then note that

$$(H'a^2)^\star_{\bar t} = H'_{\bar t-} a^2_{\bar t-} K_+ + H'_{\bar t+} a^2_{\bar t+} K_- > 0,$$

because at least one between $a_{\bar t-}K_+$ and $a_{\bar t+}K_-$ is non zero, then at least one between $a^2_{\bar t-}K_+$ and $a^2_{\bar t+}K_-$ is strictly positive, and both $H'_{\bar t+}, H'_{\bar t-}$ are strictly positive. Thus it will also follow that

$$T^n_{\bar t} \simeq \frac{-ha^\star_{\bar t}}{\sqrt{h\Delta (H'a^2)^\star_{\bar t}}} \simeq -\sqrt{\frac{h}{\Delta}}\frac{a^\star_{\bar t}}{\sqrt{H'_{\bar t}(a^2)^\star_{\bar t}}} \to \infty \cdot sgn\Big(-a^\star_{\bar t}\Big),$$

which will conclude the proof of b).

Let us now prove 2b), 3b) and then 1b). As for 2b), the difference of the terms at the two sides is

$$\int_0^T \frac{1}{h} K_s H_s' a_s^2 ds - \frac{1}{\Delta h}\sum_{i=1}^{n} K_i a_{t_{i-1}}^2 \Delta_i^2$$

$$= \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Big[K_s - K_i\Big] H_s' a_s^2 ds + \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i\Big[H_s' a_s^2 - a_{t_{i-1}}^2\frac{\Delta_i}{\Delta}\Big] ds,$$

having subtracted and added $\int_{t_{i-1}}^{t_i} K_i H_s' a_s^2 ds$ for each i: since K is Lipschitz and H' and a are bounded, the first term of the rhs above is dominated by $\frac{C}{h}\sum_{i=1}^{n}\frac{\Delta_i^2}{h} \le C\frac{\Delta_{max}}{h^2} \to 0$. We thus remain with the second term, which is split as

$$\frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i H_s'\Big[a_s^2 - a_{t_{i-1}}^2\Big] ds + \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i\Big[H_s' - \frac{\Delta_i}{\Delta}\Big] a_{t_{i-1}}^2 ds, \tag{17}$$

where the second sum is

$$\frac{1}{h}\sum_{i\in I_H^{(n)}}\int_{t_{i-1}}^{t_i} K_i\Big[H_s' - \frac{\Delta_i}{\Delta}\Big] a_{t_{i-1}}^2 ds + \frac{1}{h}\sum_{i\notin I_H^{(n)}}\int_{t_{i-1}}^{t_i} K_i\Big[H_s' - \frac{\Delta_i}{\Delta}\Big] a_{t_{i-1}}^2 ds:$$

accounting for the boundedness of $K, H', \frac{\Delta_i}{\Delta}$ and a and for the fact that $\Delta_{max} \leq C\Delta$, the latter display is dominated in absolute value by

$$\frac{C}{h} m\Delta + \frac{C}{h} \sum_{i \notin I_H^{(n)}} \sup_{s \in [t_{i-1}, t_i)} \Big| H'_s - \frac{\Delta_i}{\Delta} \Big| K_i \Delta_i,$$

$$\leq C\frac{\Delta}{h} + C \sup_{i \notin I_H^{(n)}} \sup_{s \in [t_{i-1}, t_i)} \Big| H'_s - \frac{\Delta_i}{\Delta} \Big| \frac{\sum_{i=1}^n K_i \Delta_i}{h} \stackrel{a.s.}{\to} 0,$$

having used Lemma 2 part 2). We thus remain with only the first sum in (17), whose absolute value is dominated by

$$\frac{C}{h} \sum_{i=1}^n K_i \sup_{s \in [t_{i-1}, t_i)} |a_s^2 - a_{t_{i-1}}^2| \Delta_i,$$

however note that

$$\sup_{s \in [t_{i-1}, t_i)} |a_s^2 - a_{t_{i-1}}^2| = \sup_{s \in [t_{i-1}, t_i)} |a_s - a_{t_{i-1}}||a_s + a_{t_{i-1}}| \leq C \sup_{s \in [t_{i-1}, t_i)} |a_s - a_{t_{i-1}}|$$

thus the last display is in turn dominated by

$$C \sup_{i=1,\dots,n} \sup_{s \in [t_{i-1}, t_i)} |a_s - a_{t_{i-1}}| \cdot \frac{\sum_{i=1}^n K_i \Delta_i}{h} \stackrel{a.s.}{\to} 0,$$

which concludes the proof of 2b).

If in place of **A3** (i) we assume **A3** (ii), clearly the limit above is in probability. If instead in place of **A3** (i) we assume **A3** (iii) the first sum in (17) is dealt with as follows.

$$E\left[\frac{1}{h}\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} K_i H'_s \Big[a_s^2 - a_{t_{i-1}}^2\Big] ds\right|\right] \leq \frac{C}{h} \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} E[|a_s - a_{t_{i-1}}|] ds \tag{18}$$

$$\leq \frac{C}{h} \sum_{i=1}^n K_i \Delta^{1+\rho}$$

which tends to 0. Thus again the convergence at 2b) takes place in probability.

3b) follows from Lemma 2, point 4).

1b) Writing, for each i, $\Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 = \Big(\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}}) ds + a_{t_{i-1}} \Delta_i\Big)^2$ we obtain

$$\frac{1}{\Delta h} \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 = \frac{1}{\Delta h} \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}}) ds\Big)^2 \tag{19}$$

$$+\frac{2}{\Delta h}\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})ds \cdot a_{t_{i-1}}\Delta_i + \frac{1}{\Delta h}\sum_{i=1}^{n} K_i a_{t_{i-1}}^2 \Delta_i^2,$$

and, since by 2b) and 3b) $\frac{1}{\Delta h}\sum_{i=1}^{n} K_i a_{t_{i-1}}^2 \Delta_i^2 \to \left(H'a^2\right)_{\bar{t}}^{\star} \neq 0$, it is sufficient to show that the first two sums on the right hand side above tend to 0. In both cases we use that

$$\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})ds \leq \sqrt{\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} \left(a_s - a_{t_{i-1}}\right)^2 ds}.$$

It follows that the first of the two sums is

$$\frac{1}{\Delta h}\sum_{i=1}^{n} K_i \Big(\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})ds\Big)^2 \Delta_i^2 \leq \frac{1}{\Delta h}\sum_{i=1}^{n} K_i \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})^2 ds \Delta_i^2$$

$$\leq \frac{1}{\Delta h}\sum_{i=1}^{n} K_i \sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}|^2 \Delta_i^2 \leq C \sup_{i=1,\dots,n}\sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}|^2 \frac{\sum_{i=1}^{n} K_i\Delta_i}{h},$$

which, using Lemma 1, part 2), and Assumption **A3** (i), tends a.s. to 0.

The second sum at the rhs of (19) is

$$\frac{2}{\Delta h}\sum_{i=1}^{n} K_i \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})ds \cdot a_{t_{i-1}}\Delta_i^2$$

$$\leq \frac{2}{\Delta h}\sum_{i=1}^{n} K_i \sqrt{\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} \left(a_s - a_{t_{i-1}}\right)^2 ds} \cdot |a_{t_{i-1}}|\Delta_i^2$$

$$\leq \frac{C}{\Delta h}\sum_{i=1}^{n} K_i \sqrt{\sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}|^2} \cdot \Delta_i^2$$

$$\leq C \sup_{i=1,\dots,n}\sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}| \cdot \frac{\sum_{i=1}^{n} K_i\Delta_i}{h} \xrightarrow{a.s.} 0,$$

which concludes the proof of 1b).

If in place of **A3** (i) we assume **A3** (ii), clearly the last two limits above are in probability. If instead in place of **A3** (i) we assume **A3** (iii) then

$$E\left[\frac{1}{\Delta h}\sum_{i=1}^{n} K_i \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} (a_s - a_{t_{i-1}})^2 ds \Delta_i^2\right],$$

$$E\left[\frac{2}{\Delta h}\sum_{i=1}^{n} K_i \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} |a_s - a_{t_{i-1}}| ds \cdot |a_{t_{i-1}}|\Delta_i^2\right]$$

tend to 0 because they turn out to be bounded exactly as in (18). □

The proof of Theorem 2 relies heavily on Lemmas 4, 5 and 6 stated below, and the first two in turn make use of the next Lemma 3. To allow for lean reading, Lemmas from 3 to 6 are proved in the Appendix.

Lemma 3. *Let $g : \mathbb{R} \to \mathbb{C}$ be a deterministic Lebesgue integrable function. Given a deterministic function φ defined on $\mathbb{R}_+$, assume that K satisfies* **IA1** *for φ. Then for fixed $\alpha > 0$, for any $s \in \mathbb{R}$, under (4) with $\frac{\Delta}{h^2} \to 0$, we have*
i) if K^α is Lipschitz and in $L^1(\mathbb{R})$ then

$$\sum_{i=1}^{n} \frac{K_i^\alpha}{h} \Delta_i \int_{|v| \leq \frac{K_i |s|}{\varphi(h)}} g(v) dv \to K_{(\alpha)} \int_{\mathbb{R}} g(v) dv \tag{20}$$

ii) if K is Lipschitz and $K \in L^1(\mathbb{R})$,

$$\sum_{i=1}^{n} \frac{K_i}{h} \Delta_i I_{\{\frac{|s| K_i}{\varphi(h)} > 1\}} \to K_{(1)}$$

iii) if $K^{\alpha/2}$ is Lipschitz and in $L^1(\mathbb{R})$, and $\Psi \in L^1(\mathbb{R})$ is a deterministic function then

$$\sum_{i=1}^{n} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \int_{\mathbb{R}} \Psi(u) \int_{|v| \leq \sqrt{\frac{2K_i |s|}{\varphi(h)}} |u|} g(v) dv du \to K_{(\alpha/2)} \cdot \int_{\mathbb{R}} \Psi(u) du \int_{\mathbb{R}} g(v) dv$$

Lemma 4. *Assume that K and satisfies* **IA1** *for φ_α in (10) and for $\varphi_\alpha^{(1)}(h) \doteq h^{\frac{1}{\alpha}}$. Under* **IA2, IA3**, *(4), $\Delta/h^2 \to 0$ and if K^α is Lipschitz and in $L^1(\mathbb{R})$, then, recalling the notation $\Delta_i \tilde{X} = \int_{t_{i-1}}^{t_i} \int_{|x| \leq 1} x d\tilde{\mu}$, we have*

$$\alpha \in (0,1): \quad \frac{\sum_{i=1}^n K_i \Delta_i \tilde{X}}{h} \xrightarrow{P} -a_t^\star; \quad \frac{\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|x| \leq 1} x d\mu}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha} \tag{21}$$

$$\alpha = 1 \ \& \ A_+ \neq A_-: \quad \frac{\sum_{i=1}^n K_i \Delta_i \tilde{X}}{h \log \frac{1}{h}} \xrightarrow{d} -(A_+ - A_-) K_{(1)}, \tag{22}$$

$$\alpha \in (1,2): \quad \frac{\sum_{i=1}^n K_i \Delta_i \tilde{X}}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha}. \tag{23}$$

Lemma 5. *Assume that K satisfies* **IA1** *for ψ_α, then* **IA2, IA3***, (4), $\frac{\Delta}{h^2} \to 0$ and that $K^{\alpha/2}$ is Lipschitz and in $L^1(\mathbb{R})$. In the case $\alpha = 1$ assume also $\sqrt{K}\log(K)$ bounded and $\frac{\Delta \log^2 \frac{1}{h}}{h^2} \to 0$. Then*

$$\text{if } \alpha \in (0,1) \qquad \frac{\sum_{i=1}^n K_i (\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu)^2}{h^{\frac{2}{\alpha}}} \xrightarrow{d} Z_{2,\alpha}, \tag{24}$$

$$\text{if } \alpha = 1 \qquad \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{X})^2}{h^2} \xrightarrow{d} Z_{2,\alpha}, \tag{25}$$

$$\text{if } \alpha \in (1,2) \qquad \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{X})^2}{h^{\frac{2}{\alpha}}} \xrightarrow{d} Z_{2,\alpha}, \tag{26}$$

Lemma 6. *Under* **A1**, **IA2**, **IA3** *and (4): if $\alpha \in (1,2)$, $A_+ = A_-$ and $\frac{\Delta}{h^2} \to 0$ then*

$$\left(\frac{\left(\sum_{i=1}^n K_i \Delta_i X\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i X)^2}{h^{\frac{2}{\alpha}}} \right) \xrightarrow{d} (Z_{1,\alpha}^2, Z_{2,\alpha}).$$

Remark 10. Note that under **A1** K is bounded and then also K^2 is Lipschitz and in $L^1(\mathbb{R})$.

Proof of theorem 2.

a) Since X is a càdlàg process, for fixed $\varepsilon \in (0,1)$ we have a.s. $\nu(\omega, (\varepsilon, 1] \times [0,T]) < \infty$, i.e. the jumps occurring on $[0,T]$ with size larger than ε in absolute value are only finitely many. Define now N_T^ε the a.s. finite number of jumps of X with size absolute value $|\Delta X_p| > \varepsilon$, and S_p^ε the times of such jumps, $p = 1, \ldots, N_T^\varepsilon$. For any n, for any $p = 1, \ldots, N^\varepsilon$ we call $I_p = I_p^\varepsilon$ the unique interval $(t_{i-1}, t_i] = (t_{i-1}^\varepsilon, t_i^\varepsilon]$ containing S_p^ε, and we rename its extremes $t_{i_p - 1} = t_{i_p^\varepsilon - 1}, t_{i_p} = t_{i_p^\varepsilon}$. For any $\varepsilon \in (0,1)$ we split

$$X_t = \tilde{X}_t^\varepsilon - C_t^\varepsilon + X_t^{1,\varepsilon}, \quad \text{where} \quad X_t^{1,\varepsilon} \doteq \int_0^t \int_{|x|>\varepsilon} x d\mu,$$

$$\tilde{X}_t^\varepsilon \doteq \int_0^t \int_{|x|\le\varepsilon} x d\tilde{\mu}, \quad C_t^\varepsilon \doteq \int_0^t \int_{|x|\in(\varepsilon,1]} x\lambda(x) dx ds,$$

and we proceed through the following steps.

1) For any fixed $\varepsilon \in (0,1)$, $X^{1,\varepsilon}$ is a FA jump process with piece-wise constant paths, so that, by Lemma 1 we have that, as $n \to \infty$, $F^n(X^{1,\varepsilon}) \xrightarrow{a.s.} F(X^{1,\varepsilon})$

with both $f(x) = x$ and $f(x) = x^2$, where $F(X^{1,\varepsilon})$ is finite a.s..

2) Note that as $\varepsilon \to 0$ then, for both $f(x) = x$ and $f(x) = x^2$,

$$F(X^{1,\varepsilon}) = K(0)f(\Delta X^{1,\varepsilon}_{\bar{t}}) \stackrel{a.s.}{\to} F(X) = K(0)f(\Delta X_{\bar{t}}).$$

3) Now we check that

$$\forall \eta > 0, \ \lim_{\varepsilon \to 0} \limsup_{n \to \infty} P\Big(\Big\{|F^n(X) - F^n(X^{1,\varepsilon})| > \eta\Big\}\Big) = 0. \tag{27}$$

The three properties allow to conclude (11) by Proposition 2.2.1 in [6]. We define

$$a_s(\varepsilon) \doteq \int_{|x| \in (\varepsilon,1]} x\lambda(x,s)dx, \quad \sigma^2_s(\varepsilon) \doteq \int_{|x| \leq \varepsilon} x^2\lambda(x,s)dx.$$

Note that $a(0)$ is the process a that we defined in Section 3.1, and that it has finite values only if X has finite variation jumps ($\alpha < 1$). For proving part a), note that $\int_{|x| \in (\varepsilon,1]} x^2\lambda(x,s)dx$ is bounded as (ω, s) varies, thus that for any fixed $\varepsilon > 0$ the processes $a_s(\varepsilon)$ and $\sigma^2_s(\varepsilon)$ are bounded in absolute value by constants, say A^ε, depending on ε, and Σ respectively. In fact $\sigma^2_s(\varepsilon) \leq \int_{|x| \leq 1} x^2\lambda(x,s)dx \leq \Sigma$ and

$$|a_s(\varepsilon)| \leq \int_{|x| \in (\varepsilon,1]} |x| \frac{\lambda(x,s)}{\lambda\big((\varepsilon,1],s\big)} dx \lambda\big((\varepsilon,1],s\big)$$

$$\leq \sqrt{\int_{|x| \in (\varepsilon,1]} |x|^2 \frac{\lambda(x,s)}{\lambda\big((\varepsilon,1],s\big)} dx} \ \lambda\big((\varepsilon,1],s\big) \leq \sigma_s(\varepsilon)\sqrt{\lambda\big((\varepsilon,1],s\big)} \leq A^\varepsilon.$$

Case $f(x) = x$**:** $P\Big(\Big\{|F^n(X) - F^n(X^{1,\varepsilon})| > \eta\Big\}\Big)$ is bounded by

$$P\Big(\Big\{|\sum_{i=1}^n K_i \Delta_i \tilde{X}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) + P\Big(\Big\{|\sum_{i=1}^n K_i \Delta_i C^\varepsilon| > \frac{\eta}{2}\Big\}\Big):$$

the first probability is bounded by

$$\frac{||\sum_{i=1}^n K_i \Delta_i \tilde{X}^\varepsilon||_{L^2}}{\eta/2} = \frac{\sqrt{\sum_{i=1}^n K_i^2 E[(\Delta_i \tilde{X}^\varepsilon)^2]}}{\eta/2}$$

$$= \frac{\sqrt{\sum_{i=1}^n K_i^2 E[\int_{t_{i-1}}^{t_i} \int_{|x| \leq \varepsilon} x^2\lambda(x,s)dxds]}}{\eta/2} \leq \frac{\sqrt{\Sigma \cdot \sum_{i=1}^n K_i^2 \Delta_i}}{\eta/2},$$

having used for the first equality that $K_i \Delta_i \tilde{X}^\varepsilon$ are martingale increments. Since under **A1** we have $K^2 \in L^1(\mathbb{R})$ then, from Lemma 2 point 2), as $n \to \infty$, we have

