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Abstract

We complete the investigation on the asymptotic behavior of the drift burst test statistic devised in Christensen, Oomen and Renò (2020). They analysed it for an Ito semimartingale containing a Brownian component and finite variation jumps. We also account for infinite variation jumps. We show that when there are no bursts in drift neither in volatility, explosion of the statistic only can occur in the absence of the Brownian part and when the jumps have finite variation. In that case the explosion is due to the compensator of the small jumps. We also find that the statistic could be adopted for a variety of tests useful for investigating the nature of the process, given discrete observations.

JEL classification codes: Primary 62M99, 62F05; secondary 60F17, 91B70, 60E07, 60E10.

Keywords: Test statistic, Ito semimartingale, infinite variation jumps, jump activity index, asymptotic behavior.

1 Introduction

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, we consider a càdlàg pure jump semimartingale (SM) defined by

$$X_t = \int_0^t \int_{|\delta(x,s)|\le 1} \delta(x,s)\tilde\mu(dx,ds) + \int_0^t \int_{|\delta(x,s)|>1} \delta(x,s)\mu(dx,ds),\ t\in[0,T], \tag{1}$$

for a fixed time horizon $T>0$, where $\mu(dx,ds)$ is a Poisson random measure on $(\mathbb{R}\times[0,T])$ endowed with a compensator of type $\nu(dx,dt)=\lambda(x)dxds$, and $\tilde\mu=\mu-\nu$ is the compensated Poisson random measure. Formal conditions on X are given in Section 2. The first term in (1) sums the compensated small jumps of X while the second term sums the not-compensated big jumps.

For fixed $\bar t\in(0,T)$, we focus on the asymptotic behavior of

$$T^n_{\bar t} \doteq \frac{\sum_{i=1}^n K_i\Delta_i X}{\sqrt{\sum_{i=1}^n K_i(\Delta_i X)^2}}, \tag{2}$$

where: for any integer $n>0$, $\{t_i = t_i^{(n)}, i=1,..,n\}$ gives a non-random partition of $[0,T]$; $\Delta_i X \doteq X_{t_i}-X_{t_{i-1}}; K_i = K\left(\frac{\bar t - t_{i-1}}{h}\right)$; $K:\mathbb{R}\to\mathbb{R}_+$ is a kernel continuous function and h is a bandwidth parameter. We are interested in the framework where

$$n\to+\infty \text{ while } h\to 0 \text{ in such a way that } nh\to+\infty, \tag{3}$$

and we assume that the partition asymptotically does not differ too much from the equally spaced one, in a way made explicit later.

The statistic $T^n_{\bar t}$ is devised in [5], where the considered model is an Ito semimartingale (SM) including drift and Brownian components, the jumps have finite variation (FV) and are represented as compensated

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small jumps added to not compensated large jumps. There, $T_{\bar{t}}^n$ is shown to explode any time when there is a burst in the drift larger than a burst in the volatility, while the statistic converges stably in law to a Gaussian random variable if either there are bursts and the one in volatility is larger than the one in drift, or no bursts occur at all. However one wonder what role infinite variation (IV) jumps would play, for instance, whether the explosion observed in the empirical implementation of $T_{\bar{t}}^n$ on finite samples may or may not be due to a jump component of IV, possibly present in the data generating process (DGP). Or, how the statistic would behave if the DGP did not contain any Brownian components. For this reason we specifically address a pure jump process.

Recently pure jump models for financial asset prices are being revaluated. Empirical evidence that large jumps can improve pricing models for many financial assets is documented since long time. Initially the focus was on adding large, finite activity (FA) jumps to existing models with continuous paths. By contrast, since the nineties infinite activity (IA) pure jump Ito SMs have been considered. The latter models contain large jumps, and a dense set of small jumps replace the Brownian motion to reproduce the small movements of asset prices: Eberlein and cohautors (e.g. [9]) considered Hyperbolic and Generalized Hyperbolic Lévy motions, Barndorff-Nielsen ([3]) Normal Inverse Gaussian Lévy processes, Madan and coauthors (e.g. [12]) Variance Gamma models, Carr, Geman, Madan and Yor (e.g. [4]) CGMY processes. Such models would be also economically well justified as stochastic time changed Brownian motions, where the discontinuous time change can be interpreted as a measure of the economic activity, and makes the model arbitrage free.

We now dispose of several tests to check for whether a record of an asset prices is compatible or not with the presence of a Brownian part in a SM model ([7], [2], [16], [20], [10], [13]). Note that [17] warns to correctly account for price staleness, in order to avoid possible wrong conclusions.

In any case, knowing the asymptotic behavior of $T_{\bar{t}}^n$ in a pure jump framework allows to immediately obtain its limit in a model including both Brownian motion and infinite variation jumps.

In the present pure jump framework it turns out that the behaviour of $T_{\bar{t}}^n$ is different in the two cases where $\bar{t}$ is a jump time or it is not. In fact the numerator tends (ω-wise if the jumps have FA, in probability if they have IA) to $K(0)\Delta X_{\bar{t}}$, and the denominator to $\sqrt{K(0)} \cdot |\Delta X_{\bar{t}}|$. Thus if $\Delta X_{\bar{t}} \neq 0$ the statistic has a well defined finite limit, otherwise both numerator and denominator tend to 0, and, as soon as $T_{\bar{t}}^n$ is defined, the limit is determined by the dominant terms.

The asymptotic distribution of the statistic is substantially different depending on whether the jumps have finite or infinite variation. In the former case the dominant element at both numerator and denominator is the compensator of the small jumps, which acts as a drift and determine explosion of $T_{\bar{t}}^n$. In the latter case, instead, $\left|T_{\bar{t}}^n\right|$ converges in distribution to a r.v. Z_α depending on the magnitude of the jump activity index α of X.

To get an insight into how things are going, let us mention the case where the kernel function is given by a continuous approximation of the indicator $I_{\{|x|\leq\frac{1}{2}\}}$ and the observations are evenly spaced. With FA jumps and compensator at, all the jumps are shown to have a negligible impact on $T_{\bar{t}}^n$, and, indicating by

$\simeq$ that two expressions have a.s. the same limit, we have $\sum_{i=1}^n K_i\Delta_i X \simeq \sum_{i=1}^n K_i(-a\Delta) \simeq -ah$, while $\sum_{i=1}^n K_i(\Delta_i X)^2 \simeq \sum_{i=1}^n K_i(-a\Delta)^2 \simeq a^2 h\Delta$. Since $\frac{h}{\Delta}\to\infty$, then $|T_{\bar t}^n| \to +\infty$.
For the infinite activity jump case, consider for now a model where the small jumps behave like the ones of a symmetric $\alpha-$stable Lévy process. If the jumps are of FV ($\alpha<1$) the sum of the jumps, J, contributes as follows

$$\sum_{i=1}^n K_i\Delta_i J \simeq \sum_{t_{i-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]}\Delta_i J \simeq J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}} \overset{d}{\simeq} h^{\frac{1}{\alpha}} J_1,$$

where $\overset{d}{\simeq}$ indicates that the two expressions have the same limit in distribution, and

$$\sum_{i=1}^n K_i(\Delta_i J)^2 \simeq \sum_{t_{i-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]}(\Delta_i J)^2$$
$$\simeq \left(J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}}\right)^2 - \sum_{i\neq k:\ t_{i-1},t_{k-1}\in[\bar t-\frac{h}{2},\bar t+\frac{h}{2}]}\Delta_i J\Delta_k J \overset{d}{\simeq} \left(J_{\bar t+\frac{h}{2}} - J_{\bar t-\frac{h}{2}}\right)^2 \overset{d}{\simeq} h^{\frac{2}{\alpha}} J_1^2.$$