$\Sigma \cdot \sum_{i=1}^n K_i^2 \Delta_i \simeq \Sigma h \to 0$, then $\limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i \Delta_i \tilde{X}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) = 0$ for all $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i \Delta_i \tilde{X}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) = 0.$$

As for $\sum_{i=1}^n K_i \Delta_i C^\varepsilon$, we have $\Big|\sum_{i=1}^n K_i \Delta_i C^\varepsilon\Big| \le A^\varepsilon \sum_{i=1}^n K_i \Delta_i$, which does not depend on ω and, for fixed ε, tends a.s. to 0, as $n \to \infty$, so again

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i \Delta_i C^\varepsilon| > \frac{\eta}{2}\Big\}\Big) \le \lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\Big(\Big\{A^\varepsilon \sum_{i=1}^n K_i \Delta_i > \frac{\eta}{2}\Big\}\Big) = 0$$

For the case $f(x) = x^2$ we reason similarly. In fact

$$F^n(X) - F^n(X^{1,\varepsilon}) = \sum_{i=1}^n K_i \left(\Delta_i \tilde{X}^\varepsilon\right)^2 + \sum_{i=1}^n K_i \left(\Delta_i C^\varepsilon\right)^2$$

$$+2\sum_{i=1}^n K_i \Big(\Delta_i \tilde{X}^\varepsilon \Delta_i X^{1,\varepsilon} - \Delta_i \tilde{X}^\varepsilon \Delta_i C^\varepsilon - \Delta_i X^{1,\varepsilon} \Delta_i C^\varepsilon\Big), \tag{28}$$

and we show that for fixed ε each term tends to 0 in probability as $n \to \infty$: $\sum_{i=1}^n K_i \Big(\Delta_i \tilde{X}^\varepsilon\Big)^2$ tends to 0 in probability because its L^1-norm tends to 0; and, again from Lemma 2 point 2), $\sum_{i=1}^n K_i \Delta_i \to 0$, thus we have

$$\sum_{i=1}^n K_i \left(\Delta_i C^\varepsilon\right)^2 \le \left(A^\varepsilon\right)^2 \sum_{i=1}^n K_i \Delta_i^2 \le \left(A^\varepsilon\right)^2 \Delta_{max} \sum_{i=1}^n K_i \Delta_i \stackrel{a.s.}{\to} 0.$$

Finally, the double products are all dealt with using the Schwarz inequality, and shown to be negligible:

$$\Big|\sum_{i=1}^n K_i \Delta_i Z \Delta_i V\Big| = \Big|\sum_{i=1}^n \sqrt{K_i} \Delta_i Z \sqrt{K_i} \Delta_i V\Big| \le \sqrt{\sum_{i=1}^n K_i (\Delta_i Z)^2} \sqrt{\sum_{i=1}^n K_i (\Delta_i V)^2}$$

and for each one of the three double products in (28) at least one of the square roots on the right hand side above tends to 0 in probability, while $\sum_{i=1}^n K_i (\Delta_i X^{1,\varepsilon})^2 = F^n(X^{1,\varepsilon})$ converges to the finite quantity

$$F(X^{1,\varepsilon}) = K(0)(\Delta X_{\bar{t}}^{1,\varepsilon})^2.$$

It follows that, for fixed $\varepsilon > 0$, $F^n(X) - F^n(X^{1,\varepsilon}) \stackrel{P}{\to} 0$ as $n \to \infty$, thus again

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\Big(\Big\{|F^n(X) - F^n(X^{1,\varepsilon})| > \eta\Big\}\Big) = \lim_{\varepsilon\to 0} 0 = 0.$$

b) We concentrate on the set $\{\Delta X_{\bar t}=0\}$, having probability 1. On that set both the numerator and the denominator of $T^n_{\bar t}$ tend to 0 in probability: using Lemmas 4 and 5 we reach the following speeds, as will be explained below:

$$\sum_{i=1}^n K_i\Delta_i X \stackrel{d}{\simeq} \begin{cases} -a^\star_{\bar t}h, & \text{if } \alpha\in(0,1) \\ -(A_+-A_-)K_{(1)}\cdot h\log\frac{1}{h}, & \text{if } \alpha=1 \text{ and } A_+\neq A_- \\ h^{\frac{1}{\alpha}}Z_{1,\alpha}, & \text{if } \alpha\in(1,2) \end{cases}; \quad (29)$$

$$\sum_{i=1}^n K_i(\Delta_i X)^2 \stackrel{d}{\simeq} \begin{cases} h^{\frac{2}{\alpha}}Z_{2,\alpha}+o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}), & \text{if } \alpha\in(0,1) \\ h^2 Z_{2,\alpha}, & \text{if } \alpha=1 \\ h^{\frac{2}{\alpha}}Z_{2,\alpha}, & \text{if } \alpha\in(1,2), \end{cases}. \quad (30)$$

It follows that for $\alpha\in(0,1)$ and $a^\star_{\bar t}\neq 0$ then

$$T^n_{\bar t}\stackrel{d}{\simeq}\frac{-a^\star_{\bar t}}{\sqrt{h^{\frac{2}{\alpha}-2}Z_{2,\alpha}+o_P\left(\sqrt{\frac{\Delta}{h}}\right)}}\to -sgn(a^\star_{\bar t})\cdot\infty,$$

for $\alpha=1$ then

$$T^n_{\bar t}\stackrel{d}{\simeq}-\frac{(A_+-A_-)K_{(1)}}{\sqrt{Z_{2,\alpha}}}\ \log\frac{1}{h}\stackrel{a.s.}{\to}-sgn(A_+-A_-)\cdot\infty$$

while for $\alpha\in(1,2)$ numerator and denominator of $T^n_{\bar t}$ have the same speed, thus with probability 1 the statistic $T^n_{\bar t}$ cannot diverge.

To obtain (29) from Lemma 4, we simply note that a.s. the speed of $\sum_{i=1}^n K_i\Delta_i X^1$ is $K\left(\frac{\bar t-S_{\underline{p}}}{h}\right)$, where $S_{\underline{p}}$ is the time of the jump of X^1 closest to $\bar t$ (see the proof of Theorem 1 after (15)). Since $K\left(\frac{\bar t-S_{\underline{p}}}{h}\right)=o_P(h\Delta)$ by assumption **A1.2**, $K\left(\frac{\bar t-S_{\underline{p}}}{h}\right)$ is negligible with respect to $\varphi_\alpha(h)$, for any α.

To obtain (30) from Lemmas 4 and 5 we first note that, similarly as above, $\sum_{i=1}^n K_i(\Delta_i X^1)^2$ tends to zero still at speed $K\left(\frac{\bar t-S_{\underline{p}}}{h}\right)=o_P(h\Delta)$. Then

· for $\alpha\in(0,1)$ the squared denominator of $T^n_{\bar t}$ is

$$\sum_{i=1}^n K_i(\Delta_i X)^2=\sum_{i=1}^n K_i\Big(\int_{t_{i-1}}^{t_i}\int_{|x|\le 1}xd\mu\Big)^2+\sum_{i=1}^n K_i\left(\int_{t_{i-1}}^{t_i}a_s ds\right)^2$$

$$+\sum_{i=1}^n K_i(\Delta_i X^1)^2-2\sum_{i=1}^n K_i\int_{t_{i-1}}^{t_i}a_s ds\ \int_{t_{i-1}}^{t_i}\int_{|x|\le 1}xd\mu$$

$$-2\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} a_s ds\ \Delta_i X^1 + 2\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big) \Delta_i X^1 : \qquad (31)$$

within the last term, $\sum_{i=1}^{n} \sqrt{K_i} \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big) \sqrt{K_i} \Delta_i X^1$ is dominated by $\sqrt{\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big)^2} \sqrt{\sum_{i=1}^{n} K_i (\Delta_i X^1)^2}$, and this is $o_P(h^{\frac{3}{2}} \Delta^{\frac{1}{2}})$, because, by Lemma 5, $\frac{\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big)^2}{h^{2/\alpha}} \xrightarrow{d} Z_{2,\alpha}$, then

$$h^{\frac{1}{\alpha}-1} \sqrt{\frac{\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big)^2}{h^{2/\alpha}}} \xrightarrow{P} 0,$$

while $\sum_{i=1}^{n} K_i (\Delta_i X^1)^2$ converges a.s. to zero as $K\Big(\frac{\bar{t} - S_p}{h}\Big) << h\Delta$, then a.s.

$$h^{\frac{1}{\alpha}} \sqrt{\frac{\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big)^2}{h^{\frac{2}{\alpha}}}} \sqrt{\sum_{i=1}^{n} K_i (\Delta_i X^1)^2}$$

$$\le Ch \cdot h^{\frac{1}{\alpha}-1} \sqrt{\frac{\sum_{i=1}^{n} K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \Big)^2}{h^{\frac{2}{\alpha}}}} \sqrt{K\left(\frac{\bar{t} - S_p}{h}\right)} = o_P(h^{\frac{3}{2}} \Delta^{\frac{1}{2}}).$$

Since a.s. $|\int_{t_{i-1}}^{t_i} a_s ds| \le \Delta \sup_s \int_{|x|\le 1} |x| \lambda(x,s) ds$, the term $\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} a_s ds\ \Delta_i X^1$ in (31) tends to 0 as $\Delta K\Big(\frac{\bar{t} - S_p}{h}\Big) = o_P(h\Delta)$; the term

$$\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} a_s ds \int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu \le C(\omega) \Delta h \cdot h^{\frac{1}{\alpha}-1} \frac{\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu}{h^{\frac{1}{\alpha}}} = o_P(h\Delta);$$

as already said, the term $\sum_{i=1}^{n} K_i (\Delta_i X^1)^2 = o_P(h\Delta)$; while $\sum_{i=1}^{n} K_i (\int_{t_{i-1}}^{t_i} a_s ds)^2$ tends a.s. to 0 as $h\Delta$. Thus the display in (31) is asymptotically equivalent to

$$h^{\frac{2}{\alpha}} Z_{2,\alpha} + O_P(h\Delta) + o_P(h^{\frac{3}{2}} \Delta^{\frac{1}{2}}) = h^{\frac{2}{\alpha}} Z_{2,\alpha} + o_P(h^{\frac{3}{2}} \Delta^{\frac{1}{2}}).$$

· for $\alpha = 1$ we instead split $\Delta_i X$ into $\Delta_i \tilde{X}$ and $\Delta_i X^1$ and, using again the Schwarz inequality, the mixed term within the squared denominator of $T_{\bar{t}}^n$ is shown to be dominated by

$$2\sqrt{\sum_{i=1}^{n} K_i \Big(\Delta_i \tilde{X}\Big)^2} \sqrt{\sum_{i=1}^{n} K_i (\Delta_i X^1)^2} = O_P\left(h \sqrt{K\left(\frac{\bar{t} - S_p}{h}\right)}\right) = o_P(h^{\frac{3}{2}} \Delta^{\frac{1}{2}}).$$

Thus

$$\sum_{i=1}^{n} K_i \Big(\Delta_i X\Big)^2 \overset{d}{\simeq} h^2 Z_{2,\alpha} + O_P\left(K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)\right) + o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}) \overset{d}{\simeq} h^2 Z_{2,\alpha}.$$

· for $\alpha \in (1,2)$ we again split $\Delta_i X$ into $\Delta_i \tilde{X}$ and $\Delta_i X^1$ and use the Schwarz inequality:

$$\sum_{i=1}^{n} K_i \Big(\Delta_i X\Big)^2 \overset{d}{\simeq} h^{\frac{2}{\alpha}} Z_{2,\alpha} + O_P\left(K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)\right) + o_P(h^{\frac{1}{\alpha}}\sqrt{\Delta h}) \overset{d}{\simeq} h^{\frac{2}{\alpha}} Z_{2,\alpha}.$$

c) By (29), (30) and Lemma 6 part c) is proved. □

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8. Appendix. Proofs of the other Lemmas and of Corollary 3.

This Appendix contains the detailed proofs of Lemmas 2, 3 , 4, 5 and 6, and of Corollary 3. Two further Lemmas are needed for the proofs of the last four Lemmas .

Proof of Lemma 2. As for 1), recalling form Notation 2 that $K_s = K\left(\frac{\bar{t}-s}{h}\right)$, the displayed left term coincides with

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{1}{h}(K_s - K_i) b_s^{(n)} ds,$$

whose absolute value is dominated by

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{L}{h^2} |s - t_{i-1}| C ds = O_{a.s.}\left(\frac{\Delta}{h^2}\right).$$

2) By 1) in the special case where $b^{(n)} \equiv 1$ for all n we have that $\frac{\sum_{i=1}^n K_i \Delta_i}{h}$ equals $\frac{1}{h}\int_0^T K\left(\frac{\bar{t}-s}{h}\right) ds + O_{a.s.}\left(\frac{\Delta}{h^2}\right) = \int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K(u) du + O_{a.s.}\left(\frac{\Delta}{h^2}\right) \to \int_{\mathbb{R}} K(u) du$, where for the last equality we operated the change of variable $u = (\bar{t} - s)/h$.

3) We apply 2).

4) For fixed ω the term $\int_0^T \frac{1}{h} K_s b_s ds$ coincides with $\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K(u) b_{\bar{t}-hu} du$, and

$$\left| \int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K(u) b_{\bar{t}-hu} du - b_{\bar{t}}^{\star} \right| \leq \left| \int_{\frac{\bar{t}-T}{h}}^{0} K(u) b_{\bar{t}-hu} du - b_{\bar{t}+} \cdot \int_{-\infty}^{0} K(u) du \right|$$

$$+\left|\int_0^{\frac{\bar t}{h}} K(u) b_{\bar t - hu} du - b_{\bar t -}\cdot \int_0^{+\infty} K(u)du\right|$$

$$\leq \int_{\mathbb{R}} |b_{\bar t - hu} - b_{\bar t +}|\, I_{(\frac{\bar t - T}{h},0]}(u)K(u)du + \int_{\mathbb{R}} |b_{\bar t - hu} - b_{\bar t -}|\, I_{(0,\frac{\bar t}{h}]}(u)K(u)du$$

$$+\int_{\mathbb{R}} \Big(|b_{\bar t +}| I_{(-\infty,\frac{\bar t - T}{h})}(u) + |b_{\bar t -}| I_{(\frac{\bar t}{h},+\infty)}(u)\Big)K(u)du :$$

the three terms are integrals, in the finite measure on $\mathbb{R}$ having intensity K, of bounded integrands converging to 0 point-wise as $h \to 0$. By the dominated convergence theorem the three integrals tend to 0 and 4) is proved.

5) If either (i) or (ii) holds true, the thesis follows from the fact that

$$\left|\sum_{i=1}^n \frac{1}{h} K_i \int_{t_{i-1}}^{t_i} b_s^{(n)} - b_{t_{i-1}}^{(n)} ds\right|$$

$$\leq \sup_{i=1,\dots,n} \sup_{s\in[t_{i-1},t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar t - t_{i-1}}{h}\Big)\Delta_i,$$

which tends to 0 a.s. (respectively, tends to 0 in P).
If (iii) holds true then

$$E\left[\left|\sum_{i=1}^n \frac{1}{h} K_i \int_{t_{i-1}}^{t_i} b_s^{(n)} - b_{t_{i-1}}^{(n)} ds\right|\right] \leq \frac{1}{h}\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} E[|b_s^{(n)} - b_{t_{i-1}}^{(n)}|]ds,$$

which in turn is dominated by $\frac{C}{h}\sum_{i=1}^n K_i \Delta_i^{1+\rho} \to 0$.

6) As for the first relation we have

$$\int_0^T K_u^2 \int_0^u K_s^2 ds du - \sum_{i=1}^n K_i^2\Big(\sum_{j<i} K_j^2 \Delta_j\Big)\Delta_i = \Bigg(\int_0^T K_u^2 \int_0^u K_s^2 ds du \tag{32}$$

$$-\sum_{i=1}^n K_i^2 \int_0^{t_{i-1}} K_s^2 ds \Delta_i\Bigg) + \Bigg(\sum_{i=1}^n K_i^2 \int_0^{t_{i-1}} K_s^2 ds\Delta_i - \sum_{i=1}^n K_i^2\Big(\sum_{j<i} K_j^2\Delta_j\Big)\Delta_i\Bigg).$$

Since $\int_0^{t_{i-1}} K_s^2 ds = \sum_{j<i}\int_{t_{j-1}}^{t_j} K_s^2 ds$, the latter term is dominated in absolute value by

$$\sum_{i=1}^n K_i^2 \sum_{j<i}\int_{t_{j-1}}^{t_j} |K_s^2 - K_j^2| ds\Delta_i \leq C\sum_{i=1}^n K_i^2 \sum_{j<i}\int_{t_{j-1}}^{t_j} \frac{|s-t_{j-1}|}{h} ds \Delta_i$$

$$\simeq C\sum_{i=1}^n K_i^2 \sum_{j<i}\frac{\Delta_j^2}{h}\Delta_i \leq C\Delta \frac{\sum_{i=1}^n K_i^2 \Delta_i}{h} = O(\Delta) \to 0.$$

The right hand side term in (32) equals

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_u^2\int_0^u K_s^2 ds du - \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i^2\int_0^{t_{i-1}} K_s^2 ds du$$

$$= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(K_u^2 - K_i^2\right)\int_0^{t_{i-1}} K_s^2 ds du + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_u^2\int_{t_{i-1}}^{u} K_s^2 ds du :$$

using that for any t_{i-1} we have $\int_0^{t_{i-1}} K_s^2 ds = h\int_{\frac{\bar{t}-t_{i-1}}{h}}^{\frac{\bar{t}}{h}} K^2(w)dw \le hK_{(2)}$, the first sum is dominated by

$$C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{|u-t_{i-1}|}{h}du\cdot hK_{(2)} = O(\Delta)\to 0.$$

Also for the second sum we use that $\int_{t_{i-1}}^{u} K_s^2 ds = h\int_{\frac{\bar{t}-u}{h}}^{\frac{\bar{t}-t_{i-1}}{h}} K^2(w)dw \le hK_{(2)}$, thus the sum is dominated by

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_u^2 du\cdot O(h) = \int_0^T K_u^2 du\cdot O(h) = O(h^2)\to 0.$$

As for the second relation,

$$\frac{1}{h^2}\int_0^T K_u^2\int_0^u K_s^2 ds du = \frac{1}{h^2}\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K^2(v)\int_0^{\bar{t}-vh} K_s^2 ds dv$$

$$= \int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K^2(v)\int_v^{\frac{\bar{t}}{h}} K^2(w)dwdv \to \int_{\mathbb{R}} K^2(v)\int_v^{+\infty} K^2(w)dwdv. \qquad \square$$

Proof of Lemma 3. i) Since the difference of the two terms in (20) can be written as

$$\sum_{i=1}^{n}\frac{K_i^\alpha}{h}\Delta_i\left(\int_{|v|\le\frac{K_i|s|}{\varphi(h)}} g(v)dv - \int_{\mathbb{R}} g(v)dv\right) + \int_{\mathbb{R}} g(v)dv\left(\sum_{i=1}^{n}\frac{K_i^\alpha}{h}\Delta_i - K_{(\alpha)}\right),$$

it is sufficient to show that

$$\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\left(\int_{|v|\le\frac{K_i|s|}{\varphi(h)}} g(v)dv - \int_{\mathbb{R}} g(v)dv\right)\to 0, \tag{33}$$

because using then that, similarly as in Lemma 2, 2), $\sum_{i=1}^n\frac{K_i^\alpha\Delta_i}{h}\to K_{(\alpha)}$, the proof is concluded. The absolute value of the expression in (33) is dominated by

$$\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\int_{|v|>\frac{K_i|s|}{\varphi(h)}}|g(v)|dv.$$

We split $I \doteq \{1, 2, ...n\} = I' \cup I''$, where

$$I' = \{i \in I : |\bar{t} - t_{i-1}| \le \varepsilon_h\}, \quad I'' = \{i \in I : |\bar{t} - t_{i-1}| > \varepsilon_h\}.$$

For $i \in I'$ we have $K_i \ge K\left(\frac{\varepsilon_h}{h}\right)$, thus

$$\sum_{i\in I'} \frac{K_i^\alpha \Delta_i}{h} \int_{|v|>\frac{K_i|s|}{\varphi(h)}} |g(v)|dv \le \sum_{i\in I'} \frac{K_i^\alpha \Delta_i}{h} \int_{|v|>\frac{K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}} |g(v)|dv,$$

and the latter tends to 0, because the first factor is dominated by $\sum_{i=1}^n \frac{K_i^\alpha \Delta_i}{h} \to K_\alpha$, while the second factor is an integral of $|g|$ on a vanishing region.
On the other hand,

$$\sum_{i\in I''} \frac{K_i^\alpha \Delta_i}{h} \int_{|v|>\frac{K_i|s|}{\varphi(h)}} |g(v)|dv \le \sum_{i\in I''} \frac{K_i^\alpha \Delta_i}{h} \int_{\mathbb{R}} |g(v)|dv,$$

and we show that $\sum_{i\in I''} \frac{K_i^\alpha \Delta_i}{h} \to 0$. First we have

$$\left| \sum_{i\in I''} \frac{K_i^\alpha \Delta_i}{h} - \sum_{i=1}^n \frac{K_i^\alpha}{h} \int_{t_{i-1}}^{t_i} I_{\{s:|\bar{t}-s|>\varepsilon_h\}} ds \right|$$

$$\le \sum_{i=1}^n \frac{K_i^\alpha}{h} \left| \Delta_i I_{\{i:|\bar{t}-t_{i-1}|>\varepsilon_h\}} - \int_{t_{i-1}}^{t_i} I_{\{s:|\bar{t}-s|>\varepsilon_h\}} ds \right| :$$

since $|\bar{t}-t_{i-1}|/h > \varepsilon_h/h \to \infty$, and, for all the considered numbers s, $|\bar{t}-s|/h > \varepsilon_h/h$, then the only involved K_i^α are such that $K_i^\alpha \le K^\alpha(\varepsilon_h/h) \to 0$; further $\Delta_i I_{\{i:|\bar{t}-t_{i-1}|>\varepsilon_h\}} - \int_{t_{i-1}}^{t_i} I_{\{s:|\bar{t}-s|>\varepsilon_h\}} ds$ is 0 for all that intervals $[t_{i-1}, t_i)$ but for the two containing $\bar{t}+\varepsilon$ and $\bar{t}-\varepsilon$. Thus the latter sum is dominated by $\frac{K^\alpha(\varepsilon 7h)}{h} \cdot 2\Delta \to 0$, and

$$\lim \sum_{i\in I''} \frac{K_i^\alpha \Delta_i}{h} = \lim \sum_{i=1}^n \frac{K_i^\alpha}{h} \int_{t_{i-1}}^{t_i} I_{\{s:|\bar{t}-s|>\varepsilon_h\}} ds,$$

and using Lemma 2, 1), with $b_s^n = I_{\{s:|\bar{t}-s|>\varepsilon_h\}}$, the latter limit coincides with

$$\lim \int_{s\in(0,T):|\bar{t}-s|>\varepsilon_h} \frac{K_s^\alpha}{h} ds = \lim \int_{\frac{\bar{t}-T}{h}}^{-\frac{\bar{t}_h}{h}} K^\alpha(u) I_{\{|u|>\frac{\varepsilon_h}{h}\}} du \tag{34}$$

$$= \lim \int_{\frac{\bar{t}-T}{h}}^{-\frac{\varepsilon_h}{h}} K^\alpha(u)du + \int_{\frac{\varepsilon_h}{h}}^{\frac{\bar{t}}{h}} K^\alpha(u)du = 0.$$

ii) We have that

$$\sum_{i=1}^{n} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}>1\right\}} - K_{(1)} \simeq \sum_{i=1}^{n} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}>1\right\}} - \sum_{i=1}^{n} \frac{K_i}{h} \Delta_i$$

$$= \sum_{i=1}^{n} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\leq 1\right\}},$$

and we show that the latter sum has limit 0. With I' and I'' as defined at point i), we immediately see that

$$\sum_{i\in I'} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\leq 1\right\}} \to 0,$$

in fact if $|\bar{t} - t_{i-1}| \leq \varepsilon_h$ then $K\left(\frac{|\bar{t}-t_{i-1}|}{h}\right) \geq K\left(\frac{\varepsilon_h}{h}\right)$, thus

$$\sum_{i\in I'} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\leq 1\right\}} \leq \sum_{i\in I'} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)}\leq 1\right\}} = I_{\left\{\frac{|s|K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)}\leq 1\right\}} \sum_{i\in I'} \frac{K_i}{h} \Delta_i :$$

since the first factor tends to 0 and the second one is bounded, the latter product tends to 0.

In order to check that also

$$\sum_{i\in I''} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\leq 1\right\}} \to 0,$$

note that

$$\sum_{i\in I''} \frac{K_i}{h} \Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\leq 1\right\}} \leq \sum_{i\in I''} \frac{K_i}{h} \Delta_i = \sum_{i=1}^{n} \frac{K_i}{h} \Delta_i I_{\left\{|\bar{t}-t_{i-1}|>\varepsilon_h\right\}},$$

and the latter is shown to tend to 0 as in (34).

As for iii), the proof is substantially the same as for i), we only point out some details. It is sufficient to prove that

$$\sum_{i=1}^{n} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \left(\int_{\mathbb{R}} \Psi(u) \int_{|v|\leq\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|} g(v)dvdu - \int_{\mathbb{R}} \Psi(u)du \int_{\mathbb{R}} g(v)dv \right)$$

$$= \sum_{i=1}^{n} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \int_{\mathbb{R}} \Psi(u) \int_{|v|>\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|} g(v)dvdu \to 0, \tag{35}$$

because as in lemma 2, 3), we have $\sum_{i=1}^{n} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \to K_{(\alpha/2)}$. The sum in (35) is again split into the sum of the terms with $i \in I'$ and the sum of the ones with

$i \in I''$: since for $i \in I'$ we have $\{|v| > \sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|\} \subset \{|v| > \sqrt{\frac{2K(\frac{\varepsilon_h}{h})|s|}{\varphi(h)}}|u|\}$, the absolute value of the first sum is dominated by

$$\sum_{i \in I'} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \int_{\mathbb{R}} \Psi(u) \int_{|v| > \sqrt{\frac{2K(\frac{\varepsilon_h}{h})|s|}{\varphi(h)}}|u|} |g(v)| dv du,$$

where for any u we have $\int_{|v| > \sqrt{\frac{2K(\frac{\varepsilon_h}{h})|s|}{\varphi(h)}}|u|} |g(v)| dv \to 0$ and

$$\Psi(u) \int_{|v| > \sqrt{\frac{2K(\frac{\varepsilon_h}{h})|s|}{\varphi(h)}}|u|} |g(v)| dv \leq C\Psi(u) \in L^1(\mathbb{R}),$$

here $C = \int_{\mathbb{R}} |g(v)| dv$, thus by the dominated convergence theorem the sum over $i \in I'$ tends to 0. On the other hand,

$$\sum_{i \in I''} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \int_{\mathbb{R}} \Psi(u) \int_{|v| > \sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|} |g(v)| dv du \leq \sum_{i \in I''} \frac{K_i^{\frac{\alpha}{2}}}{h} \Delta_i \int_{\mathbb{R}} \Psi(u) du \int_{\mathbb{R}} |g(v)| dv$$

where, as in (34), the first factor tends to 0. $\square$

A useful procedure to extend results for α-stable processes to semimartingales. This procedure is explained in [5], sec. 12.4, we report it with some adjustments needed in our framework.

Let us consider a one sided martingale $\tilde{X}_t^+ = \int_0^t \int_{0<x\leq 1} x d\tilde{\mu}^+, \quad t \in [0,T]$, where the jump measure μ^+ has Lévy measure

$$\lambda^+(x,s) = \frac{A_+ g(x,s)}{x^{1+\alpha}} I_{0<x\leq 1} dx.$$

In our application $\tilde{X}^+$ is the component of X involving the positive small jumps.

Since $g(x,s) \leq 1$, then $A_+(1-g(x,s)) \geq 0$ and $\left[I_{0<x\leq 1} A_+(1-g(x,s))/(x^{1+\alpha})\right]$ dx can represent a Lévy measure. Consider the Skorohod space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t\in[0,T]})$ of the càdlàg functions starting from state 0 at time 0. For any fixed $\omega \in \Omega$ we define

$$\nu'^+_\omega(\omega', dx, ds) = \left[\frac{A_+}{x^{1+\alpha}} I_{0<x\leq 1} - \lambda^+(x,s)\right] dxds = I_{0<x\leq 1} \frac{A_+(1-g(x,s))}{x^{1+\alpha}} dxds,$$

and we put on Ω' the unique probability Q_ω under which the canonical process, that we call $\tilde{X}'^+$, is a SM with characteristics $(0,0,\nu'^+_\omega)$: $\tilde{X}'^+_t = \int_0^t \int_{0<x\leq 1} x d\tilde{\mu}'^+$,

$\tilde{\mu}'^{+} = \mu'^{+} - \nu'^{+}_{\omega}$.

Since ν'^{+}_{ω} keeps fixed as ω' varies on Ω', then $\tilde{X}'^{+}$ on Ω' has independent increments. Further, ν'^{+}_{ω} is measurable as a function of ω, because $I_{0<x\leq 1}A_{+}(1 - g(x,s)))/(x^{1+\alpha})$ is such, then $Q_{\omega}(d\omega')$ is a transition probability from $(\Omega, \mathcal{F})$, to $(\Omega', \mathcal{F}')$, and we can enlarge $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ to $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in[0,T]}, \bar{P})$, where $\bar{\Omega} = \Omega \times \Omega', \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \bar{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}'_t$, $\bar{P}(d\omega, d\omega') = P(d\omega)Q_{\omega}(d\omega')$. We extend $\tilde{X}^{+}, \tilde{X}'^{+}$ to $\bar{\Omega}$ by keeping $\tilde{X}^{+}(\omega)$ constant as ω' varies and $\tilde{X}'^{+}(\omega')$ constant as ω varies. In order to simplify notations we keep the same name for an object originally defined on Ω or on Ω' and which was extended on $\bar{\Omega}$. Only ν'^{+}_{ω} undergoes a slight name change in what follows.

The above enlargement turns out to be a very good extension ([6], p.36), which ensures that $\tilde{X}^{+}$ and $\tilde{X}'^{+}$ are still martingales on $\bar{\Omega}$, with respective characteristics

$$\nu^{+}\big((\omega,\omega'), dx, ds\big) = \lambda^{+}(x,s)dxds, \quad \nu'^{+}\big((\omega,\omega'), dx, ds\big) = \nu'^{+}_{\omega}(\omega', dx, ds).$$

Now, $\tilde{X}$ and $\tilde{X}'^{+}$ turn out not to have common jumps. In fact if at time τ we have $\Delta\tilde{X}_{\tau} \neq 0$, then τ depends on ω and not on ω'. $\tilde{X}'^{+}$ has absolutely continuous characteristics $(0, 0, \nu'^{+}_{\omega})$, thus it is an Ito SM. But then, since $\tau(\omega)$ is fixed on Ω', and $\tilde{X}'^{+}$ cannot have on Ω' fixed times of discontinuity, thus $\Delta\tilde{X}'^{+}_{\tau(\omega)} = 0$. This implies that the number of jumps of $\tilde{X}^{+} + \tilde{X}'^{+}$ on any subset of $\bar{\Omega} \times [0, T]$ is the sum of the number of jumps of the two terms on the same subset, i.e.

$$\begin{aligned}\nu^{\tilde{X}^{+}+\tilde{X}'^{+}}\big((\omega,\omega'), dx, ds\big) &= \nu^{+}\big((\omega,\omega'), dx, ds\big) + \nu'^{+}\big((\omega,\omega'), dx, ds\big)\\ &= \frac{A_{+}}{x^{1+\alpha}} I_{0<x\leq 1} dxds,\end{aligned}$$

but then $\tilde{X}^{+} + \tilde{X}'^{+}$ is on $\bar{\Omega}$ a martingale made of compensated jumps smaller than 1 and having one sided α stable law. Thus we identify $\tilde{X}^{+} + \tilde{X}'^{+}$ with a martingale, say $\tilde{J}^{+}$, represented by the compensated small jumps of an α-stable process. In the following we denote by $\tilde{J}^{+}$ either the compensated small jumps of an α-stable process on Ω or the compensated small jumps of an α-stable process on $\bar{\Omega}, \bar{P}$.

From one sided to two sided. The model (2) we are dealing with has possibly two sided small jumps. By applying the same reasoning above also to the side $\tilde{X}^-$ of the process having negative jumps, we end up with a connection of $\tilde{X} = \tilde{X}^+ + \tilde{X}^-$ with a possibly non-symmetric martingale $\tilde{J} = \tilde{J}^+ + \tilde{J}^-$ representing the compensated small jumps of an α stable process. With $\tilde{X}' = \tilde{X}'^+ + \tilde{X}'^-$ we obtain

$$\tilde{X} + \tilde{X}' = \tilde{J}, \tag{36}$$

where

$$\tilde{X}_t = \int_0^t \int_{|x|\le 1} x(d\mu - d\nu),\ \tilde{X}'_t = \int_0^t \int_{|x|\le 1} x(d\mu' - d\nu'),\ \tilde{J}_t = \int_0^t \int_{|x|\le 1} x(d\mu^J - d\nu^J),$$

$$\nu\big((\omega,\omega'), dx, ds\big) = \lambda(x,s)dxds,\ \nu'\big((\omega,\omega'), dx, ds\big) = \lambda'(x,s)dxds,$$

$$\nu^J\big((\omega,\omega'), dx, ds\big) = \lambda^J(x)dxds,$$

with $\lambda(x,s)$ as in **AI3**,

$$\lambda'(x,s) = I_{0<x\le 1}\frac{A_+(1-g(x,s))}{x^{1+\alpha}} + I_{-1\le x<0}\frac{A_-(1-g(x,s))}{x^{1+\alpha}},$$

$$\lambda^J(x) = \frac{A_+}{x^{1+\alpha}} I_{0<x\le 1} + \frac{A_-}{|x|^{1+\alpha}} I_{-1\le<x<0}.$$

The big advantage of this approach is that we now have a useful expression linking expectations of functionals of $\tilde{J}$ under $\bar{P}$ and expectations of functionals of $\tilde{X}$ under P (see Lamma 7). This allows us to firstly prove our results for the small jumps of an α stable process and then to extend the results to the process in (2).