The compensator part of the model, instead, contributes as a drift, as in the previous case. Then at the numerator of $|T_{\bar t}^n|$ the contribution of the compensator dominates and tends to 0 at speed h, while the denominator tends to 0 more quickly, and again the statistic explodes.
In the case of IV jumps, instead, we cannot separate the jumps from the compensator, and it turns out that $\sum_{i=1}^n K_i\Delta_i X \overset{d}{\simeq} h^{\frac{1}{\alpha}} Z_{1,\alpha}$ and $\sum_{i=1}^n K_i(\Delta_i X)^2 \overset{d}{\simeq} h^{\frac{2}{\alpha}} Z_{2,\alpha}$, with given r.v.s $Z_{1,\alpha}, Z_{2,\alpha}$, and, as mentioned, $\left|T_{\bar t}^n\right|$ converges in distribution.

The finite activity jump case is dealt with under more general conditions on the partitions choice and on the jump sizes. For the infinite activity case, instead, we assume evenly spaced observations and that the small jumps behave like the ones of an (not necessarily symmetric) α-stable Lévy process. In the latter case we separately studied the asymptotic behavior for the characteristic functions of the statistic numerator and squared denominator, and, for $\alpha>1$ also the characteristic function of the joint law of squared numerator and squared denominator. We obtained closed form expressions for the limit characteristic functions.

Our results are consistent with the ones in [5]: in our case σ is zero (no volatility burst), and when the jumps have finite variation the compensator of the small jumps makes $|T_{\bar t}^n|$ to explode. Such a compensator can be interpreted as a bursting drift with respect to the absent Brownian part.
If we add a non-zero Brownian term to our model X then $T_{\bar t}^n$ never explodes: it is asymptotically normal in all cases, because the leading terms at numerator and denominator are all dominated by the Brownian part. Now the picture given in [5], that was missing the case of IV jumps ($\alpha\geq 1$), is complete. Further, we have a new potential test for the presence of a Brownian motion in a DGP.

Actually, $T_{\bar t}^n$ could be exploited for many different tests. Assuming model (1) possibly added with a Brownian part, we firstly check whether $T_{\bar t}^n$ is asymptotically Gaussian or not. In the first case the DGP contains a BM, while in the second case it is a pure jump SM, and if $\left|T_{\bar t}^n\right| \to \infty$ then the DGP has FV jumps, otherwise $\left|T_{\bar t}^n\right| \overset{d}{\to} Z_\alpha$, and then the DGP has IV jumps. In the former case, $|T_{\bar t}^n|$ offers a potential test for whether a jump occurred at $\bar t$ (in which case $|T_{\bar t}^n| \to \sqrt{K(0)}$) or not (in which case $|T_{\bar t}^n| \to +\infty$). Assessment on whether through $T_{\bar t}^n$ we can further distinguish FA from IA jumps is on going.

The paper is organized as follows: Section 2 describes the details about the considered model and sets some notation; Section 3 deals with the case in which the process only has finite activity jumps: the necessary assumptions are set and the first main theorem is stated. Section 4 deals with the case of infinite activity jumps: further assumptions are set and the second main result of the paper is stated. Section 5 accounts for the behaviour of $T_{\bar{t}}^n$ in a SM including also a Brownian component. Section 6 contains the proofs of the Theorems and the necessary Lemmas.

2 Setting

We start with introducing our setting and some notation. We assume that the density λ within the compensator ν in model (1) does not depend on ω, nor on s. For any (x, s), $\delta(x, s) = \delta(\omega, x, s)$ is the random jump size occurring when $\mu(\omega, \{x\}, \{s\}) = 1$, and we assume that $\delta(\omega, x, s)$, from $\Omega \times \mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}$, is a predictable function, i.e. it is measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the predictable σ-algebra of $\Omega \times \mathbb{R}_+$ andLévy $\mathcal{B}(\mathbb{R})$ is the Borelian σ-algebra of $\mathbb{R}$.[2] Further, we assume that $\int_{x,s:|\delta(x,s)|\le 1} \delta^2(x, s)\lambda(x)dx$ is locally bounded, and that if $\mu(\omega, \mathbb{R}, \{s\}) \neq 0$ then $\int_{\mathbb{R}} \delta(\omega, x, s)\mu(dx, \{s\}) \neq 0$.

The measurability conditions above are required to make $\int_0^t \int_{|x|\le 1} \delta(x, s)\tilde{\mu}(dx, ds)$ and $\int_0^t \int_{|x|>1} \delta(x, s)\lambda(x)\cdot dxds$ well defined.

The local boundedness assumption is fulfilled e.g. each time when δ does not depend on s nor on ω, in fact since $\int_0^T \int \delta^2(x, s) \wedge 1\lambda(x)dxds$ is a.s. finite for any semimartingale, then $\int \delta^2(x) \wedge 1\lambda(x)dx$ is finite. That is the case, for instance, of any Lévy process, where $\delta(x) = x$. Actually, for the IA jump case we will restrict to α-stable Lévy processes.

The last requirement above simply means that if a jump occurs at s then the size is non-zero.

Notation 1. $\cdot$ $K_+ \doteq \int_0^{+\infty} K(u)du$, $K_- \doteq \int_{-\infty}^0 K(u)du$;

$\cdot$ for any random process b,

$$b_{\bar{t}}^\star \doteq b_{\bar{t}-} \cdot K_+ + b_{\bar{t}+} \cdot K_-; \tag{4}$$

$\cdot$ when X has FV jumps, we define $a_s \doteq \int_{|\delta(x,s)|\le 1} \delta(x, s)\lambda(x)dx$.

For fixed $\bar{t} \in (0, T)$ the statistic $T_{\bar{t}}^n$ of our interest is well defined when the denominator is non-zero. As it will be clear from the proofs of our Lemmas, this is the case at least when X jumps at $\bar{t}$ or when X has IA Lévy jumps (in which case in any small interval some jumps occur). When no jumps occurr at $\bar{t}$ and X has FA jumps, the statistic is well defined at least when $a_{\bar{t}}^\star \neq 0$ (see (15)).

Defined $\Delta = \Delta_n = \frac{T}{n}$ and $\Delta_{max} = \Delta_{max,n} = \max_{i=1..n} |t_i - t_{i-1}|$ we assume that

$$\Delta_{max} \le C\Delta$$

for a fixed constant C, which means that the partition should not differ too much, asymptotically, from the equally spaced one. The framework (3), under which we look for our asymptotic results, implies that $\Delta \to 0$ and $\frac{\Delta}{h} \to 0$.