Lemma 7. *Let f_n be a sequence of deterministic functionals to be applied to either a process $\bar{V}$ on $\bar{\Omega}$ or to a process V on Ω, and let $g_n = g_n(\omega,\cdot)$ a sequence of functionals, possibly depending on ω, to be applied to a process V' on Ω' such that the processes V, V' extended on $\bar{\Omega}$ satisfy $\bar{V} = V + V'$.*
Let, for all n, $|f_n|$ and $|f_n||g_n|$ be bounded, $f_n\big(\bar{V}\big) = f_n\big(V\big)g_n\big(V'\big)$ and let

$$\forall\omega,\quad g_n\big(V'\big) \xrightarrow{Q_\omega} 1.$$

Then, denoting $\bar{E} = E^{\bar{P}}$, as $n \to \infty$, we obtain

$$\lim_n \bar{E}\Big[f_n(\bar{V})\Big] = \lim_n E^P\Big[f_n(V)\Big].$$

Proof. Since $f_n(V)$ only depends on ω and not on ω',

$$\bar{E}[f_n(V)] = E^P[E^{Q_\omega}[f_n(V)]] = E^P[f_n(V)E^{Q_\omega}[1]] = E^P[f_n(V)].$$

Then

$$\bar{E}\Big[f_n(\bar{V})\Big] - E^P\Big[f_n(V)\Big] = \bar{E}\Big[f_n(\bar{V}) - f_n(V)\Big] = \bar{E}\,\Big[f_n(V)\Big[g_n(V') - 1\Big]\Big].$$

Since $\forall\omega$, $g_n(V') \xrightarrow{Q_\omega} 1$ then $g_n(V') \xrightarrow{\bar{P}} 1$, moreover $|f_n(V)|$ is bounded, thus $f_n(V)\Big[g_n(V') - 1\Big] \xrightarrow{\bar{P}} 0$, and is bounded because also $f_n g_n$ is bounded. Therefore, by the dominated convergence Theorem, the latter display tends to 0, and the Lemma is proved. $\square$

Remark 11. We use this Lemma for the second steps of Lemmas 4, 5, 6. For instance for Lemma 5 when $\alpha \geq 1$ (eq. (63)), with the notation in (36), we have $\bar{V} = \tilde{J}$, $V = \tilde{X}$, $V' = \tilde{X}'$, and $f_n(\tilde{X}) = e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}(\Delta_j\tilde{X})^2}$, while $g_n(\omega, \tilde{X}') = e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}(\Delta_j\tilde{X}')^2 - 2s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\Delta_j\tilde{X}\Delta_j\tilde{X}'}$.

Lemma 8. *Let $r < \alpha \leq 1$ be such that $\int_{|x|\leq 1}|x|^r\lambda'(x,s)dx \leq C$ for any $(\omega, s) \in \Omega \times [0,T]$; and let the kernel satisfy $K^r \in L^1(\mathbb{R})$. Then for all ω, on Ω' we have*

$$\frac{\sum_{j=1}^n K_j \int_{t_{i-1}}^{t_i}\int_{|x|\leq 1} x d\mu'}{h^{\frac{1}{\alpha}}} \overset{Q_\omega}{\to} 0.$$

Proof. It is sufficient to show that

$$E^{Q_\omega}\left[\left|\frac{\sum_{j=1}^n K_j \int_{t_{i-1}}^{t_i}\int_{|x|\leq 1} x d\mu'}{h^{\frac{1}{\alpha}}}\right|^r\right] \to 0.$$

Now note that, due to the fact that $r < 1$, the left term in the above display is dominated by

$$E^{Q_\omega}\left[\frac{\sum_{j=1}^n K_j^r \left|\int_{t_{i-1}}^{t_i}\int_{|x|\leq 1} x d\mu'\right|^r}{h^{\frac{r}{\alpha}}}\right] = \frac{\sum_{j=1}^n K_j^r E^{Q_\omega}\left[\left|\int_{t_{i-1}}^{t_i}\int_{|x|\leq 1} x d\mu'\right|^r\right]}{h^{\frac{r}{\alpha}}}$$

so by [6] (2.1.40), recalling that ν'_ω does not depend on ω', the latter term is dominated by

$$\frac{\sum_{j=1}^n K_j^r \int_{t_{i-1}}^{t_i}\int_{|x|\le 1}|x|^r d\nu'_\omega}{h^{\frac{r}{\alpha}}}.$$

Recalling that $d\nu'_\omega = \lambda'_\omega(x,s)dxds$, since by assumption $\int_{|x|\le 1}|x|^r\lambda'_\omega(x,s)dx \le C$, the above display is upper bounded by

$$C\frac{\sum_{j=1}^n K_j^r \Delta}{h}\cdot h^{1-\frac{r}{\alpha}} \to 0. \qquad \square$$

Proof of Lemma 4.

First step. We start by proving the results when the small jumps of X are the ones of an α stable process J, i.e. $g(x,t)\equiv 1$ and $\lambda(\omega,x,s) \equiv= \lambda(x)$. In this case for any $\bar t$ we have

$$a^\star_{\bar t} = a = \int_{|x|\le 1} x\lambda(x)dx = \frac{A_+ - A_-}{1-\alpha}.$$

To distinguish the stable case we replace $\Delta_i\tilde X$ with $\Delta_i\tilde J$. We now prove that under the assumptions of the Lemma we have

$$\begin{cases} \text{if } \alpha\in(0,1) & \frac{\sum_{i=1}^n K_i\Delta_i\tilde J}{h} \xrightarrow{d} -a \text{ and } \frac{\sum_{i=1}^n K_i\int_{t_{i-1}}^{t_i}\int_{|x|\le 1} x d\mu}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha} \\ \text{if } \alpha = 1 \text{ and } A_+\ne A_- & \frac{\sum_{i=1}^n K_i\Delta_i\tilde J}{h\log\frac{1}{h}} \xrightarrow{d} -(A_+ - A_-)K_{(1)}, \\ \text{if } \alpha\in(1,2) & \frac{\sum_{i=1}^n K_i\Delta_i\tilde J}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha}. \end{cases}$$

For each $\alpha\in(0,2)$, defined $Z_n \doteq \frac{\sum_{i=1}^n K_i\Delta_i\tilde J}{\varphi_\alpha(h)}$, we proceed by showing that the characteristic functions $E[e^{isZ_n}]$ converge to the characteristic function of the limit shown in the statement of the Lemma.

Since $\tilde J$ is a Lévy process,

$$E[e^{isZ_n}] = E\left[\prod_{j=1}^n e^{is\frac{K_j\Delta_j\tilde J}{\varphi_\alpha(h)}}\right] = \prod_{j=1}^n E\Big[e^{is\frac{K_j\Delta_j\tilde J}{\varphi_\alpha(h)}}\Big]$$

$$= \prod_{j=1}^n e^{\Delta\int_{|x|\le 1} e^{is\frac{K_j}{\varphi_\alpha(h)}x} - 1 - is\frac{K_j}{\varphi_\alpha(h)}x\,\lambda(x)dx}$$

With $z \doteq s\frac{K_j}{\varphi_\alpha(h)}$, the integral at exponent is

$$A_+ \int_{0<x\leq 1} \left(e^{izx} - 1 - izx\right) x^{-1-\alpha} dx + A_- \int_{-1\leq x<0} \left(e^{izx} - 1 - izx\right) |x|^{-1-\alpha} dx \tag{37}$$

$$= (A_+ + A_-) \int_0^1 \frac{cos\Big(zx\Big) - 1}{x^{1+\alpha}} dx + i(A_+ - A_-) \int_0^1 \frac{sin\Big(zx\Big) - zx}{x^{1+\alpha}} dx. \tag{38}$$

By changing variable $v = |z|x$ that becomes

$$|z|^\alpha \left[(A_+ + A_-) \int_{0<v\leq |z|} \frac{cos(v) - 1}{v^{1+\alpha}} dv + i(A_+ - A_-) sgn(s) \int_{0<v\leq |z|} \frac{sin(v) - v}{v^{1+\alpha}} dv \right]$$

so that $E[e^{isZ_n}]$ is given by

$$e^{\sum_{j=1}^n \Delta \left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha \left[(A_+ + A_-) \int_{0<v\leq \frac{|s|K_j}{\varphi_\alpha(h)}} \frac{cos(v)-1}{v^{1+\alpha}} dv + i(A_+ - A_-) sgn(s) \int_{0<v\leq \frac{|s|K_j}{\varphi_\alpha(h)}} \frac{sin(v)-v}{v^{1+\alpha}} dv \right]} \tag{39}$$

In each of the three cases $\alpha < 1, \alpha = 1, \alpha > 1$ the right speed is the $\varphi_\alpha(h)$ such that the exponent in the above expression converges to a finite quantity.

In the case $\alpha \in (0,1)$ we have $\varphi_\alpha(h) = h$, $\frac{cos(v)-1}{v^{1+\alpha}}$, $\frac{\sin(v)}{v^{1+\alpha}} \in L^1(R_+)$, while

$$\left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha sgn(s) \int_{0<v\leq \frac{|s|K_j}{h}} \frac{v}{v^{1+\alpha}} dv = \frac{sK_j}{h} \frac{1}{1-\alpha}.$$

It follows from (39) that $E[e^{isZ_n}]$ equals the exponential of

$$\sum_{j=1}^n \Delta \left|\frac{sK_j}{h}\right|^\alpha (A_+ + A_-) \left[\int_{0<v\leq \frac{|s|K_j}{h}} \frac{cos(v) - 1}{v^{1+\alpha}} dv + i\beta sgn(s) \int_{0<v\leq \frac{|s|K_j}{h}} \frac{sin(v)}{v^{1+\alpha}} dv \right]$$

$$-i \sum_{j=1}^n \Delta \frac{sK_j}{h} \frac{A_+ - A_-}{1-\alpha}$$

Recall that (from [8], Lemma 14.11)

$$\int_{\mathbb{R}_+} \frac{cos(v) - 1}{v^{1+\alpha}} dv = \begin{cases} \Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right), & \alpha \in (0,1) \cup (1,2) \\ -\frac{\pi}{2}, & \alpha = 1, \end{cases} \tag{40}$$

$$\begin{cases} \int_0^{+\infty} \frac{\sin(v)}{v^{1+\alpha}} dv = -\Gamma(-\alpha) \sin\left(\frac{\pi\alpha}{2}\right), & \text{if } \alpha \in (0,1) \\ \int_0^1 \frac{\sin(v) - v}{v^2} dv + \int_1^{+\infty} \frac{\sin(v)}{v^2} dv < +\infty, & \text{if } \alpha = 1 \end{cases} \tag{41}$$

$$\begin{cases} \int_0^{+\infty} \frac{e^{ir}-1-ir}{r^{1+\alpha}} dr = \Gamma(-\alpha) e^{-i\pi\frac{\alpha}{2}}, & \text{if } \alpha \in (1,2) \\ \\ \int_0^{+\infty} \frac{e^{-ir}-1+ir}{r^{1+\alpha}} dr = \Gamma(-\alpha) e^{i\pi\frac{\alpha}{2}}, & \text{if } \alpha \in (1,2) \end{cases} \tag{42}$$

Thus, since the two integrals in the above exponent of $E[e^{isZ_n}]$ are dominated by constants, $|s|^\alpha \sum_{j=1}^n \Delta \frac{K_j^\alpha}{h^\alpha} = |s|^\alpha \frac{\sum_{j=1}^n \Delta K_j^\alpha}{h} \cdot h^{1-\alpha} \to 0$, and $a = \int_{|x|\le 1} x\lambda(x)dx = \frac{A_+ - A_-}{1-\alpha}$, we have

$$E[e^{isZ_n}] \to e^{-is\frac{A_+ - A_-}{1-\alpha}} = e^{-isa},$$

where the limit is the characteristic function of the constant random variable $-a$.

If we do not compensate the small jumps and only consider

$$Y_n \doteq \frac{\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu}{h^{1/\alpha}},$$

then

$$E[e^{iz\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu}] = e^{\Delta \int_{|x|\le 1}(e^{izx}-1)\lambda(x)dx},$$

thus, with $z = \frac{sK_j}{h^{1/\alpha}}$,

$$E[e^{isY_n}] = \prod_{j=1}^n e^{\Delta \int_{|x|\le 1}(e^{izx}-1)\lambda(x)dx},$$

and each integral at exponent differs from expression (38) because the last term $-zx/x^{1+\alpha}$ there is absent here. Thus $E[e^{isY_n}]$ coincides with

$$e^{\sum_{j=1}^n \Delta \left|\frac{sK_j}{h^{1/\alpha}}\right|^\alpha \left[(A_+ + A_-)\int_{0<v\le\frac{|s|K_j}{h^{1/\alpha}}} \frac{cos(v)-1}{v^{1+\alpha}}dv + i(A_+ - A_-)sgn(s)\int_{0<v\le\frac{|s|K_j}{h^{1/\alpha}}} \frac{sin(v)}{v^{1+\alpha}}dv\right]} \tag{43}$$

and by Lemma 3 i) we have

$$\sum_{j=1}^n \frac{K_j^\alpha}{h} \Delta \int_{0<v\le\frac{|s|K_j}{h^{\frac{1}{\alpha}}}} \frac{cos(v)-1}{v^{1+\alpha}} dv \to K_{(\alpha)}\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right),$$

$$\sum_{j=1}^n \frac{K_j^\alpha}{h} \Delta \int_{0<v\le\frac{|s|K_j}{h^{\frac{1}{\alpha}}}} \frac{sin(v)}{v^{1+\alpha}} dv \to -K_{(\alpha)}\Gamma(-\alpha)\sin\left(\frac{\pi\alpha}{2}\right).$$

Thus

$$E[e^{isY_n}] \to e^{|s|^\alpha K_{(\alpha)}\Gamma(-\alpha)\left((A_+ + A_-)\cos\left(\frac{\pi\alpha}{2}\right) - i\, sgn(s)(A_+ - A_-)\sin\left(\frac{\pi\alpha}{2}\right)\right)}:$$

by collecting $(A_+ + A_-)\cos\left(\frac{\pi\alpha}{2}\right)$ and recalling that $\beta = \frac{A_+ - A_-}{A_+ + A_-}$ and that $\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) < 0$, we obtain that the above display coincides with $E[e^{isZ_{1,\alpha}}]$, having used notation (9).

If $\alpha = 1$, with $\varphi_\alpha(h) = h\log\frac{1}{h}$ and $z_j = \frac{sK_j}{h\log\frac{1}{h}}$, from (39) we have

$$E[e^{isZ_n}] = e^{\sum_{j=1}^n \Delta|z_j|\left[(A_+ + A_-)\int_0^{|z_j|}\frac{cos(v)-1}{v^2}dv + i(A_+ - A_-)sgn(z_j)\int_0^{|z_j|}\frac{sin(v)-v}{v^2}dv\right]} \tag{44}$$

The exponent above is

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\left[|s|(A_+ + A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}}\frac{cos(v)-1}{v^2}dv + \right.$$
$$\left. is(A_+ - A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}}\frac{sin(v)-v}{v^2}dv\right]$$

which is shown to tend to $-is(A_+ - A_-)$: the first integrand $\frac{cos(v)-1}{v^2}I_{v>0}$ is in $L^1(\mathbb{R})$, thus, applying Lemma 3 i) we obtain that

$$|s|\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}(A_+ + A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}}\frac{cos(v)-1}{v^2}dv \to 0.$$

The second integral is written as

$$\int_0^{|z_j|}\frac{\sin(v)-v}{v^2}dvI_{|z_j|\le 1} + \left[\int_0^1\frac{\sin(v)-v}{v^2}dv + \int_1^{|z_j|}\frac{\sin(v)}{v^2}dv - \log(|z_j|)\right]I_{|z_j|>1} \tag{45}$$

where $\frac{sin(v)-v}{v^2} \in L^1((0,1))$, and $\frac{sin(v)}{v^2}I_{v\in(1,+\infty)} \in L^1(\mathbb{R})$. Note that if $s = 0$ we directly find that $E[e^{isZ_n}] = 1$, we thus only concentrate on a fixed $s \neq 0$. We have that

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\left(\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}}\left|\frac{sin(v)-v}{v^2}\right|dvI_{|z_j|\le 1} + \int_0^1\left|\frac{sin(v)-v}{v^2}\right|dv\right) \le$$
$$\frac{\sum_{j=1}^n \Delta K_j}{h}2\int_0^1\left|\frac{sin(v)-v}{v^2}\right|dv\frac{1}{\log\frac{1}{h}} \le \frac{\sum_{j=1}^n \Delta K_j}{h}\frac{C}{\log\frac{1}{h}} \to 0,$$

and

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\int_1^{\frac{|s|K_j}{h\log\frac{1}{h}}}\frac{sin(v)}{v^2}dvI_{|z_j|>1} \le \frac{C}{\log\frac{1}{h}}\frac{\sum_{j=1}^n \Delta K_j}{h} \to 0.$$

Finally, recalling that K is bounded (by **IA1**),

$$-is(A_+ - A_-)\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\log\left(\frac{|s|K_j}{h\log\frac{1}{h}}\right) I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}} \to -is(A_+ - A_-)K_{(1)},$$

since within

$$\frac{\sum_{j=1}^n K_j\Delta}{h\log\frac{1}{h}}\left[\log(|s|) + \log(K_j) + \log(\frac{1}{h}) - \log\left(\log\frac{1}{h}\right)\right] I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}}$$

the first two terms are bounded in absolute value by

$$\frac{1}{\log\frac{1}{h}}\left[\frac{\sum_{j=1}^n |K_j\log(K_j)|\Delta}{h} + \frac{C\sum_{j=1}^n K_j\Delta}{h}\right] \to 0,$$

the third term converges by Lemma 3 i):

$$\sum_{j=1}^n \frac{K_j}{h}\Delta I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}} \to K_{(1)};$$

and the fourth one

$$\sum_{j=1}^n \frac{K_j}{h}\Delta I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}}\frac{\log\left(\log\frac{1}{h}\right)}{\log\frac{1}{h}} \to 0.$$

Thus the statement is proved.

If $\alpha \in (1, 2)$ we can directly use the relations in (42) In fact, from (37), where $z_j = s\frac{K_j}{\varphi_\alpha(h)} = s\frac{K_j}{h^{1/\alpha}}$, we change variable $v = |z_j|x$ in the first integral, while in the second one we firstly change in $y = -x$, then in $v = |z_j|y$, and we reach

$$|z_j|^\alpha\Bigg[A_+\int_0^{|z_j|}\frac{e^{iv\cdot sgn(z_j)} - 1 - iv\cdot sgn(z_j)}{v^{1+\alpha}}dv + \tag{46}$$

$$A_-\int_0^{|z_j|}\frac{e^{-iv\cdot sgn(z_j)} - 1 + iv\cdot sgn(z_j)}{v^{1+\alpha}}dv\Bigg]$$

With $g(v) = \frac{e^{iv}-1-iv}{v^{1+\alpha}}I_{v>0} \in L^1(\mathbb{R})$, and $\bar{g}$ its complex conjugate, the above equals

$$|z_j|^\alpha\left(A_+\int_0^{|z_j|} g(v)I_{z_j>0} + \bar{g}(v)I_{z_j<0}\,dv + A_-\int_0^{|z_j|}\bar{g}(v)I_{z_j>0} + g(v)I_{z_j<0}\,dv\right)$$

thus

$$E[e^{isZ_n}] = e^{\sum_{j=1}^n \Delta \left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha \left[I_{z_j>0}\int_0^{|z_j|} A_+ g(v)+A_-\bar g(v)\ dv + I_{z_j<0}\int_0^{|z_j|} A_+\bar g(v)+A_- g(v)dv\right]}$$

With $\varphi_\alpha(h) = h^{\frac{1}{\alpha}}$, by Lemma 3 i), the exponent

$$\frac{|s|^\alpha \sum_{j=1}^n \Delta K_j^\alpha}{h}\left[I_{s>0}\int_0^{|z_j|} A_+ g(v) + A_-\bar g(v)dv + I_{s<0}\int_0^{|z_j|} A_+\bar g(v) + A_- g(v)dv\right]$$

tends to

$$|s|^\alpha K_{(\alpha)}\Gamma(-\alpha)\left(I_{s>0}\Big(A_+ e^{-i\pi\frac{\alpha}{2}} + A_- e^{i\pi\frac{\alpha}{2}}\Big) + I_{s<0}\Big(A_+ e^{i\pi\frac{\alpha}{2}} + A_- e^{-i\pi\frac{\alpha}{2}}\Big)\right).$$

By developing and simplifying, the above expression becomes

$$-|s|^\alpha K_{(\alpha)}\ c\left(1 - i\beta\tan\left(\frac{\alpha\pi}{2}\right) sgn(s)\right),$$

where $c = -\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)(A_+ + A_-), \beta = \frac{A_+ - A_-}{A_+ + A_-}$, the statement is proved and the first step concluded.

Second step. We come back to the small jumps of the form described at **IA3**. Also for X we look at the characteristic functions of the quantities in the statement of the Lemma, and at their asymptotic behavior. Now we employ (36) and show that the contribution from X' is negligible, because X' has jump activity index less than α.

On the enlarged space $(\bar\Omega, \bar{\mathcal F}, \{\bar{\mathcal F}_t\}_{t\in[0,T]}, \bar P\big)$ we have

$$\bar E\left[f_n(\tilde J)\right] = \bar E\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde J}\right] = \bar E\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde X} e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde X'}\right]$$

$$= \bar E\left[f_n(\tilde X) g_n(\tilde X')\right] = E^P\left[f_n(\tilde X)E^{Q_\omega}[g_n(\tilde X')]\right],$$

where $f_n(\tilde X) \doteq e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde X}, g_n(\tilde X') \doteq e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde X'}$ and we recall that the Lévy measure $\nu'\big((\omega,\omega'), dx, ds\big)$ of X', given in (36), does not depend on ω'.

Case $\alpha > 1$: under **IA3**, X' has FV, since $\int_0^T\int_{|x|\le 1}|x|\lambda'(s,x)dxds < \infty$, thus

$$E^{Q_\omega}\left[\left|\sum_{j=1}^n \frac{K_j}{h^{1/\alpha}}\Delta_j\tilde X'\right|\right] \le \sum_{j=1}^n \frac{K_j}{h^{1/\alpha}} E^{Q_\omega}\left[\left|\Delta_j\tilde X'\right|\right]:$$

by (2.1.36) in [6] with $p=1$, and using **IA3** the latter is dominated by

$$C\sum_{j=1}^n \frac{K_j}{h^{1/\alpha}} \int_{t_{j-1}}^{t_j}\int_{|x|\le 1} |x|\lambda'(s,x)dx = C\sum_{j=1}^n \frac{K_j\Delta}{h}\ h^{1-1/\alpha} \to 0,$$

thus, for any fixed ω, $\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde{X}'$ tends to 0 in probability wrt Q_ω, so $g_n(\tilde{X}') \stackrel{Q_\omega}{\to} 1$ and we can apply Lemma 7 and conclude that

$$\lim_n E^P\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde{X}}\right] = \lim_n \bar{E}\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde{J}}\right].$$

Since under $\bar{P}$ the process $\tilde{J}$ is α-stable, the first step of this proof applies, thus

$$\lim_n \bar{E}\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde{J}}\right] = \bar{E}\left[e^{is\bar{Z}_{1,\alpha}}\right],$$

where $\bar{Z}_{1,\alpha}$ has under $\bar{P}$ the same law as $Z_{1,\alpha}$ under $\bar{P}$, so

$$\bar{E}\left[e^{is\bar{Z}_{1,\alpha}}\right] = E\left[e^{isZ_{1,\alpha}}\right].$$

It follows that

$$\lim_n E^P\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\Delta_j\tilde{X}}\right] = E^P\left[e^{isZ_{1,\alpha}}\right],$$

and (23) is proved.

Case $\alpha=1$: now the fact that

$$E^{Q_\omega}\left[\left|\sum_{j=1}^n \frac{K_j}{h^1}\Delta_j\tilde{X}'\right|\right] \le C\sum_{j=1}^n \frac{K_j}{h}\int_{t_{j-1}}^{t_j}\int_{|x|\le 1}|x|\lambda'(s,x)dx \le C\sum_{j=1}^n \frac{K_j}{h}\Delta,$$

allows to state that for any ω

$$\sum_{j=1}^n \frac{K_j}{h\log\frac{1}{h}}\Delta_j\tilde{X}' \xrightarrow{Q_\omega} 0,$$

and again, for any ω, $g_n(\tilde{X}') \stackrel{Q_\omega}{\to} 1$, so by Lemma 7 we have

$$\lim_n E^P\left[e^{\sum_{j=1}^n is\frac{K_j}{h\log\frac{1}{h}}\Delta_j\tilde{X}}\right] = \lim_n \bar{E}\left[e^{\sum_{j=1}^n is\frac{K_j}{h\log\frac{1}{h}}\Delta_j\tilde{J}}\right] = e^{-is(A_+-A_-)K_{(1)}},$$

and (22) is proved.

If $\alpha \in (0,1)$, the jumps of $\tilde{X}$, $\tilde{J}$ and $\tilde{X}'$ have FV and we can separately deal with the not compensated small jumps and the compensator. Further, now the jump activity index of X', by assumption **IA3**, is $\alpha' \le r$.
Let us first consider the not compensated jumps: defining $V \doteq \int_0^\cdot \int_{|x|\le 1} x d\mu$ on Ω and analogously $\bar{V}$ on $\bar{\Omega}$ and V' on Ω', we have

$$\bar{E}\left[f_n(\bar{V})\right] = \bar{E}\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int\int x d\mu^J}\right] = \bar{E}\left[f_n(V)g_n(V')\right],$$

where $\int\int x d\mu$ stands for $\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu$, $f_n(\bar{V}) \doteq e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int\int x d\mu}$ and $g_n(V') \doteq e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int\int x d\mu'}$. Using that $\int_{|x|\le 1}|x|^r\lambda'(x,s)dx \le C$ (assumption **IA3**) and $K^r \in L^1(\mathbb{R})$ (assumption of Lemma 4), by Lemma 8 we obtain that for all ω

$$g_n(V') = e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu'} \xrightarrow{Q_\omega} 1,$$

so, by Lemma 7 and the first step of this proof,

$$\lim_n E^P\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int\int x d\mu}\right] = \lim_n \bar{E}\left[e^{\sum_{j=1}^n is\frac{K_j}{h^{1/\alpha}}\int\int x d\mu^J}\right] = E\left[e^{isZ_{1,\alpha}}\right]$$

and the second part of (21) is proved.
We now analyze the first part of (21) directly for X. Since we just proved that, on Ω, $\sum_{j=1}^n \frac{K_j}{h^{1/\alpha}}\int\int x d\mu \xrightarrow{d} Z_{1,\alpha}$, then

$$\sum_{j=1}^n \frac{K_j}{h}\int\int x d\mu = h^{\frac{1}{\alpha}-1}\cdot\sum_{j=1}^n \frac{K_j}{h^{1/\alpha}}\int\int x d\mu \xrightarrow{P} 0.$$

On the other hand we have $\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x\lambda(x,v)dxdv = \int_{t_{j-1}}^{t_j} a_v dv$, and a_v satisfies **A3**, thus by Lemma 2, parts 1) and 4), we have

$$\frac{\sum_{j=1}^n \frac{K_j}{h}\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x\lambda(x,v)dxdv}{h} \xrightarrow{P} a_{\bar{t}}^\star,$$

therefore

$$\sum_{j=1}^n \frac{K_j}{h}\Delta_j\tilde{X} = \sum_{j=1}^n \frac{K_j}{h}\Big[\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu - \int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x\lambda(x,v)dxdv\Big] \xrightarrow{P} -a_{\bar{t}}^\star,$$

and also the first part of (21) is done. $\square$

Proof of Lemma 5.

First step: α-stable J. We show that the Laplace transforms of either $\frac{\sum_{i=1}^n K_i(\int_{t_{i-1}}^{t_i}\int_{|x|\le 1} x d\mu)^2}{\psi_\alpha(h)}$ when $\alpha<1$, or $\frac{\sum_{i=1}^n K_i(\Delta_i \tilde{J})^2}{\psi_\alpha(h)}$ when $\alpha \ge 1$, converge to the Laplace transform of the limit shown in the statement of this Lemma (see [3], theorem 6.6.3 for the properties of the Laplace transforms limit). For that, since the law density of J is not available in explicit form, we are going to use the characteristic function as follows. For a r.v. U on $\mathbb{R}$ with law density $u(x)$ and for a given $v>0$, it is possible to compute $E\Big[e^{-vU^2}\Big] = \int_{\mathbb{R}} e^{-vx^2}u(x)dx$ by interpreting e^{-vx^2} as the characteristic function $E[e^{ixW}]$ of a Gaussian random variable W, with mean 0, variance $\sigma^2 \doteq 2v$ and density $\phi(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$, and to obtain $E\Big[e^{-vU^2}\Big]$ only using the characteristic function of U. In fact

$$E\Big[e^{-vU^2}\Big] = \int_{\mathbb{R}} e^{-vx^2}u(x)dx = \int E[e^{ixW}]u(x)dx$$

$$= \int\int e^{ixz}\phi(z)dz\ u(x)dx = \int\phi(z)\int e^{ixz}u(x)dx\ dz = \int\phi(z)E[e^{izU}]dz.$$

We will apply this in the following way: $v = v_j \doteq \frac{sK_j}{\psi_\alpha(h)}$ and:

$\cdot$ when $\alpha<1$, $U=U_j=\int_{t_{i-1}}^{t_i}\int_{|x|\le 1} x d\mu$ and $E[e^{izU_j}] = e^{\Delta\int_{|r|\le 1} e^{izr}-1\ \lambda(dr)}$;

$\cdot$ when $\alpha\in[1,2)$, $U=U_j=\Delta_j\tilde{J} = \int_{t_{i-1}}^{t_i}\int_{|x|\le 1} x d\tilde{\mu}$ and

$E[e^{izU_j}] = e^{\Delta\int_{|r|\le 1} e^{izr}-1-izr\ \lambda(dr)}$.

Then

$$E[e^{-\sum_{j=1}^n v_j U_j^2}] = \prod_{j=1}^n E\Big[e^{-v_jU_j^2}\Big] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{z^2}{2\sigma_j^2}}}{\sigma_j\sqrt{2\pi}} E[e^{izU_j}]dz, \tag{47}$$

with $\sigma_j^2 = 2v_j$. The latter display, with $u \doteq \frac{z}{\sigma_j}$, becomes

$$\prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot E[e^{i\sigma_j u U_j}]du. \tag{48}$$

Case $\alpha\in(0,1)$. Let $V_n \doteq \frac{\sum_{i=1}^n K_i(\int_{t_{i-1}}^{t_i}\int_{|x|\le 1} x d\mu)^2}{\psi_\alpha(h)}$, then, with $s>0$ and $v_j = sK_j/\psi_\alpha > 0$,

$$E[e^{-sV_n}] \doteq E[e^{-\sum_{j=1}^n v_j U_j^2}] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot e^{\Delta\int_{|r|\le 1} e^{i\sigma_j ur}-1\ \lambda(dr)}du. \tag{49}$$

Similarly as when from (37) we obtained (39) and then (43), with $z = \frac{sK_j}{h^{1/\alpha}}$ there replaced by $\sigma_j u = \sqrt{2v_j} \cdot u$ here, we have

$$\int_{|r|\leq 1} e^{i\sigma_j ur} - 1 \, \lambda(dr) = \sigma_j^\alpha |u|^\alpha (A_+ + A_-) \int_0^{\sigma_j|u|} \left[\frac{cos(w) - 1}{w^{1+\alpha}} + i\beta sgn(u) \frac{\sin(w)}{w^{1+\alpha}} \right] dw$$

$$\doteq \sigma_j^\alpha |u|^\alpha \int_0^{\sigma_j|u|} f_u(w) dw \doteq \sigma_j^\alpha |u|^\alpha g_j(u), \tag{50}$$

then we are left with

$$E[e^{-sV_n}] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta \sigma_j^\alpha |u|^\alpha g_j(u)} du.$$

By developing $e^y = \sum_{k=0}^{+\infty} \frac{y^k}{k!}$, we obtain $\prod_{j=1}^n \left(1 + \theta_j^{(n)}\right) \doteq$

$$\prod_{j=1}^n \left[\int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du + \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \Delta \sigma_j^\alpha |u|^\alpha g_j(u) du + \sum_{k\geq 2} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\Delta^k \left(\sigma_j^\alpha |u|^\alpha g_j(u)\right)^k}{k!} du \right] \tag{51}$$

We are now going to show that

- $(c1)$ $\forall j = 1, \ldots, n, \theta_j^{(n)} \to 0$ and $\max_{j=1,\ldots,n} |\theta_j^{(n)}| \to 0$
- $(c2)$ $\sum_{j=1}^n |\theta_j^{(n)}| \leq M < \infty$
- $(c3)$ $\sum_{j=1}^n \theta_j^{(n)} \to \theta,$

where M does not depend on n, and

$$\theta \doteq s^{\frac{\alpha}{2}} 2^\alpha K_{(\alpha/2)} (A_+ + A_-) \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{\sqrt{\pi}} \Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) < 0.$$

That allows to conclude ([3], Lemma p.199) that

$$E[e^{-sV_n}] = \prod_{j=1}^n \left(1 + \theta_j^{(n)}\right) \to e^\theta,$$

which is the Laplace transform of the law of the $Z_{2,\alpha}$ in the notations, and the stated result follows.