[2] It is well known that we can equivalently write $X_t = \int_0^t \int_{|x|\le 1} x\tilde{\mu}'(dx, ds) + \int_0^t \int_{|x|>1} x\mu'(dx, ds)$, where μ' is a random counting measure with compensator $\nu'(dx, ds) = F_s(dx)ds$ and $F_s(dx) = F_s(\omega, dx)$ is random (see [14], Sec. 2.1.4).

As mentioned in the Introduction, it turns out that for a fixed ω the behaviour of $T^n_{\bar{t}}$ is different in the two cases where $\bar{t}$ is a jump time or it is not, and the statistic asymptotic distribution is substantially different depending on whether the jumps have finite or infinite variation. We tackle the finite activity jump case first, while the infinite activity case is dealt with in Section 4.

Notation 2. $\cdot$ C always indicates a constant. Within the algebraic expressions we keep the same name C even if for the two sides of an equality we have different constants.

$\cdot$ Given two functions f, g, then $f(h) \simeq g(h)$ indicates that $\lim_{h\to 0} f(h) = \lim_{h\to 0} g(h)$, while $f(h) \sim g(h)$ indicates that $\lim_{h\to 0} \frac{f(h)}{g(h)} = C$, $f(h) \ll g(h)$ indicates asymptotic negligibility of f w.r.t. g, i.e. $\lim_{h\to 0} \frac{f(h)}{g(h)} = 0$; given two sequences T^n, U^n of random variables, $T^n \overset{d}{\simeq} U^n$ means that they have the same limit in distribution.

$\cdot$ ΔX_t indicates the size of the jump possibly occurred at t (under our framework $\Delta X_t = 0$ iff $\mu(\omega, \mathbb{R}, \{t\}) = 0$)

$\cdot$ $K_s \doteq K\left(\frac{\bar{t}-s}{h}\right)$

$\cdot$ For any $\alpha > 0$, $K_{(\alpha)} \doteq \int_{\mathbb{R}} K^\alpha(u)du$

$\cdot$ $\mathbb{R}_+ = (0, +\infty)$, $\quad \mathbb{R}_- = (-\infty, 0)$

$\cdot$ $\lambda(\mathbb{R}) \doteq \int_{\mathbb{R}} \lambda(x)dx$;

$\cdot$ $\mu(dx, ds), \tilde{\mu}(dx, ds)$ can be abbreviated using $d\mu, d\tilde{\mu}$, respectively;

$\cdot$ sometimes we write δ in place of $\delta(x, s)$.

3 Finite activity jumps

We now consider the case in which $\int_0^T \int_{\mathbb{R}} 1\nu(dx, ds) = T\int_{\mathbb{R}} \lambda(x)dx < \infty$. Then we have that

$$\left| \int_0^t \int_{x,s:|\delta(x,s)|\le 1} \delta(x,s)\nu(dx,ds) \right| \le \int_0^t \int_{x,s:|\delta(x,s)|\le 1} \lambda(x)dxds \le \lambda(\mathbb{R})T$$

is finite, and then X can be written as

$$X_t = \int_0^t \int_{\mathbb{R}} \delta(x,s)\mu(dx,ds) - \int_0^t \int_{x,s:|\delta(x,s)|\le 1} \delta(x,s)\nu(dx,ds).$$

The latter term, $-\int_0^t \int_{|\delta(x,s)|\le 1} \delta(x,s)\lambda(x)dxds$, is a random drift also named $-\int_0^t a_s ds$, and its absolute value is bounded by $\lambda(\mathbb{R})t$. On the other hand $\int_0^t \int_{\mathbb{R}} \delta(x,s)\mu(dx,ds)$ coincides with $\sum_{p=1}^{N_t} c_p$ for any $t \in [0,T]$, where N is the process counting the finitely many jumps, occurring at some random times $S_1(\omega), ..., S_{N_T(\omega)}(\omega)$ on $[0,T]$, and $c_p = c_p(\omega) \doteq \int_{\mathbb{R}} \delta(\omega, x, S_p)\mu(dx, \{S_p\}) = \delta(\omega, x_p, S_p)$ is the random finite size of the jump at S_p. Thus we also can write X as

$$X_t = \sum_{p=1}^{N_t} c_p - \int_0^t a_s ds \doteq J_t - \int_0^t a_s ds.$$

Assumption A1. Kernel function.

A1.1 $K : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant L and satisfies $\lim_{x\to+\infty} K(x) = 0, \lim_{x\to-\infty} K(x) = 0$ and $\int_{\mathbb{R}} K(x)dx = 1$.

A1.2 K satisfies the following:

$\cdot$ if $a < b$ then $K(\frac{b}{h}) << K(\frac{a}{h})$

$\cdot$ for any fixed $x \neq 0$, $K(\frac{x}{h}) << h\Delta$, as $h \to 0$, under (3).

Remark 1. *i) The Gaussian kernel* $K(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ *satisfies Assumption* **A1** *for instance with* $h = \Delta^\gamma$ *with* $\gamma \in (0,1)$. *This is the case if for instance* $h = k_n\Delta$ *with* $k_n = C\sqrt{\Delta}$.

ii) To know how T_t^n *behaves asymptotically if the kernel was an indicator function, one can use our results where the kernel is a Lipschitz continuous approximation of the indicator function.*

Assumption A2. Partitions of $[0,T]$**.** Defined

$$H_t^{(n)} \doteq \frac{1}{\Delta}\sum_{t_i \le t} \Delta_i^2,$$

we assume that:

$\cdot$ for any $t \in (0,T]$ the $\lim_{n\to+\infty} H_t^{(n)} \doteq H_t > 0$ exists and is finite,

$\cdot$ H is Lebesgue differentiable in $(0,T)$ except for a finite and fixed number $m \ge 0$ of points $\tau_1, .., \tau_m$, and H' is bounded,

$\cdot$ defined $I_H^{(n)} = \{i : \exists k, \tau_k \in [t_{i-1}, t_i)\}$, then $\sup_{\{i \notin I_H^{(n)}\}} \sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 0$, as $n \to \infty$.

Remark 2. *The previous Assumption* **A2** *is similar to Assumption 2.2 in [18] but less restrictive. When we have equally spaced observations all the* Δ_i *coincide with* $\frac{T}{n}$ *and* $H' \equiv 1$. *When the observations are more (less) concentrated around t, we have* $H_t < 1$ *(*$H_t > 1$*).*
Note that, where it is defined, $H' \ge 0$, *however if e.g. we had* $n \cdot \min_i \Delta_i \to C$ *then* $H' > 0$.