Let us now evaluate the numbers $\theta_j^{(n)}$. Denoted

$$\theta_{j,1}^{(n)} \doteq \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta \sigma_j^\alpha |u|^\alpha g_j(u) du \tag{52}$$

$$= \int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha u^\alpha (A_+ + A_-) \int_0^{\sigma_j u} \left[\frac{\cos(w) - 1}{w^{1+\alpha}} + i\beta \frac{\sin(w)}{w^{1+\alpha}} \right] dwdu$$

$$+ \int_{\mathbb{R}_-} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha (-u)^\alpha (A_+ + A_-) \int_0^{\sigma_j \cdot (-u)} \left[\frac{\cos(w) - 1}{w^{1+\alpha}} - i\beta \frac{\sin(w)}{w^{1+\alpha}} \right] dwdu, \tag{53}$$

by changing variable $y = -u$, the second integral in du becomes

$$\int_{\mathbb{R}_+} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha y^\alpha (A_+ + A_-) \int_0^{\sigma_j y} \frac{\cos(w) - 1}{w^{1+\alpha}} - i\beta \frac{\sin(w)}{w^{1+\alpha}} dwdy :$$

by renaming u the variable y of the latter integral, in (53) the sin function simplifies, and we obtain

$$\theta_{j,1}^{(n)} = 2 \int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha u^\alpha (A_+ + A_-) \int_0^{\sigma_j u} \frac{\cos(w) - 1}{w^{1+\alpha}} dwdu. \tag{54}$$

We preliminarily show that

$(c4)$ $\sum_{j=1}^n \theta_{j,1}^{(n)} \to \theta$

$(c5)$ $\sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| \to 0.$

Note that the function $\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^{\alpha k}$ is in $L^1(\mathbb{R})$ for any integer k, with

$$\int_{\mathbb{R}_+} e^{-\frac{u^2}{2}} |u|^{\alpha k} du = 2^{\frac{\alpha k - 1}{2}} \Gamma\left(\frac{\alpha k + 1}{2}\right). \tag{55}$$

As for (c4), using the notation in (50), Lemma 3 iii), (55) and (40) and with $\sigma_j = \sqrt{2w_j} = \sqrt{2\frac{sK_j}{\psi_\alpha(h)}}$ we have that $\sum_{j=1}^n \theta_{j,1}^{(n)}$ coincides with

$$\sum_{j=1}^n \Delta\sigma_j^\alpha \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha g_j(u) du = s^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}} \Delta}{h} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha \int_0^{\sigma_j |u|} f(w) dwdu$$

$$= s^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}} \Delta}{h} 2(A_+ + A_-) \int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha \int_0^{\sigma_j |u|} \frac{\cos(w) - 1}{w^{1+\alpha}} dvdu \to \theta. \tag{56}$$

As for (c5), since for all $j = 1, \ldots, n$, $|g_j(u)| \le C \int_{\mathbb{R}_+} \frac{|cos(w) - 1|}{w^{1+\alpha}} + \frac{|\sin(w)|}{w^{1+\alpha}} dw < \infty$, $g_j(u)$ is bounded uniformly in j and u, thus we have that $\sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}|$ is dominated by

$$\sum_{j=1}^n \sum_{k \ge 2} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \frac{\Delta^k \left(C\sigma_j^\alpha |u|^\alpha \right)^k}{k!} du = \sum_{j=1}^n \sum_{k \ge 2} C^k \left(\frac{\Delta}{h} \right)^k \frac{2^{\frac{\alpha k}{2}} K_j^{\frac{\alpha k}{2}}}{k!} 2^{\frac{\alpha k - 1}{2}} \Gamma\left(\frac{\alpha k + 1}{2} \right), \tag{57}$$

where the term $s^{\frac{\alpha}{2}}$ within σ_j has been included into C, because s is fixed. Since the kernel K is bounded, the above is dominated by

$$\left(\frac{\Delta}{h}\right)^2 n \sum_{k\geq 2} \left(\frac{\Delta}{h}\right)^{k-2} C^k \frac{2^{\alpha k - \frac{1}{2}}}{k!} \Gamma\left(\frac{\alpha k+1}{2}\right). \tag{58}$$

Since for large n we have $\Delta/h < 1$, in the the series above, for sufficiently small Δ and large k, we have

$$\frac{a_{k+1}}{a_k} = \frac{C}{k+1}\frac{\Delta}{h}\frac{\Gamma\left(\frac{\alpha k+\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha k+1}{2}\right)} = \frac{C}{k+1}\frac{\Delta}{h}\frac{\alpha k+\alpha+1}{2}\frac{\Gamma\left(\frac{\alpha k+\alpha-1}{2}\right)}{\Gamma\left(\frac{\alpha k+1}{2}\right)}$$
$$< \frac{C}{k+1}\frac{\Delta}{h}\frac{\alpha k+\alpha+1}{2} < \frac{C}{k+1}\frac{\Delta}{h}\left(\frac{k}{2}+1\right) < C\frac{\Delta}{h} < 1,$$

because $0 < \alpha < 1$ and for large k the the argument of the Gamma function is positive, so the function is increasing. Thus by the quotient criterion the series is absolutely convergent, and (58) is $O\left(\frac{\Delta}{h^2}\right)$, therefore it tends to 0, and (c5) is verified.

It follows that, since $\theta_{j,1}^{(n)} = \Delta\sigma_j^\alpha \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}|u|^\alpha g_j(u)du$, where $\sigma_j^\alpha \leq C\frac{K(0)^{\frac{\alpha}{2}}}{h}$ and $g_j(u)$ is uniformly bounded, thus $|\theta_{j,1}^{(n)}| \leq C\Delta/h$ uniformly in j, and

$$\max_{j=1,\dots,n}|\theta_j^{(n)}| \leq \max_{j=1,\dots,n}|\theta_j^{(n)} - \theta_{j,1}^{(n)}| + \max_{j=1,\dots,n}|\theta_{j,1}^{(n)}| \leq \sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| + C\frac{\Delta}{h} = O\left(\frac{\Delta}{h^2}\right)$$

and thus tends to 0, which solves (c1).

As for (c2), using again Lemma 3 iii), we have that $\sum_{j=1}^n |\theta_{j,1}^{(n)}|$ is dominated by

$$\sum_{j=1}^n \sigma_j^\alpha \Delta \int_{\mathbb{R}} \Psi(u)|g_j(u)|du \leq C\sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}}}{h}\Delta\int_{\mathbb{R}}\Psi(u)\int_0^{|u|\sqrt{\frac{2|s|K_j}{h^{2/\alpha}}}} |f(w)|dw\, du \to C,$$

thus using also that (57) is $O(\Delta/h^2)$ we reach

$$\sum_{j=1}^n |\theta_j^{(n)}| \leq \sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| + \sum_{j=1}^n |\theta_{j,1}^{(n)}| \leq C\frac{\Delta}{h^2} + C \leq M.$$

Finally (c3) follows directly from (c4) and (c5).

Case $\alpha \in [1,2)$**.** Let now $\tilde V_n \doteq \frac{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}{\psi_\alpha(h)}$, then, with $v_j = sK_j/\psi_\alpha > 0$,

$$E[e^{-s\tilde V_n}] = E[e^{-\sum_{j=1}^n v_j U_j^2}] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta \int_{|r|\le 1} e^{i\sigma_j ur} - 1 - i\sigma_j ur \ \lambda(dr)} du.$$

The integral in $\lambda(dr)$ is given by

$$\int_0^1 (A_+ + A_-)\frac{cos(\sigma_j ur) - 1}{r^{1+\alpha}} + i(A_+ - A_-)\frac{\sin(\sigma_j ur) - \sigma_j ur}{r^{1+\alpha}} \ dr =$$

$$\sigma_j^\alpha |u|^\alpha \int_0^{\sigma_j|u|} (A_+ + A_-)\frac{cos(w) - 1}{w^{1+\alpha}} + i(A_+ - A_-)sgn(u)\frac{\sin(w) - w}{w^{1+\alpha}} \ dw \doteq \sigma_j^\alpha |u|^\alpha \tilde g_j(u)$$

Thus

$$E[e^{-s\tilde V_n}] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta \sigma_j^\alpha |u|^\alpha \tilde g_j(u)} du$$

$$= \prod_{j=1}^n \left(1 + \sum_{k\ge 1} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \frac{\left(\Delta \sigma_j^\alpha |u|^\alpha \tilde g_j(u)\right)^k}{k!} du \right) \doteq \prod_{j=1}^n \left(1 + \tilde\theta_j^{(n)}\right). \qquad (59)$$

Again, we show that $\tilde\theta_{j,1}^{(n)} \doteq \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta \sigma_j^\alpha |u|^\alpha \tilde g_j(u) du$ turns out to be the leading term of $\tilde\theta_j^{(n)}$, and that the conditions (c1) to (c5) above are satisfied also for $\tilde\theta_j^{(n)}$, which allows to conclude the proof. Note that for any $\alpha \in [1,2)$, similarly as from (52) to (54),

$$\tilde\theta_{j,1}^{(n)} = 2\int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta \sigma_j^\alpha u^\alpha \int_0^{\sigma_j u} (A_+ + A_-)\frac{cos(w) - 1}{w^{1+\alpha}} dw du,$$

which is the same expression of $\theta_{j,1}^{(n)}$ in (52), thus $\sum_{j=1}^n \tilde\theta_{j,1}^{(n)}$ coincides exactly with the left part of the last line in (56). By Lemma 3 iii), using (55) and the relations in (40) we obtain that for $\alpha = 1$ then $\sum_{j=1}^n \tilde\theta_{j,1}^{(n)} \to \tilde\theta \doteq -s^{\frac{\alpha}{2}} 2^{\alpha-1}\sqrt{\pi} K_{(\alpha/2)}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)$, while for $\alpha \in (1,2)$ then $\sum_{j=1}^n \tilde\theta_{j,1}^{(n)} \to \theta$, and a condition of type (c4) is satisfied in any case.

As for (c5), we need to bound differently $|\tilde\theta_j^{(n)} - \tilde\theta_{j,1}^{(n)}|$ in the two cases $\alpha = 1$, $\alpha \in (1,2)$.

If $\alpha = 1$, splitting as in (45), we write

$$\tilde g_j(u) = (A_+ + A_-)\int_0^{\sigma_j|u|} \frac{cos(w) - 1}{w^2} dw + i(A_+ - A_-)sgn(u)\int_0^{\sigma_j|u|} \frac{\sin(w) - w}{w^2} dw I_{\sigma_j|u|\le 1}$$

$$+i(A_+ - A_-)sgn(u)\left[\int_0^1 \frac{\sin(w)-w}{w^2}dw + \int_1^{\sigma_j|u|}\frac{\sin(w)}{w^2}dw - \log(\sigma_j|u|)\right] I_{\sigma_j|u|>1},$$

where $\log(\sigma_j|u|) = \frac{1}{2}\log(2s) + \frac{1}{2}\log(K_j) + \log\left(\frac{1}{h}\right) + \log(|u|)$, thus

$$\tilde{g}_j(u) \doteq \ell_j(u) - i(A_+ - A_-)sgn(u)\left[\tfrac{1}{2}\log(K_j) + \log\left(\tfrac{1}{h}\right) + \log(|u|)\right] I_{\sigma_j|u|>1},$$

where $\ell_j(u)$ is uniformly bounded in j and u. Using that $|u\log(|u|)| \le |u|^2 I_{|u|>1}$ $+\frac{1}{e}I_{0<|u|<1}$, then for any triplet of positive quantities A_1, A_2, A_3 with $A = A_1 + A_2 + A_3$, we have

$$|u|^k\left[A + |\log|u||\right]^k \le |u|^k 2^k\left[A^k + |\log|u||^k\right] = 2^k\Big(|u|^k A^k + (|u\log|u||)^k\Big) \le$$

$$2^k\Big(|u|^k A^k + (u^2 + C)^k\Big) \le 8^k\Big(|u|^k(A_1^k + A_2^k + A_3^k) + u^{2k} + C^k\Big).$$

Thus

$$\left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right| \le \sum_{k\ge 2}\frac{\Delta^k}{h^k}C^k\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}|u|^k\left[C + |\log(K_j)| + \log\left(\frac{1}{h}\right) + |\log|u||\right]^k du$$

$$\le \sum_{k\ge 2}\frac{\Delta^k}{h^k}C^k\frac{K_j^{\frac{k}{2}}}{k!}\cdot 2\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\left[u^k C^k + u^k|\log(K_j)|^k + u^k\log^k\left(\frac{1}{h}\right) + u^{2k} + C^k\right]du:$$

similarly as above,

$$\sum_{k\ge 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = \frac{\Delta^2}{h^2}\sum_{k\ge 2}C^k\frac{\Delta^{k-2}}{h^{k-2}}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\frac{\Delta^2}{h^2}\right),$$

$$\sum_{k\ge 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^{2k}du = O\left(\frac{\Delta^2}{h^2}\right), \quad \sum_{k\ge 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!} = O\left(\frac{\Delta^2}{h^2}\right);$$

since $\sqrt{K}|\log(K)|$ is bounded, also

$$\sum_{k\ge 2}\frac{\Delta^k}{h^k}C^k\frac{\left(K_j^{\frac{1}{2}}|\log(K_j)|\right)^k}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\frac{\Delta^2}{h^2}\right).$$

Finally,

$$\sum_{k\ge 2}\left(\frac{\Delta\log\left(\frac{1}{h}\right)}{h}\right)^k C^k\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\left(\frac{\Delta\log\left(\frac{1}{h}\right)}{h}\right)^2\right),$$

thus $\sum_{j=1}^n\left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right| = O\left(\frac{\Delta\log^2\left(\frac{1}{h}\right)}{h^2}\right) \to 0$, and (c5) for $\tilde{\theta}_j^{(n)}$ is proved.

Thus (c1), (c2) and (c3) for $\tilde{\theta}_j^{(n)}$ follow analogously as for $\theta_j^{(n)}$.

If $\alpha \in (1,2)$**,** due to (42), $\tilde{g}_j(u)$ is uniformly bounded in j and u, thus $\sum_{j=1}^n \left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right|$ is dealt exactly as in (57), thus it is $O\left(\frac{\Delta}{h^2}\right) \to 0$, and (c5) is done. From (c4) and (c5) then the properties (c1) to (c3) again follow as above, and now the proof of the first step is complete.

Second step. We use (36).

$\alpha \in (0,1)$:

$$\bar{E}\left[f_n(\bar{V})\right] \doteq \bar{E}\left[e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}(\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu^J)^2}\right] = \bar{E}\left[f_n(V)g_n(V')\right],$$

where

$$f_n(V) \doteq e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}(\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu)^2}, \tag{60}$$

$$g_n(V') \doteq e^{\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\left(\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu'\right)^2 - 2s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu \int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu'}.$$

We are going to apply Lemma 7, so it is sufficient we check that for all ω, $g_n(\omega, V') \stackrel{Q_\omega}{\to} 1$. We start by showing that

$$\sum_{j=1}^n \frac{K_j}{h^{\frac{2}{\alpha}}}\left(\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu'\right)^2 \xrightarrow{Q_\omega} 0. \tag{61}$$

In fact we pick $\gamma \in \left(\frac{r}{2}, \frac{\alpha}{2}\right)$, so $\gamma < 1$. and we can say

$$E^{Q_\omega}\left[\left|\sum_{j=1}^n \frac{K_j}{h^{\frac{2}{\alpha}}}\left(\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} x d\mu'\right)^2\right|^\gamma\right] \le \sum_{j=1}^n \frac{K_j^\gamma}{h^{\frac{2\gamma}{\alpha}}} E^{Q_\omega}\left[\left|\int_{|x|\le 1} x d\mu'\right|^{2\gamma}\right], \tag{62}$$

moreover $2\gamma < 1$ and we can apply (2.1.40) in [6] and upper bound with

$$\sum_{j=1}^n \frac{K_j^\gamma}{h^{\frac{2\gamma}{\alpha}}} E^{Q_\omega}\left[\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} |x|^{2\gamma}\lambda'(x,s)dxds\right].$$

Since $2\gamma > r$ it follows from **IA3** that $\int_{|x|\le 1}|x|^{2\gamma}\lambda'(x,s)dx \le C$, and then the above is upper bounded by

$$C\sum_{j=1}^n \frac{K_j^\gamma \Delta}{h} \cdot h^{1-\frac{2\gamma}{\alpha}} \to 0,$$

since $2\gamma/\alpha < 1$. Thus (61) is proved. From it we obtain that for all ω

$$\sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \int_{t_{j-1}}^{t_j} \int_{|x|\le 1} x d\mu \int_{t_{j-1}}^{t_j} \int_{|x|\le 1} x d\mu' \xrightarrow{Q_\omega} 0,$$

because, with $\iint$ understanding $\int_{t_{j-1}}^{t_j} \int_{|x|\le 1}$,

$$\left| \sum_{j=1}^{n} \sqrt{\frac{K_j}{h^{2/\alpha}}} \iint x d\mu \sqrt{\frac{K_j}{h^{2/\alpha}}} \iint x d\mu' \right| \le \sqrt{\sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\iint x d\mu \right)^2} \sqrt{\sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\iint x d\mu' \right)^2},$$

so for all ω the above display tends to 0 in Q_ω-probability, and thus $g_n(V') \overset{\bar{Q}_\omega}{\to} 1$. We now can apply Lemma 7 to (60) and obtain

$$E^P \left[e^{-sZ_{2,\alpha}} \right] = \lim_n \bar{E} \left[e^{-s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\iint x d\mu^J \right)^2} \right] = \lim_n E^P \left[e^{-s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\iint x d\mu \right)^2} \right],$$

which concludes the proof of (24).

$\alpha = 1$: we have

$$\bar{E} \left[f_n(\tilde{J}) \right] = \bar{E} \left[e^{-s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\Delta_j \tilde{J} \right)^2} \right] = \bar{E} \left[f_n(\tilde{X}) g_n(\tilde{X}') \right] \tag{63}$$

where

$$f_n(\tilde{X}) \doteq e^{-s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\Delta_j \tilde{X} \right)^2}, g_n(\tilde{X}') \doteq e^{-s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \left(\Delta_j \tilde{X}' \right)^2 - 2s \sum_{j=1}^{n} \frac{K_j}{h^{2/\alpha}} \Delta_j \tilde{X} \Delta_j \tilde{X}'}$$

and again we show that for all ω, $g_n(\tilde{X}') \overset{Q_\omega P}{\to} 1$ and apply Lemma 7. Since $r < 1$,

$$E^{Q_\omega} \left[\left| \sum_{j=1}^{n} \frac{K_j}{h^2} \left(\Delta_j \tilde{X}' \right)^2 \right|^{\frac{r}{2}} \right] \le \sum_{j=1}^{n} \frac{K_j^{\frac{r}{2}}}{h^r} E^{Q_\omega} \left[\left| \Delta_j \tilde{X}' \right|^r \right] \tag{64}$$

$$\le C \sum_{j=1}^{n} \frac{K_j^{\frac{r}{2}}}{h^r} \Delta = C \sum_{j=1}^{n} \frac{K_j^{\frac{r}{2}} \Delta}{h} \cdot h^{1-r} \to 0.$$

Further, for all ω

$$\left| \sum_{j=1}^{n} \sqrt{\frac{K_j}{h^2}} \Delta_j \tilde{X} \sqrt{\frac{K_j}{h^2}} \Delta_j \tilde{X}' \right| \le \sqrt{\sum_{j=1}^{n} \frac{K_j}{h^2} \left(\Delta_j \tilde{X} \right)^2} \sqrt{\sum_{j=1}^{n} \frac{K_j}{h^2} \left(\Delta_j \tilde{X}' \right)^2} \overset{Q_\omega}{\to} 0. \tag{65}$$

By Lemma 7 we obtain

$$E^P\left[e^{-sZ_{2,\alpha}}\right] = \lim_n \bar{E}\left[e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\left(\Delta_j \tilde{J}\right)^2}\right] = \lim_n E^P\left[e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\left(\Delta_j \tilde{X}\right)^2}\right],$$

which concludes the proof of (25).