As an example, consider the sequence of partitions where the amplitude of the first $[n/2]$ *intervals* $[t_{i-1}, t_i)$ *is* 2Φ *and the one of the remaining* $n - [n/2]$ *is* Φ. *Then* $\Phi = \frac{T}{n}\frac{1}{1+[\frac{n}{2}]\frac{1}{n}}$ *and, for any* $t \in (0,T]$, $H_t = \frac{4t}{3}I_{t\le\tau_1} + (\frac{4T}{9} + \frac{2t}{3})I_{t>\tau_1}$ *where* $\tau_1 = 2T/3$. *This function H is not differentiable at* τ_1, *so* $m = 1$ *and for any n,* $I_H^{(n)}$ *is the only i for which* $[t_{i-1}, t_i)$ *contains* τ_1. *Further, the interval* $[t_{i-1}, t_i)$ *for which* $i \in I_H^{(n)}$ *is the first interval having length* Φ. *As for the third condition in Assumption* **A2**, *for any n we have that if* $t_{i-1} \le \tau_1 < t_i$ *then* $\sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 2/3$, *but if both* t_{i-1}, t_i *are on the same side of* τ_1 *(thus* $i \notin I_H^{(n)}$*) then* $\sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| \to 0$. *Further,* $\sup_{\{i \notin I_H^{(n)}\}} \sup_{s\in[t_{i-1},t_i)} |H'_s - \frac{\Delta_i}{T/n}| = |\frac{4}{3} - \frac{2}{1+[\frac{n}{2}]\frac{1}{n}}| \to 0$, *and Assumption* **A2** *is satisfied.*

Assumption A3. Jump sizes. For $\delta(\omega, x, s)$, with $a_s = \int_{|\delta(x,s)|\le 1} \delta(x,s)\lambda(x)dx$, at least one of the following conditions holds true:

(i) a.s. $\sup_{i=1,..,n} \sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}| \to 0$;

(ii) $\sup_{i=1,..,n} \sup_{s\in[t_{i-1},t_i)} |a_s - a_{t_{i-1}}| \xrightarrow{P} 0$;

(iii) there exists $\rho > 0$: $\forall s, u$ such that $|s - u| \le \Delta$ then $E[|a_s - a_u|] \le C\Delta^\rho$.

Remark 3. *i) The above requires regularity of the paths of the drift coefficient* a.

ii) Condition (i) amounts to requiring that a has a.s. continuous paths. In fact, if a has continuous paths then on $[0,T]$ each path is uniformly continuous in t, and then (i) is satisfied, while as soon as on a path of a some jumps occur, then (i) is not satisfied.

iii) If δ does not depend on s then a_t collapses on the r.v. $a \equiv \int_{|\delta(x)|\leq 1} \delta(x)\lambda(x)dx$ for any t, and trivially all the three conditions (i) - (iii) are satisfied. In particular **A3** *is satisfied if X is an α-stable process, any $\alpha \in (0,2)$ is.*

iv) If, rather than through a truncation function $I_{\{|x|\leq 1\}}$, X is represented as $X_t = \int_0^t \int_{\mathbb{R}} \kappa(\delta(x,s))\tilde{\mu}(dx,ds) + \int_0^t \int_{\mathbb{R}} \kappa'(\delta(x,s))\mu(dx,ds)$, where $\kappa(x)$ is a deterministic continuous function of $x \in \mathbb{R}$, bounded, with compact support, with $\kappa(x) = x$ in a neighbourhood B of 0 and $\kappa'(x) \doteq x - \kappa(x)$, then **A3** *(i) is satisfied, in this framework of finite activity jumps, as soon as, for any x, $\delta(x,s)$ is a.s. continuous in s, with $a_s = \int_{\delta(x,s)\in B} \delta(x,s)\lambda(x)dx$.*

v) Condition (ii) amounts to saying that the sequence of processes $G_s^{(n)} \doteq \sum_{i=1}^n (a_s - a_{t_{i-1}})I_{s\in[t_{i-1},t_i)}$ tends to 0 ucp.

vi) Condition (iii) is similar to a requirement given at Assumption 2.1 in [18].

The following definition helps to focus on the asymptotic behavior of $T_{\bar{t}}^n$: given a deterministic function $f(x)$ we set

$$F^n(X) \doteq \sum_{i=1}^n K_i f(\Delta_i X). \tag{5}$$

With $f(x) = x$ we obtain the numerator of $T_{\bar{t}}^n$, with $f(x) = x^2$ the squared denominator. Note that here we only are interested in the r.v. $F^n(X)$ (rather than in a process), which is computed using *all* the increments $\Delta_i X$ with t_i from t_1 to t_n. The next Lemma describes the asymptotic behavior of $F^n(X)$.

Lemma 1. *If $\lambda(\mathbb{R}) < \infty$ and $J \doteq \left(\int_0^t \int \delta(x,s)\mu(dx,ds)\right)_{t\geq 0}$, then under (3), if K is continuous at 0 and $\lim_{x\to\pm\infty} K(x) = 0$, then for any real function $f(x)$ continuous on $\mathbb{R}$ we have*

$$F^n(J) \stackrel{a.s.}{\to} F(J) \doteq K(0)f(\Delta J_{\bar{t}}).$$

From the Lemma, the limit of $T_{\bar{t}}^n$ is almost immediately obtained if $\Delta J_{\bar{t}} \neq 0$. On the other hand, if $\Delta J_{\bar{t}} = 0$ both the numerator and the denominator of $T_{\bar{t}}^n$ tend to 0, and we need some work to catch the leading terms. The behavior of $T_{\bar{t}}^n$ in this framework is as follows.

Theorem 1. *Under model (1) and conditions (3),*

a) If K satisfies Assumption **A1.1** *and $\frac{\Delta}{h^2} \to 0$, we have a.s. that if $\bar{t}$ is a jump time then*

$$T_{\bar{t}}^n \to \sqrt{K(0)} \cdot sgn(\Delta X_{\bar{t}}).$$

b) Under Assumptions **A1**, **A2** *and* **A3***(i), under $\frac{\Delta}{h^2} \to 0$ and if $(a_s)_{s\geq 0}$ is làdlàg then we have that a.s., if $\Delta X_{\bar{t}} = 0$ but $a_{\bar{t}}^\star \neq 0$ and $H'_{\bar{t}\pm} > 0$, then*

$$T_{\bar{t}}^n \to sgn(-a_{\bar{t}}^\star) \cdot \infty,$$

where $a^\star$ is defined as in (4).

If, within b), Assumption **A3***(i) is replaced by either Assumption* **A3***(ii) or Assumption* **A3***(iii) then the result is in probability.*

Remark 4. *i) If, on ω, a is continuous at $\bar t$ then $a^\star_{\bar t} = a_{\bar t}$.*

ii) Note that, since our process X is an Ito semimartingale, it has "no fixed times of discontinuities," namely $P\{\Delta X_{\bar t} \neq 0\} = 0$. That notwithstanding, point a) of the theorem is relevant from the practical point of view, because we only have at hand one specific path $\{X_s(\omega), s \in [0,T]\}$, on which at $\bar t$ a jump could well be occurred.

Remark 5. *If the jump process is represented in the form*

$$J_t = \sum_{p=1}^{N_t} c_p,$$

without compensation, then the drift coefficient $a_s \equiv 0$, and part b) of the theorem above does not apply. However, the limit behavior of $T^n_{\bar t}(J)$ does not change if $\bar t$ is a jump time, because for small Δ we have (with the notation given within the proof of the Theorem)

$$T^n_{\bar t}(J) = \frac{\sum_{p=1}^{N_T} K_{i_p} c_p}{\sqrt{\sum_{p=1}^{N_T} K_{i_p} c_p^2}} \simeq \frac{K(0) c_{\bar t}}{\sqrt{K(0) c_{\bar t}^2}} = \sqrt{K(0)} \cdot sgn(c_{\bar t}).$$

In the case where $\bar t$ is not a jump time, the absence of a drift in J could imply that $T^n_{\bar t}(J)$ is not defined. This is the case for instance when $N_T = 0$; or when $N_T \geq 1$ but the support of K is bounded. If e.g. $K(x)$ is a Lipschitz continuous approximation of $I_{\{|x| \leq \frac{1}{2}\}}$ then for sufficiently small h we have that both $\sum_{i=1}^n K_i \Delta_i X = 0$ and $\sum_{i=1}^n K_i (\Delta_i X)^2 = 0$, thus $T^n_{\bar t}(J)$ is not defined.
Note that it is always true that if $\sum_{i=1}^n K_i (\Delta_i X)^2 = 0$ then also $\sum_{i=1}^n K_i \Delta_i X = 0$.
If $N_T \geq 1$ and $spt(K) = \mathbb{R}$, then $T^n_{\bar t}(J) \to 0$. In fact, let us indicate: by $[t_{i_p-1}, t_{i_p}[$ the unique interval of the partition containing the time of the p-th jump; and by $\underline{p}$ the number such that $|\bar t - S_{\underline{p}}| \doteq \min_p |\bar t - S_p| > 0$. Then, for small Δ,

$$T^n_{\bar t}(J) = \frac{\sum_{p=1}^{N_T} K_{i_p} c_p}{\sqrt{\sum_{p=1}^{N_T} K_{i_p} c_p^2}} \simeq \frac{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right) c_{\underline{p}}}{\sqrt{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right) c_{\underline{p}}^2}} = \sqrt{K\left(\frac{\bar t - S_{\underline{p}}}{h}\right)} \cdot sign(c_{\underline{p}}) \to 0.$$

Note that in this framework of FA jumps $T^n_{\bar t}$ could offer a test for the presence of a drift part in the DGP: if a drift $\int a_s ds$ is present in X then either $|T^n_{\bar t}| \to \sqrt{K(0)}$ or $|T^n_{\bar t}| \to \infty$; if not then $T^n_{\bar t} \to 0$. We comment of the potential use of $T^n_{\bar t}$ as a test for a jump at $\bar t$ in the next Section.

In this paper we conduct our analysis for model (1), which coincides with the jump component in [5], and is always well defined. On the contrary, dealing with only the jump process J is not possible when jumps have IV, and when we apply the test statistic to some data we do not know whether the jumps of the DGP are of FV or of IV, so we do not know whether we can separate the jumps from the compensator part.

4 Infinite activity jumps

When the jumps have infinite activity, it turns out that if $\Delta X_{\bar{t}} \neq 0$ (again an event of zero probability), then $T^n_{\bar{t}}$ has the same limit as in the FA jumps case. While when $\Delta X_{\bar{t}} = 0$, as above both the numerator and the denominator tend to 0 in probability, and the freneticity of the small jumps activity is crucial in determining how quickly they converge. For that we need to account for a jump activity index, and it is natural to focus on the very representative case where the compensated small jumps of X behave like the ones of a Lévy α-stable processes X. Note that the large jumps are always of FA, thus their jump activity index is 0 and they do not contribute in determining the convergence speeds. For the stable processes, α coincides with the Blumenthal-Getoor jump activity index, so that the higher the α the wilder the jump activity. In particular we show that the speed of convergence of numerator and denominator of $T^n_{\bar{t}}$ heavily depends on α, in particular the limit of $T^n_{\bar{t}}$ is different when $\alpha < 1$ (finite variation jumps) or $\alpha > 1$ (infinite variation jumps).

In this part, for the cases when $\Delta X_{\bar{t}} = 0$ we specify the α-stable assumption **IA3** on the compensated small jumps and for simplicity we concentrate on the case of equally spaced observations (assumption **IA2**). Further, we add the technical requirement **IA1** on the Kernel function, which is satisfied at least in the Gaussian kernel case.

Assumption IA1. Kernel. Given a deterministic function φ defined on $\mathbb{R}_+$, we say that K satisfies **IA1** for φ if K is monotonically non-decreasing on $\mathbb{R}_-$ and non-increasing on $\mathbb{R}_+$ and there exists a deterministic function ε_h such that as $h \to 0$

$$\varepsilon_h \to 0, \quad \frac{\varepsilon_h}{h} \to +\infty \quad \text{and} \quad \frac{K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)} \to +\infty. \tag{6}$$

Remark 6. *For instance, with φ equal to any one of the speed functions $\varphi_\alpha(h)$ or $\psi_\alpha(h)$ at (9) below, with the Gaussian kernel, and with the function*

$$\varepsilon_h \doteq h\sqrt{\log\log\frac{1}{h}} \tag{7}$$

the above conditions (6) are satisfied for any $\alpha \in (0,2)$.

Assumption IA2. Partitions. We take $\Delta_i = \Delta$ for all n, for all $i = 1, .., n$.

Assumption IA 3. Small jumps. The compensated jumps of X, with size smaller than 1 in absolute value, are α-stable, that is

$$X = \tilde{J} + J^1, \qquad \text{where} \qquad \tilde{J}_t = \int_0^t \int_{|x|\le 1} x\tilde{\mu}(dx,ds), \quad J^1_t = \int_0^t \int_{|\delta(x,s)|>1} \delta(x,s)\mu(dx,ds),$$

where the compensating measure of the jumps smaller than 1 has the form $\nu(dx,ds) = \lambda(x)dxds$, with

$$\lambda(x) = \frac{A_+}{x^{1+\alpha}} I_{\{0<x\le 1\}} + \frac{A_-}{|x|^{1+\alpha}} I_{\{-1\le x<0\}},$$

where $A_+, A_- > 0$ and $\alpha \in (0,2)$, while $\delta(\omega,x,s)I_{|\delta(\omega,x,s)|>1}$ is a predictable function as in Section 1.

Remark 7. *i) Assumption* **IA3** *requires in particular that the* jump activity index *of X defined in [1] (p.2) is constant with respect to t and ω. The prototypical example of process having constant jump activity index α is the α-stable process. In [1], Assumption 2, the jump activity index is constant but λ is replaced by a richer $F_t(\omega, x)$ where $A_+ I_{\{x>0\}}, A_- I_{\{x<0\}}$ are replaced by $\Big(1 + |x|^\gamma f(t,x)\Big) a_t^{(+)} I_{\{x\in(0,z_t^{(+)}]\}}$ and $\Big(1 + |x|^\gamma f(t,x)\Big) a_t^{(-)} I_{\{x\in[-z_t^{(-)},0)\}}$ where $f(\cdot, x), a^{(\pm)}$ and also the boundaries $z^{(\pm)}$ of the jump sizes are random processes, and $\gamma > 0$. The latter processes however are uniformly bounded and the boundaries are also bounded away from 0, while the contribution of $|x|^\gamma f(t,x)$ vanishes when x approaches 0. Thus we expect that if the compensated small jumps obeyed such assumptions our results would be substantially the same.*

ii) We would obtain the same results if we chose to model as α-stable jumps the ones of X having size smaller than any boundary $c > 0$ in place of 1. We recall that α-stable processes necessarily have $\alpha \in (0, 2]$ and the only 2-stable process is the Brownian motion.

Notation 3. $\cdot$ $E_{i-1}[Z] = E[Z|\mathcal{F}_{t_{i-1}}]$.