$\alpha > 1$: consider again (63), and repeat a similar reasoning as above. We have

$$\sum_{j=1}^n \frac{K_j}{h^{\frac{2}{\alpha}}}\left(\Delta_j \tilde{X}'\right)^2 \xrightarrow{Q_\omega} 0. \tag{66}$$

In fact we pick $\gamma \in (\frac{1}{2}, \frac{\alpha}{2})$, so that the conditions we need below are ensured: $K^\gamma \in L^1(\mathbb{R})$; $2\gamma \in (1,2)$; $1 - 2\gamma/\alpha > 0$. Since $\gamma < 1$, we obtain

$$E^{Q_\omega}\left[\left|\sum_{j=1}^n \frac{K_j}{h^{\frac{2}{\alpha}}}\left(\Delta_j \tilde{X}'\right)^2\right|^\gamma\right] \le \sum_{j=1}^n \frac{K_j^\gamma}{h^{\frac{2\gamma}{\alpha}}} E^{Q_\omega}\left[\left|\Delta_j \tilde{X}'\right|^{2\gamma}\right]. \tag{67}$$

Since $2\gamma \in (1,2)$ we can apply (2.1.36) in [6] and upper bound with

$$\sum_{j=1}^n \frac{K_j^\gamma}{h^{\frac{2\gamma}{\alpha}}} E^{Q_\omega}\left[\int_{t_{j-1}}^{t_j}\int_{|x|\le 1} |x|^{2\gamma}\lambda'(x,s)dxds\right]$$

Since $2\gamma > 1$ then $\int_{|x|\le 1} |x|^{2\gamma}\lambda'(x,s)dx \le \int_{|x|\le 1}|x|\lambda'(x,s)dx \le C$, and the above is bounded by

$$C\sum_{j=1}^n \frac{K_j^\gamma \Delta}{h} \cdot h^{1-2\gamma/\alpha} \to 0.$$

Similarly as in (65), from (66) it follows that for all ω

$$\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}} \Delta_j \tilde{X} \Delta_j \tilde{X}' \xrightarrow{Q_\omega} 0. \tag{68}$$

Again, Lemma 7 applies to (63), and gives

$$E^P\left[e^{-sZ_{2,\alpha}}\right] = \lim_n \bar{E}\left[e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\left(\Delta_j \tilde{J}\right)^2}\right] = \lim_n E^P\left[e^{-s\sum_{j=1}^n \frac{K_j}{h^{2/\alpha}}\left(\Delta_j \tilde{X}\right)^2}\right],$$

which concludes the proof of (26). $\square$

Proof of Lemma 6. First step: X is an α stable process, and we name it $J = \tilde{J} + J^1$, where $\tilde{J} = \int_0^\cdot \int_{|x|\le 1} x d\tilde{\mu}$, $J^1 = \int_0^\cdot \int_{|x|>1} dd\mu$. We write $\Delta_i J =$

$\Delta_i A + \Delta \cdot B$, where A is self-similar, and B is a constant:

$$A_t \doteq \int_0^t \int_{\mathbb{R}} x d\tilde{\mu}, \quad B \doteq \int_{|x|>1} x\lambda(x)dx < \infty,$$

and proceed through the following steps:

1) due to the negligibility of the contribution of the terms ΔB and $\Delta_i J^1$, we show that a.s.

$$\left(\frac{\sum_{i=1}^n K_i \Delta_i J}{h^{\frac{1}{\alpha}}}\right)^2 \simeq \left(\frac{\sum_{i=1}^n K_i \Delta_i A}{h^{\frac{1}{\alpha}}}\right)^2, \quad \frac{\sum_{i=1}^n K_i \Big(\Delta_i J\Big)^2}{h^{\frac{2}{\alpha}}} \simeq \frac{\sum_{i=1}^n K_i \Big(\Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}} \tag{69}$$

After that, it is sufficient to prove the convergence in distribution of

$$\left(\frac{\Big(\sum_{i=1}^n K_i \Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i A)^2}{h^{\frac{2}{\alpha}}}\right).$$

2) We develop

$$\frac{\Big(\sum_{i=1}^n K_i \Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}} = \frac{\sum_{i=1}^n \Big(K_i \Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}} + \frac{\sum_{i,j=1..n: i\neq j} K_i K_j \Delta_i A \Delta_j A}{h^{\frac{2}{\alpha}}}$$

and we show that, since $A_+ = A_-$ then $\frac{\sum_{i\neq j} K_i K_j \Delta_i A \Delta_j A}{h^{\frac{2}{\alpha}}} \xrightarrow{P} 0$, so the stated limit in distribution is the same as for

$$\left(\frac{\sum_{i=1}^n \Big(K_i \Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i A)^2}{h^{\frac{2}{\alpha}}}\right)$$

3) Again by the negligibility of the contribution of the terms ΔB and $\Delta_i J^1$,

$$\frac{\sum_{i=1}^n \Big(K_i \Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}} \simeq \frac{\sum_{i=1}^n \Big(K_i \Delta_i \tilde{J}\Big)^2}{h^{\frac{2}{\alpha}}}, \quad \frac{\sum_{i=1}^n K_i \Big(\Delta_i A\Big)^2}{h^{\frac{2}{\alpha}}} \simeq \frac{\sum_{i=1}^n K_i \Big(\Delta_i \tilde{J}\Big)^2}{h^{\frac{2}{\alpha}}}, \tag{70}$$

then we only have to deal with

$$\left(\frac{\sum_{i=1}^n \Big(K_i \Delta_i \tilde{J}\Big)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}}\right).$$

4) For $s_1, s_2 > 0$ we show that as $n \to +\infty$

$$\mathcal{L}_n(s_1, s_2) \doteq E\left[e^{-s_1 \frac{\sum_{i=1}^n \left(K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}} - s_2 \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}}}\right] \to \tag{71}$$

$$E\left[e^{-s_1 Z_{1,\alpha}^2 - s_2 Z_{2,\alpha}}\right] \doteq \mathcal{L}(s_1, s_2),$$

which concludes the proof of the first step.

Let us start by 1). Note that
1.1) $\frac{\sum_{i=1}^n K_i \Delta B}{h^{\frac{1}{\alpha}}} = B \frac{\sum_{i=1}^n K_i \Delta}{h} h^{1-\frac{1}{\alpha}} \to 0,$
and from this we immediately have that

$$\frac{\sum_{i=1}^n K_i \Delta_i J}{h^{\frac{1}{\alpha}}} \simeq \frac{\sum_{i=1}^n K_i \Delta_i A}{h^{\frac{1}{\alpha}}}$$

which gives the first asymptotic equality at (69).
1.2) $\frac{\sum_{i=1}^n K_i \Delta^2 B^2}{h^{\frac{2}{\alpha}}} = B^2 \frac{\sum_{i=1}^n K_i \Delta}{h} \frac{\Delta}{h} h^{2-\frac{2}{\alpha}} \to 0$
1.3) $\frac{\sum_{i=1}^n K_i \Delta_i J^1}{h^{\frac{1}{\alpha}}} \xrightarrow{P} 0$. In fact recalling that the probability that $\Delta J^1_{\bar{t}} \neq 0$ is zero, for the convergence in distribution we can focus on those ω where there is no jump at $\bar{t}$. For any fixed ω such that $\Delta J^1_{\bar{t}} = 0$, using the notation at the proof of Lemma 1, part b), and recalling that J^1 has FA jumps, c $\bar{t} - S_{\underline{p}}$ is a fixed quantity, and

$$\frac{\sum_{i=1}^n K_i \Delta_i J^1}{h^{\frac{1}{\alpha}}} \simeq \frac{K\left(\frac{\bar{t}-S_p}{h}\right)}{h^{\frac{1}{\alpha}}}:$$

by assumption $K\left(\frac{\bar{t}-S_p}{h}\right) = o(h)$, and since $\alpha > 1$ then $h = o(h^{\frac{1}{\alpha}})$, thus the above display tends a.s. to 0.
1.4) $\frac{\sum_{i=1}^n K_i \Delta_i A}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha}$. In fact

$$\frac{\sum_{i=1}^n K_i \Delta_i A}{h^{\frac{1}{\alpha}}} = \frac{\sum_{i=1}^n K_i \Delta_i \tilde{J}}{h^{\frac{1}{\alpha}}} + \frac{\sum_{i=1}^n K_i \Delta_i J^1}{h^{\frac{1}{\alpha}}} + \frac{\sum_{i=1}^n K_i \Delta B}{h^{\frac{1}{\alpha}}} \tag{72}$$

and by 1.1), 1.3) and Lemma 4 we have the result.
1.5) $\frac{\sum_{i=1}^n K_i \Delta_i A \Delta B}{h^{\frac{2}{\alpha}}} \xrightarrow{P} 0$, since the left side is given by

$$B \frac{\sum_{i=1}^n K_i \Delta_i A}{h^{\frac{1}{\alpha}}} \frac{\Delta}{h} h^{1-\frac{1}{\alpha}},$$

and 1.4) is used.
At this point the second asymptotic equality at (69) follows from

$$\frac{\sum_{i=1}^n K_i (\Delta_i J)^2}{h^{\frac{2}{\alpha}}} = \frac{\sum_{i=1}^n K_i (\Delta_i A)^2}{h^{\frac{2}{\alpha}}} + \frac{\sum_{i=1}^n K_i \Delta^2 B^2}{h^{\frac{2}{\alpha}}} + 2 \frac{\sum_{i=1}^n K_i \Delta_i A \Delta B}{h^{\frac{2}{\alpha}}},$$

1.2) and 1.5).

As for 2), for any $\eta > 0$ we have

$$P\left\{\frac{|\sum_{i,j:i\neq j} K_i\Delta_i A K_j \Delta_j A|}{h^{\frac{2}{\alpha}}} > \eta\right\} = P\left\{\frac{|\sum_{i,j:i\neq j} K_i\Delta^{\frac{1}{\alpha}} A_{1i} K_j \Delta^{\frac{1}{\alpha}} A_{1j}|}{h^{\frac{2}{\alpha}}} > \eta\right\},$$

where, by selfsimilarity, each $\Delta_i A$ has the same distribution as $\Delta^{\frac{1}{\alpha}} A_1$, and since $\Delta_i A$ and $\Delta_j A$ are independent, $\Delta_i A \Delta_j A \stackrel{d}{=} \Delta^{\frac{2}{\alpha}} A_{1i} A_{1j}$, where A_{1i}, A_{1j} are independent copies of A_1.

Now, as in [5], we localize the space Ω in such a way that, on any considered stochastic interval, process A has bounded jumps. Namely, for any $M > 0$ we take $T_M(\omega)$ such that for any $t \leq T_M$ we have $|\Delta A_t(\omega)| \leq M$. Since A has jumps in $\mathbb{R}$, then a.s. $T_M \stackrel{P}{\to} +\infty$ as $M \to \infty$ (a.s. $T_M(\omega)$ is increasing with M, then the sequence has a limit. If the limit was $\ell(\omega) < \infty$ then $|\Delta A_{\ell(\omega)}(\omega)| > M$ for any M, thus $|\Delta A_{\ell(\omega)}(\omega)| = +\infty$). In this way the second moment of $A_{t\wedge T_M}$ is finite, and we write the above display as

$$P\left\{\frac{|\sum_{i,j:i\neq j} K_i\Delta^{\frac{1}{\alpha}} A_{1i} K_j \Delta^{\frac{1}{\alpha}} A_{1j}|}{h^{\frac{2}{\alpha}}} > \eta, T_M \leq 1\right\} +$$

$$P\left\{\frac{|\sum_{i,j:i\neq j} K_i\Delta^{\frac{1}{\alpha}} A_{1i} K_j \Delta^{\frac{1}{\alpha}} A_{1j}|}{h^{\frac{2}{\alpha}}} > \eta, T_M > 1\right\}:$$

the first term is dominated by $P\{T_M \leq 1\}$ which tends to 0 as $M \to \infty$, while the second one is dominated by

$$\frac{1}{\eta^2} E\left[\left(\frac{\sum_{i=1}^n \sum_{j:j\neq i} K_i K_j \Delta^{\frac{2}{\alpha}} A_{1i} A_{1j}}{h^{\frac{2}{\alpha}}}\right)^2 I_{T_M > 1}\right].$$

On $\{T_M > 1\}$ the variables A_{1i}, A_{1j} have jumps bounded by M, and, since $A_+ = A_-$, the compensator of the big jumps (the jumps bigger than 1 in absolute value) is null. Thus, on $\{T_M > 1\}$, A_{1i}, A_{1j} can be written as copies of $\bar{A}_1 \doteq \int_0^1 \int_{|x|\leq M} x d\tilde{\mu}$, and the above display is bounded from above by

$$\frac{1}{\eta^2} E\left[\left(\frac{\sum_{i=1}^n \sum_{j:j\neq i} K_i K_j \Delta^{\frac{2}{\alpha}} \bar{A}_{1i} \bar{A}_{1j}}{h^{\frac{2}{\alpha}}}\right)^2\right] =$$

$$\frac{1}{\eta^2} \frac{4\sum_{i=1}^n \sum_{j:j\neq i} K_i^2 K_j^2 \Delta^{\frac{4}{\alpha}} E[\bar{A}_{1i}^2 \bar{A}_{1j}^2]}{h^{\frac{4}{\alpha}}} +$$

$$\frac{1}{\eta^2}\frac{N\sum_{i,j,\ell,m:(i,j)\neq(\ell,m)}\Delta^{\frac{4}{\alpha}}K_iK_jK_\ell K_m E[\bar{A}_{1i}\bar{A}_{1j}\bar{A}_{1\ell}\bar{A}_{1m}]}{h^{\frac{4}{\alpha}}},$$

where $N = n\sum_{j=1}^{n-2} j+((n-1)^2-1)\sum_{j=1}^{n-1} j$, and within each term $\bar{A}_{1i}\bar{A}_{1j}\bar{A}_{1\ell}\bar{A}_{1m}$ at least one increment is raised to power 1 only. Since for $i \neq j$ the variables $\bar{A}_{1i}, \bar{A}_{1j}$ are independent, have the same law and are centered, the second term in the above display is 0, while in the first term $E[\bar{A}_{1i}^2\bar{A}_{1j}^2] = E^2[\bar{A}_{1i}^2] = C < \infty$, so we remain with

$$C\frac{\sum_{i=1}^n\sum_{j:j\neq i}K_i^2K_j^2\Delta^{\frac{4}{\alpha}}}{h^{\frac{4}{\alpha}}} = C\frac{\sum_{i=1}^n\sum_{j:j\neq i}K_i^2K_j^2\Delta^2}{h^2}\left(\frac{\Delta}{h}\right)^{\frac{4}{\alpha}-2}:$$

by Lemma 2, point 6), since $4/a > 2$, the latter term tends to 0 as $n \to 0$.