$\cdot$ For each $\alpha \in (0, 2)$ let $Z_{i,\alpha}$, $i = 1, 3$, be stable random variables characterized by

$$E[e^{isZ_{1,\alpha}}] = e^{-|s|^\alpha K_{(\alpha)} \left|\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)\right| \cdot (A_+ + A_-)\left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right) sign(s)\right)}; \tag{8}$$

where $\beta = \frac{A_+ - A_-}{A_+ + A_-}$;

$$Z_{2,\alpha} \geq 0, E[e^{-sZ_{2,\alpha}}] = \begin{cases} e^{-s^{\frac{\alpha}{2}} \cdot \frac{2^\alpha}{\sqrt{\pi}} K_{(\alpha/2)}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\left|\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right|}, & \alpha \in (0,1) \cup (1,2) \\ e^{-s^{\frac{\alpha}{2}} \cdot 2^{\alpha-1}\sqrt{\pi} K_{(\alpha/2)}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)}, & \alpha = 1 \end{cases}.$$

$\cdot$ For each $\alpha \in (0, 2)$ let us define on $\mathbb{R}_+$ the speed functions of our interest

$$\varphi_\alpha(h) \doteq \begin{cases} h & \text{if } \alpha \in (0,1), \\ h \log \frac{1}{h} & \text{if } \alpha = 1, \\ h^{\frac{1}{\alpha}} & \text{if } \alpha \in (1,2); \end{cases} \qquad \psi_\alpha(h) \doteq h^{\frac{2}{\alpha}}, \tag{9}$$

where φ_α is shown to be the speed (of convergence to 0 when $\Delta X_{\bar{t}} = 0$) of the numerator of $T_{\bar{t}}^n$ and ψ_α the speed of the squared denominator.

Remark 8. *The random variable $Z_{1,\alpha}$ is α-stable of type $S_\alpha(c, \beta, 0)$, with scale parameter $c = K_{(\alpha)} \left|\Gamma(-\alpha)\right| \cdot \left|\cos\left(\frac{\alpha\pi}{2}\right)\right| (A_+ + A_-)$, skewness parameter β and zero shift parameter (parametrization of [19], thm 14.15).*

By contrast, the law of $Z_{2,\alpha}$ cannot be stable, in that $Z_{2,\alpha}$ is non-negative with positive jump sizes, so it would have to be $\beta = 1$ but then the characteristic function of an $S_{\alpha/2}(c, 1, 0)$ would be not compatible with the above Laplace transform. $Z_{2,\alpha}$ comes from the leading term of a squared α-stable random variable in Lemma 5, but nor does it have the law of a squared α-stable random variable.

Note that $\Gamma(-\alpha) < 0$ and $\cos\left(\frac{\pi\alpha}{2}\right) > 0$ for $\alpha \in (0, 1)$, while $\Gamma(-\alpha) > 0$ and $\cos\left(\frac{\pi\alpha}{2}\right) < 0$ for $\alpha \in (1, 2)$.

The following Theorem provides the asymptotic behavior of the drift burst test statistic $T_{\bar{t}}^n$ in the absence of a Brownian component in X.

Theorem 2. *a) Under Assumption* **A1** *and (3) we still have that*

$$F^n(X) \stackrel{P}{\to} F(X) \doteq K(0)f(\Delta X_{\bar t}), \tag{10}$$

having used the notation in (5).

b) Let the kernel satisfy **A1** *and be such that* $K^{\alpha/2}$ *is Lipschitz and in* $L^1(\mathbb{R})$. *Assume that* K *satisfies* **IA1** *for both the functions* φ_α *and* ψ_α *in (9), and assume* **IA2, IA3**, *the asymptotics (3) and* $\frac{\Delta}{h^2} \to 0$.

In the case $\alpha \le 1$ *let further be* $a^\star \neq 0$.

In the case $\alpha = 1$ *let further* $\sqrt{K}\log K$ *be bounded and* $\frac{\Delta}{h^2}\log^2\frac{1}{h} \to 0$.

Then we have

$$\text{if } \alpha \in (0,1], \quad T^n_{\bar t} \stackrel{P}{\to} sgn(-(A_+ - A_-))\cdot\infty,$$

$$\text{if } \alpha \in (1,2), \quad |T^n_{\bar t}| \stackrel{d}{\to} Z_\alpha \doteq \frac{|Z_{1,\alpha}|}{\sqrt{Z_{2,\alpha}}}.$$

Remarks.

i) Result a) above implies that if on the given path, ω, X has a jump at $\bar t$ then $T^n_{\bar t} \stackrel{P}{\to} \sqrt{K(0)}\cdot sgn(\Delta X_{\bar t})$. However $P\{\Delta X_{\bar t} \neq 0\} = 0$.

ii) Note that under **IA3**, which is assumed at point b), and in the case $\alpha < 1$ we have $a^\star = a = \int_{|x|\le 1} x\lambda(x)dx = \frac{A_+ - A_-}{1-\alpha} < \infty$. Thus when $\alpha < 1$ and $a \neq 0$, $sgn(a^\star) = sgn(A_+ - A_-)$, and the above result is in continuity with Theorem 1, part b).

iii) The asymptotic law of $T^n_{\bar t}$ does not depend on $\bar t$, nor on T, because even if $T^n_{\bar t}$ is substantially constructed with the increments of X within a window of length h around $\bar t$, under our framework such increments are i.i.d., and have the same law for any $\bar t$ and any T.

iv) From the proof of Lemma 6, the two random variables $Z_{1,\alpha}, Z_{2,\alpha}$ turn out not to be independent, because as soon as $\alpha < 2$ the joint Laplace transform of $(Z^2_{1,\alpha}, Z_{2,\alpha})$ cannot be factorized.

v) It is never the jumps to cause $T^n_{\bar t}$ to explode: when the jumps have FV ($\alpha < 1$) then the explosion is due to the compensator (drift part of the model); when the jumps have IV ($\alpha > 1$) then $T^n_{\bar t}$ converges to a finite r.v.. This corroborates the results in [5].

vi) It is not clear whether or not it is possible to construct confidence intervals for Z_α starting from the Laplace transform of $(Z^2_{1,\alpha}, Z_{2,\alpha})$.

In case, $T^n_{\bar t}$ would offer a test for FV jumps (in which case $|T^n_{\bar t}| \to +\infty$) against IV jumps (in which case $|T^n_{\bar t}| \to Z_\alpha$), or a test for whether a jump occurred at $\bar t$ (in which case $|T^n_{\bar t}| \to \sqrt{K(0)}$) or not (either $|T^n_{\bar t}| \to +\infty$ or $|T^n_{\bar t}| \to Z_\alpha$).

vii) In practice, financial asset price models use CGMY processes in place of $\alpha-$stable processes. The former are Lévy processes where the small jumps behave exactly as the ones of stable processes, while the large jumps have smaller size, so allowing the increments of X to have finite moments. The Lévy density of a CGMY model is of type

$$\lambda(x) = \frac{Ce^{-Mx}}{x^{1+Y}} I_{\{x>0\}} + \frac{Ce^{-G|x|}}{|x|^{1+Y}} I_{\{x<0\}},$$

where $C, G, M > 0$. Under this model, for $Y \in (0,2)$ the same results of the current Section would have substantially to hold, because they only depend on the behavior of the small jumps. However probably the constants G, M appear within the limit laws of $Z_{1,\alpha}, Z_{2,\alpha}$, and possibly could multiply the speed functions $\varphi_\alpha, \psi_\alpha$ of numerator and squared denominator of $T^n_{\bar{t}}$. Note that $e^{-G|x|}$ can be written as $1 - G|x|f(x)$, so the CGMY model falls into the framework in [1].