As for 4), we have

$$\mathcal{L}_n(s_1, s_2) = E\left[e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}(\Delta_i\tilde{J})^2}\right] = \prod_{i=1}^n E\left[e^{-u_i(\Delta_i\tilde{J})^2}\right],$$

having set $u_i \doteq \frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}} > 0$. The above display is the same as in (47), with u_i in place of $v_j = \frac{sK_j}{h^{2/\alpha}}$, $\Delta_i\tilde{J}$ in place of U_j and $(\sigma_i)^\alpha = (2u_i)^{\frac{\alpha}{2}} = \frac{2^{\frac{\alpha}{2}}(s_1K_i^2+s_2K_i)^{\frac{\alpha}{2}}}{h}$ in place of $(\sigma_j)^\alpha = (2v_j)^{\frac{\alpha}{2}} = \frac{(2sK_j)^{\frac{\alpha}{2}}}{h}$. Thus (48) applies, and proceeding similarly as in the proof of Lemma 5 for the case $\alpha > 1$, we obtain that the above display coincides with the last term in (59), i.e.

$$\prod_{i=1}^n E\left[e^{-u_i(\Delta_i\tilde{J})^2}\right] = \prod_{j=1}^n\left(1+\tilde{\theta}_i^{(n)}\right),$$

where within each

$$\tilde{\theta}_{i,1}^{(n)} = 2\Delta\sigma_i^\alpha\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot u^\alpha\int_0^{\sigma_i u}(A_+ + A_-)\frac{cos(v)-1}{v^{1+\alpha}}dvdu$$

we have to plug in the value of σ_i which is pertinent here. Since

$$\sum_{i=1}^n\Delta\sigma_i^\alpha \simeq 2^{\frac{\alpha}{2}}\int_0^T[s_1K_r^2+s_2K_r]^{\frac{\alpha}{2}}\frac{dr}{h} = 2^{\frac{\alpha}{2}}\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}}[s_1K^2(u)+s_2K(u)]^{\frac{\alpha}{2}}du$$

tends to $2^{\frac{\alpha}{2}}\int_{\mathbb{R}}[s_1K^2(u)+s_2K(u)]^{\frac{\alpha}{2}}du$ then, similarly as for Lemma 3, part iii), we have that $\sum_{i=1}^n\tilde{\theta}_{i,1}^{(n)}$ tends to

$$2\cdot 2^{\frac{\alpha}{2}}\int_{\mathbb{R}}[s_1K^2(u)+s_2K(u)]^{\frac{\alpha}{2}}du\cdot\frac{2^{\frac{\alpha-1}{2}}}{\sqrt{2\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\cdot(A_+ + A_-)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)$$

and, similarly as in Lemma 5,

$$\prod_{j=1}^{n}\left(1+\tilde{\theta}_i^{(n)}\right) \simeq \prod_{j=1}^{n}\left(1+\tilde{\theta}_{i,1}^{(n)}\right) \to e^{\underline{\theta}} \doteq \mathcal{L}_\infty(s_1,s_2),$$

where

$$\underline{\theta} \doteq \frac{2^\alpha}{\sqrt{\pi}}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int_{\mathbb{R}}[s_1K^2(u)+s_2K(u)]^{\frac{\alpha}{2}}du.$$

The function $\mathcal{L}_\infty$ is the Laplace transform of a probability law (because $\mathcal{L}_\infty(0,0)$ $= 1$ and the function is continuous at (0,0)), and we see that it is the one of a proper joint law having marginals $Z_{1,\alpha}^2$ and $Z_{2,\alpha}$. In fact, with $s_2=0$ we have

$$e^{\frac{2^\alpha}{\sqrt{\pi}}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int_{\mathbb{R}}[s_1K^2(u)]^{\frac{\alpha}{2}}du} = \mathcal{L}_\infty(s_1,0)$$

$$= \lim_n \mathcal{L}_n(s_1,0) = \lim_n E\left[e^{-s_1\frac{\sum_{i=1}^n(K_i\Delta_i\tilde{J})^2}{h^{2/\alpha}}}\right]:$$

$\frac{\sum_{i=1}^n(K_i\Delta_i\tilde{J})^2}{h^{2/\alpha}} \overset{d}{\simeq} \left(\frac{\sum_{i=1}^n K_i\Delta_i\tilde{J}}{h^{1/\alpha}}\right)^2$, as we saw above at 2), and, by Lemma 4, the latter term converges in distribution to $Z_{1,\alpha}^2$.

On the other hand, with $s_1=0$ we have

$$e^{\frac{2^\alpha}{\sqrt{\pi}}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int_{\mathbb{R}}[s_2K(u)]^{\frac{\alpha}{2}}du} = \mathcal{L}_\infty(0,s_2)$$

$$= \lim_n \mathcal{L}_n(0,s_2) = \lim_n E\left[e^{-s_2\frac{\sum_{i=1}^n K_i(\Delta_i\tilde{J})^2}{h^{2/\alpha}}}\right]$$

and, by Lemma 5, $\frac{\sum_{i=1}^n K_i(\Delta_i\tilde{J})^2}{h^{2/\alpha}} \xrightarrow{d} Z_{2,\alpha}$. Thus $\mathcal{L}_\infty$ describes a specific joint law of $\left(Z_{1,\alpha}^2, Z_{2,\alpha}\right)$.

Second step. Let us now consider a process X as in **IA3**. Again we refer to (36) and use that $J = X + X'$ is a Levy stable process on $\bar{\Omega}$. Since the contribution of X^1 is negligible then we have

$$\left(\frac{\left(\sum_{i=1}^n K_i\Delta_iX\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i(\Delta_iX)^2}{h^{\frac{2}{\alpha}}}\right) \overset{d}{\simeq} \left(\frac{\left(\sum_{i=1}^n K_i\Delta_i\tilde{X}\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i(\Delta_i\tilde{X})^2}{h^{\frac{2}{\alpha}}}\right).$$

We now show that

$$\frac{\sum_{i,j:i\neq j} K_i\Delta_i\tilde{X}K_j\Delta_j\tilde{X}}{h^{\frac{2}{\alpha}}} \xrightarrow{P} 0. \tag{73}$$

Firstly, the processes $J = \tilde{J} + J^1, B, A$ mentioned at point 2) of the first step are now an α- stable process, the compensator the big jumps, and a self-similar α-stable process on $\bar{\Omega}$. Thus, by point 2) of the first step,

$$\frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{J}\ K_j \Delta_j \tilde{J}}{h^{\frac{2}{\alpha}}} \simeq \frac{\sum_{i,j:i\neq j} K_i \Delta_i A\ K_j \Delta_j A}{h^{\frac{2}{\alpha}}} \xrightarrow{\bar{P}} 0.$$

However now $\Delta_i \tilde{J} = \Delta_i \tilde{X} + \Delta_i \tilde{X}'$, thus the left side above is

$$\frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{J}\ K_j \Delta_j \tilde{J}}{h^{\frac{2}{\alpha}}} = \frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X} K_j \Delta_j \tilde{X}}{h^{\frac{2}{\alpha}}} + \frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X}' K_j \Delta_j \tilde{X}'}{h^{\frac{2}{\alpha}}}$$
$$+ \frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X}' K_j \Delta_j \tilde{X}}{h^{\frac{2}{\alpha}}} + \frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X} K_j \Delta_j \tilde{X}'}{h^{\frac{2}{\alpha}}},$$

and we show that the last 3 terms tend to 0 in $\bar{P}$-probability, therefore also the first one necessarily does. The process X' has finite variation and on Ω' has independent increments, thus for any ω we have $E^{Q_\omega}\left[|\Delta_i \tilde{X}' \Delta_j \tilde{X}'|\right] = E^{Q_\omega}\left[|\Delta_i \tilde{X}'|\right] E^{Q_\omega}\left[|\Delta_j \tilde{X}'|\right] \leq \Delta^2 C$, therefore

$$E^{Q_\omega}\left[\frac{|\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X}' K_j \Delta_j \tilde{X}'|}{h^{\frac{2}{\alpha}}}\right] \leq \frac{\sum_{i,j:i\neq j} K_i K_j E^{Q_\omega}\left[|\Delta_i \tilde{X}' \Delta_j \tilde{X}'|\right]}{h^{\frac{2}{\alpha}}}$$
$$\leq C \frac{\sum_{i,j:i\neq j} K_i K_j \Delta^2}{h^2} h^{2-\frac{2}{\alpha}} \to 0.$$

Thus

$$\bar{E}\left[\frac{|\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X}' K_j \Delta_j \tilde{X}'|}{h^{\frac{2}{\alpha}}}\right] = E^P\left[E^{Q_\omega}\left[\frac{|\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X}' K_j \Delta_j \tilde{X}'|}{h^{\frac{2}{\alpha}}}\right]\right]$$
$$\leq E^P\left[C \frac{\sum_{i,j:i\neq j} K_i K_j \Delta^2}{h^2} h^{2-\frac{2}{\alpha}}\right] \to 0.$$

As for the mixed products, we split $\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X} K_j \Delta_j \tilde{X}' / h^{\frac{2}{\alpha}}$ into

$$\frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X} K_j \int_{t_{j-1}}^{t_j} \int_{|x|\leq 1} x d\mu'}{h^{\frac{2}{\alpha}}} - \frac{\sum_{i,j:i\neq j} K_i \Delta_i \tilde{X} K_j \int_{t_{j-1}}^{t_j} \int_{|x|\leq 1} x d\lambda'}{h^{\frac{2}{\alpha}}} : \tag{74}$$

the second term

$$\frac{\sum_{i=1}^n K_i \Delta_i \tilde{X}}{h^{\frac{1}{\alpha}}} \frac{\sum_{j=1}^n K_j \int_{t_{j-1}}^{t_j} \int_{|x|\leq 1} x d\lambda'}{h^{\frac{1}{\alpha}}} - \frac{\sum_{i=1}^n K_i^2 \Delta_i \tilde{X}}{h^{\frac{1}{\alpha}}} \frac{\int_{t_{i-1}}^{t_i} \int_{|x|\leq 1} x d\lambda'}{h^{\frac{1}{\alpha}}}$$

has absolute value dominated by

$$C\frac{\sum_{j=1}^n K_j\Delta}{h}h^{1-\frac{1}{\alpha}}\left|\frac{\sum_{i=1}^n K_i\Delta_i\tilde{X}}{h^{\frac{1}{\alpha}}}\right|+\frac{\Delta C}{h^{\frac{1}{\alpha}}}\left|\frac{\sum_{i=1}^n K_i^2\Delta_i\tilde{X}}{h^{\frac{1}{\alpha}}}\right|\stackrel{\bar{P}}{\to}0,$$

because K is bounded, $1/\alpha<1$, and the last factors of the two terms converge in P-distribution by Lemma 4 with kernel either K or K^2, and thus also in $\bar{P}$-distribution. Similarly, the first term of (74)

$$\frac{\sum_{j=1}^n K_j\int_{t_{j-1}}^{t_j}\int_{|x|\le 1}xd\mu'}{h^{\frac{1}{\alpha'}}}h^{\frac{1}{\alpha'}-\frac{1}{\alpha}}\frac{\sum_{i=1}^n K_i\Delta_i\tilde{X}}{h^{\frac{1}{\alpha}}}-\frac{\sum_{i=1}^n K_i^2\int_{t_{i-1}}^{t_i}\int_{|x|\le 1}xd\mu'\Delta_i\tilde{X}}{h^{\frac{2}{\alpha}}}$$

tends to 0 in $\bar{P}$-probability, because X' has jump index $\alpha'\le 1<\alpha$, by Lemma 4 we know that $\frac{\sum_{j=1}^n K_j\int_{t_{j-1}}^{t_j}\int_{|x|\le 1}xd\mu'}{h^{\frac{1}{\alpha'}}}$ and $\frac{\sum_{i=1}^n K_i\Delta_i\tilde{X}}{h^{\frac{1}{\alpha}}}$ converge in $\bar{P}$-distribution, while $h^{1/\alpha'-1/\alpha}\to 0$; moreover

$$\frac{\sum_{i=1}^n K_i^2\int_{t_{i-1}}^{t_i}\int_{|x|\le 1}xd\mu'\Delta_i\tilde{X}}{h^{\frac{2}{\alpha}}}\le\sqrt{\frac{\sum_{i=1}^n K_i^2(\int_{t_{i-1}}^{t_i}\int_{|x|\le 1}xd\mu')^2}{h^{\frac{2}{\alpha'}}}h^{\frac{2}{\alpha'}-\frac{2}{\alpha}}}\sqrt{\frac{\sum_{i=1}^n K_i^2(\Delta_i\tilde{X})^2}{h^{\frac{2}{\alpha}}}},$$

which tends to 0 in $\bar{P}$-probability by Lemma 5 with K^2 in palce of K.

From (73) it follows that

$$\left(\frac{\left(\sum_{i=1}^n K_i\Delta_i\tilde{X}\right)^2}{h^{\frac{2}{\alpha}}},\frac{\sum_{i=1}^n K_i(\Delta_i\tilde{X})^2}{h^{\frac{2}{\alpha}}}\right)\stackrel{d}{\simeq}\left(\frac{\sum_{i=1}^n K_i^2(\Delta_i\tilde{X})^2}{h^{\frac{2}{\alpha}}},\frac{\sum_{i=1}^n K_i(\Delta_i\tilde{X})^2}{h^{\frac{2}{\alpha}}}\right).$$

It remains to show the generalisation of point 4) at the first step. From there we know that, since J is α stable on $\bar{\Omega}$, then as $n\to\infty$,

$$\bar{E}\left[f_n(\tilde{J})\right]\doteq\bar{\mathcal{L}}_n(s_1,s_2)=\bar{E}\left[e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}(\Delta_i\tilde{J})^2}\right]\to\mathcal{L}_\infty(s_1,s_2).$$

On the other hand,

$$\bar{E}\left[f_n(\tilde{J})\right]=\bar{E}\left[e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}(\Delta_i\tilde{J})^2}\right]=\bar{E}\left[e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}(\Delta_i\tilde{X})^2}\cdot\right.$$

$$\left.e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}\left[(\Delta_i\tilde{X}')^2+2\Delta_i\tilde{X}\Delta_i\tilde{X}'\right]}\right]=\bar{E}\left[f_n(\tilde{X})g_n(\tilde{X}')\right].$$

Similarly as in (66) and (68), naming $z_i\doteq s_1K_i^2+s_2K_i$, we have

$$\sum_{i=1}^n\frac{z_i}{h^{2/\alpha}}(\Delta_i\tilde{X}')^2\stackrel{Q_\omega}{\to}0\text{ and }\sum_{i=1}^n\frac{z_i}{h^{2/\alpha}}\Delta_i\tilde{X}\Delta_i\tilde{X}'\stackrel{\bar{P}}{\to}0,$$

then $g_n(\tilde{X}') \xrightarrow{\bar{P}} 1$ and by Lemma 7

$$\lim_n E^P\left[f_n(\tilde{X})\right] \doteq \lim_n E\left[e^{-\sum_{i=1}^n \frac{z_i}{h^{2/\alpha}}(\Delta_i\tilde{X})^2}\right] = \lim_n \bar{E}\left[f_n(\tilde{J})\right] = \mathcal{L}_\infty(s_1, s_2).$$

□

Remark 12. The Laplace transform of the joint law of $\left[Z^2_{1,\alpha}, Z_{2,\alpha}\right]$ under P is an exponential of the expression $-C\int_{\mathbb{R}}[s_1K^2(u)+s_2K(u)]^{\frac{\alpha}{2}}du$, with $C>0$, having no linear part in s_1, s_2, thus in the path representation of the bivariate random variable there are no drift terms. The law could resemble a bidimensional $\alpha/2$-stable, however this is not the case, because it is concentrated on a parabola (if $x_2 = K(u)$ then $x_1 = x_2^2$) rather than on the unit sphere (see [8], Thm 14.10).

Proof of Corollary 3. Let us split $Y = Y' + \tilde{X}$, where $Y'_t \doteq Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + X^1_t$, then $T^n_{\bar{t}}$ equals

$$\frac{\sum_{i=1}^n K_i\Delta_i Y}{\sqrt{\sum_{i=1}^n K_i(\Delta_i Y)^2}} = \frac{\sum_{i=1}^n K_i\Delta_i Y' + \sum_{i=1}^n K_i\Delta_i\tilde{X}}{\sqrt{\sum_{i=1}^n K_i(\Delta_i Y')^2 + \sum_{i=1}^n K_i(\Delta_i\tilde{X})^2 + 2\sum_{i=1}^n K_i\Delta_i Y'\Delta_i\tilde{X}}}$$

with $S_n \doteq \sum_{i=1}^n K_i(\Delta_i Y')^2$, the above equals

$$\frac{\frac{\sum_{i=1}^n K_i\Delta_i Y'}{\sqrt{S_n}} + \frac{\sum_{i=1}^n K_i\Delta_i\tilde{X}}{\sqrt{S_n}}}{\sqrt{1 + \frac{\sum_{i=1}^n K_i(\Delta_i\tilde{X})^2}{S_n} + 2\frac{\sum_{i=1}^n K_i\Delta_i Y'\Delta_i\tilde{X}}{S_n}}},$$

and we show that the last display tends to $\mathcal{N}(0,1)$ in distribution.

In fact first of all note that with probability 1 there is no jump at $\bar{t}$, and when $\Delta X_{\bar{t}} = 0$ the leading term of S_n is $\sum_{i=1}^n K_i(\int_{t_{i-1}}^{t_i} b_s ds + \int_{t_{i-1}}^{t_i}\sigma_s dW_s)^2 \sim h\sigma^\star_{\bar{t}}$ ([7], thm 2.7) because $\sum_{i=1}^n K_i(\Delta_i X^1)^2 \sim K(\frac{\bar{t}-S_p}{h}) = o(\Delta h)$. Thus, with probability 1, $S_n \sim h$.

Then, the first quotient of the above numerator tends in distribution to a standard Gaussian r.v. because Y' has finite variation jumps, so the result in [2] applies. We now show that all the other terms tend to 0.

If $\alpha \in (1,2)$, by Lemma 4, $\sum_{i=1}^n K_i\Delta_i\tilde{X}$ tends to 0 at speed $h^{1/\alpha} << h^{1/2}$, thus the second quotient at numerator tends to 0; by Lemma 5 the second term at denominator

$$\frac{\sum_{i=1}^n K_i(\Delta_i\tilde{X})^2}{S_n} \sim \frac{h^{\frac{2}{\alpha}}}{h} \to 0$$

and the third one

$$\frac{\sum_{i=1}^n K_i \Delta_i Y' \Delta_i \tilde{X}}{S_n} \leq \frac{\sqrt{\sum_{i=1}^n K_i (\Delta_i \tilde{X})^2}\sqrt{S_n}}{S_n} \sim \frac{h^{\frac{1}{\alpha}}}{\sqrt{h}} \to 0.$$

If instead $\alpha = 1$, the second quotient at numerator is

$$\frac{\sum_{i=1}^n K_i \Delta_i \tilde{X}}{\sqrt{S_n}} \sim \frac{h \log \frac{1}{h}}{\sqrt{h}} \to 0,$$

the second term at denominator

$$\frac{\sum_{i=1}^n K_i (\Delta_i \tilde{X})^2}{S_n} \sim \frac{h^2}{h} \to 0$$

and the third one

$$\frac{\sum_{i=1}^n K_i \Delta_i Y' \Delta_i \tilde{X}}{S_n} \leq \frac{\sqrt{\sum_{i=1}^n K_i (\Delta_i \tilde{X})^2}\sqrt{S_n}}{S_n} \sim \frac{h}{\sqrt{h}} \to 0. \qquad \square$$