5 In the presence of a Brownian component

It is natural now to wonder what is the behavior of $T^n_{\bar{t}}$ when X contains both a Brownian part and infinite variation jumps. In [5] it is proved that in the presence of a Brownian part, when the jumps have finite variation, corresponding here to the case $\alpha < 1$, and there is no drift burst, then $T^n_{\bar{t}} \xrightarrow{d} \mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ denotes the law of a standard normal r.v.. The following corollary certifies that the same result holds also when the jumps have infinite variation, because the Brownian part introduces the leading terms both at the numerator and at the denominator of $T^n_{\bar{t}}$. It follows that

(a) In the presence of a never vanishing volatility component we have

- $T^n_{\bar{t}} \xrightarrow{d} \mathcal{N}(0,1)$ when there is no drift burst (whatever the variation of the jumps)
- $|T^n_{\bar{t}}| \xrightarrow{P} +\infty$ when there is drift burst at $\bar{t}$

(b) In the absence of a Brownian component and of drift burst then

- $|T^n_{\bar{t}}| \xrightarrow{d} Z_\alpha$ if $\alpha \in (1,2)$, while
- $|T^n_{\bar{t}}| \xrightarrow{P} +\infty$ if $\alpha \in (0,1]$.

As mentioned in the Introduction, tests based on discrete observations are available for assessing whether in a SM model without drift bursts a Brownian component is needed to better explain the data. Potentially $|T^n_{\bar{t}}|$ offers a further test.

Corollary 1. *Let Y evolve following $dY_t = b_t dt + \sigma_t dW_t + dX_t$, Y_0 being $\mathcal{F}_0$-measurable, where $\{b_t\}_{t\ge 0}$ is a locally bounded and predictable drift process, $\{\sigma_t\}_{t\ge0}$ is an adapted, càdlàg positive volatility process bounded away from zero: a.s., for any $t > 0, \sigma_t \ge \underline{\Sigma} > 0$; $\{W_t\}_{t\ge0}$ is a standard Brownian motion and $X = \tilde{J} + J^1$ is a pure-jump process for which the compensated small jumps behave like the ones of an α-stable process with $\alpha \in [1,2)$.*

Let the assumptions of Theorem 2, part b), be fulfilled. Then

$$T^n_{\bar{t}}(Y) = \frac{\sum_{i=1}^n K_i \Delta_i Y}{\sqrt{\sum_{i=1}^n K_i (\Delta_i Y)^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

6 Proofs

The following preliminary Lemma gathers properties of the kernel function that are used numerous times. Some results stated in the Lemma are known, but the proof is reported to ascertain that under the assumptions of this paper everything works correctly.

Lemma 2. *Whatever $\bar t \in (0,T)$ is, under (3), the following hold true:*

1) [Lemma A.1 (i) in [18]]. For a sequence of processes $b^{(n)}$ bounded by the same constant C, for any Lipschitz function $K(x)$ with Lipschitz constant L and $\frac{\Delta}{h^2} \to 0$ then

$$\int_0^T \frac{1}{h} K\Big(\frac{\bar t - s}{h}\Big) b_s^{(n)} ds - \sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar t - t_{i-1}}{h}\Big) \int_{t_{i-1}}^{t_i} b_s^{(n)} ds = O_{a.s.}\left(\frac{\Delta}{h^2}\right)$$

2) If K is Lipschitz, $K \in L^1(\mathbb{R})$ and $\frac{\Delta}{h^2} \to 0$ then $\frac{\sum_{i=1}^n K_i \Delta_i}{h} \to K_{(1)} = \int_{\mathbb{R}} K(u)du$.

3) If K^2 is Lipschitz, has $K_{(2)} = \int_{\mathbb{R}} K^2(x)dx < \infty$ and $\frac{\Delta}{h^2} \to 0$ then $\frac{\sum_{i=1}^n K_i^2 \Delta_i}{h} \to K_{(2)}$.

4) For a làdlàg bounded process b and any density function $K(x)$ on $\mathbb{R}$ we have a.s.

$$\int_0^T \frac{1}{h} K\Big(\frac{\bar t - s}{h}\Big) b_s ds \to b_{\bar t}^{\star}.$$

5) If K is Lipschitz, $K \in L^1(\mathbb{R})$, $\frac{\Delta}{h^2} \to 0$ and $b^{(n)}$ are processes for which

(i) a.s. $\sup_{i=1,..,n} \sup_{s\in[t_{i-1},t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \to 0$,

then a.s.

$$\sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar t - t_{i-1}}{h}\Big) b_{t_{i-1}}^{(n)} \Delta_i \simeq \sum_{i=1}^n \frac{1}{h} K\Big(\frac{\bar t - t_{i-1}}{h}\Big) \int_{t_{i-1}}^{t_i} b_s^{(n)} ds.$$

If the last assumption is replaced by either

(ii) $\sup_{i=1,..,n} \sup_{s\in[t_{i-1},t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \xrightarrow{P} 0$

or

(iii) there exists $\rho > 0:$ $\forall s, u$ such that $|s-u| \le \Delta$ then $E[|b_s^{(n)} - b_u^{(n)}|] \le C\Delta^\rho$,

then the above result holds in probability rather than a.s..

6) If K^2 is Lipschitz and in $L^1(\mathbb{R})$, then under (3) and $\frac{\Delta}{h^2} \to 0$

$$\sum_{i=1}^n \sum_{j<i} K_i^2 K_j^2 \Delta_j \Delta_i \simeq \int_0^T K_u^2 \int_0^u K_s^2 ds du.$$

Proof of Lemma 2. As for 1), the displayed left term coincides with

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{1}{h} (K_s - K_i) b_s^{(n)} ds,$$

whose absolute value is dominated by

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{L}{h^2} |s - t_{i-1}| C ds = O_{a.s.}\left(\frac{\Delta}{h^2}\right).$$

2) By 1) in the special case where $b^{(n)} \equiv 1$ for all n we have $\frac{\sum_{i=1}^n K_i \Delta_i}{h} = \frac{1}{h}\int_0^T K\left(\frac{\bar t - s}{h}\right) ds + O_{a.s.}\left(\frac{\Delta}{h^2}\right) = \int_{\frac{\bar t - T}{h}}^{\frac{\bar t}{h}} K(u)du + O_{a.s.}\left(\frac{\Delta}{h^2}\right) \to \int_{\mathbb{R}} K(u)du$, where for the last equality we operated the change of variable $u = (\bar t - s)/h$.

3) We apply 2).

4) For fixed ω the term $\int_0^T \frac{1}{h} K_s b_s ds$ coincides with $\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K(u) b_{\bar{t}-hu} du$, and

$$\left| \int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} K(u) b_{\bar{t}-hu} du - b_{\bar{t}}^{\star} \right| \leq \left| \int_{\frac{\bar{t}-T}{h}}^{0} K(u) b_{\bar{t}-hu} du - b_{\bar{t}+} \cdot \int_{-\infty}^{0} K(u) du \right|$$

$$+ \left| \int_{0}^{\frac{\bar{t}}{h}} K(u) b_{\bar{t}-hu} du - b_{\bar{t}-} \cdot \int_{0}^{+\infty} K(u) du \right|$$

$$\leq \int_{\mathbb{R}} |b_{\bar{t}-hu} - b_{\bar{t}+}| \, I_{(\frac{\bar{t}-T}{h}, 0]}(u) K(u) du + \int_{\mathbb{R}} |b_{\bar{t}-hu} - b_{\bar{t}-}| \, I_{(0, \frac{\bar{t}}{h}]}(u) K(u) du$$

$$+ \int_{\mathbb{R}} \Big(|b_{\bar{t}+}| I_{(-\infty, \frac{\bar{t}-T}{h})}(u) + |b_{\bar{t}-}| I_{(\frac{\bar{t}}{h}, +\infty)}(u) \Big) K(u) du :$$

the three terms are integrals, in the finite measure on $\mathbb{R}$ having intensity K, of bounded integrands converging to 0 point-wise as $h \to 0$. By the dominated convergence theorem the three integrals tend to 0 and 4) is proved.

5) If either (i) or (ii) holds true, the thesis follows from the fact that

$$\left| \sum_{i=1}^{n} \frac{1}{h} K_i \int_{t_{i-1}}^{t_i} b_s^{(n)} - b_{t_{i-1}}^{(n)} ds \right| \leq \sup_{i=1,..,n} \sup_{s \in [t_{i-1}, t_i)} |b_s^{(n)} - b_{t_{i-1}}^{(n)}| \sum_{i=1}^{n} \frac{1}{h} K\Big(\frac{\bar{t} - t_{i-1}}{h} \Big) \Delta_i,$$

which tends to 0 a.s. (respectively, tends to 0 in P).
If (iii) holds true then

$$E\left[\left| \sum_{i=1}^{n} \frac{1}{h} K_i \int_{t_{i-1}}^{t_i} b_s^{(n)} - b_{t_{i-1}}^{(n)} ds \right| \right] \leq \frac{1}{h} \sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} E[|b_s^{(n)} - b_{t_{i-1}}^{(n)}|] ds \leq \frac{C}{h} \sum_{i=1}^{n} K_i \Delta_i^{1+\rho} \to 0.$$

6) We have

$$\int_0^T K_u^2 \int_0^u K_s^2 ds du - \sum_{i=1}^{n} K_i^2 \Big(\sum_{j<i} K_j^2 \Delta_j \Big) \Delta_i = \Bigg(\int_0^T K_u^2 \int_0^u K_s^2 ds du \tag{11}$$

$$- \sum_{i=1}^{n} K_i^2 \int_0^{t_{i-1}} K_s^2 ds \Delta_i \Bigg) + \Bigg(\sum_{i=1}^{n} K_i^2 \int_0^{t_{i-1}} K_s^2 ds \Delta_i - \sum_{i=1}^{n} K_i^2 \Big(\sum_{j<i} K_j^2 \Delta_j \Big) \Delta_i \Bigg).$$

Since $\int_0^{t_{i-1}} K_s^2 ds = \sum_{j<i} \int_{t_{j-1}}^{t_j} K_s^2 ds$, the latter term is dominated in absolute value by

$$\sum_{i=1}^{n} K_i^2 \sum_{j<i} \int_{t_{j-1}}^{t_j} |K_s^2 - K_j^2| ds \Delta_i \leq C \sum_{i=1}^{n} K_i^2 \sum_{j<i} \int_{t_{j-1}}^{t_j} \frac{|s - t_{j-1}|}{h} ds \Delta_i$$

$$\simeq C \sum_{i=1}^{n} K_i^2 \sum_{j<i} \frac{\Delta_j^2}{h} \Delta_i \leq C\Delta \frac{\sum_{i=1}^{n} K_i^2 \Delta_i}{h} = O(\Delta) \to 0.$$

The right hand side term in (11) equals

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K_u^2 \int_0^u K_s^2 ds du - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K_i^2 \int_0^{t_{i-1}} K_s^2 ds du$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big(K_u^2 - K_i^2 \Big) \int_0^{t_{i-1}} K_s^2 ds du + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K_u^2 \int_{t_{i-1}}^{u} K_s^2 ds du :$$

using that for any t_{i-1} we have $\int_0^{t_{i-1}} K_s^2 ds = h\int_{\frac{\bar{t}-t_{i-1}}{h}}^{\frac{\bar{t}}{h}} K^2(w)dw \leq hK_{(2)}$, the first sum is dominated by

$$C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{|u-t_{i-1}|}{h}du\cdot hK_{(2)} = O(\Delta)\to 0.$$

Also for the second sum we use that $\int_{t_{i-1}}^{u} K_s^2 ds = h\int_{\frac{\bar{t}-u}{h}}^{\frac{\bar{t}-t_{i-1}}{h}} K^2(w)dw \leq hK_{(2)}$, thus the sum is dominated by

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_u^2 du\cdot O(h) = \int_0^T K_u^2 du\cdot O(h) = O(h^2)\to 0.$$ □

Proof of Lemma 1. For fixed ω, for any given jump time $S_p = S_p(\omega)$ of J and any integer n, let $i_p = i_p(\omega)$ be the right extreme of the unique interval $[t_{i-1}, t_i)$ containing S_p.

For the fixed ω, $\sum_{p=1}^{N_t} c_p$ is a step-wise constant function of t, so each increment $\Delta_i J$ either is 0, if $[t_{i-1}, t_i)$ does not contain jump times, or is $\sum_{p=1}^{\Delta_i N} c_p$, if $[t_{i-1}, t_i)$ contains some instants S_p. Since the time horizon T is finite and fixed, for sufficiently small Δ we have $0 \leq \Delta_i N \leq 1$ for all $i = 1,..,n$, thus $\Delta_i J$ either is 0 or reduces to a single $c_p \in \mathbb{R} - \{0\}$, and $\sum_{i=1}^n K_i f\Big(\sum_{p=1}^{\Delta_i N} c_p\Big)$ reduces to $\sum_{p=1}^{N_T} K_{i_p} f(c_p)$.

a) When $\bar{t}$ is a jump time then it coincides with one of the S_p, say $S_{\bar{p}} \doteq \bar{t}$, while for the other indices p we have $\Delta S \doteq \min_{p\neq\bar{p}} |S_p - \bar{t}| > 0$. For $\Delta\to 0$ we have that, for all $p = 1,..,N_T$, $t_{i_p-1}\to S_p$, so that $\bar{t} - t_{i_{\bar{p}}-1}\to 0$, and since $|\bar{t} - t_{i_{\bar{p}}-1}| \leq \Delta_{i_p} \leq \Delta$, we have $\frac{|\bar{t}-t_{i_{\bar{p}}-1}|}{h} \leq \frac{\Delta}{h} \to 0$, thus $K_{i_{\bar{p}}} f(c_{\bar{p}}) \to K(0)f(c_{\bar{p}}) = K(0)f(\Delta J_{\bar{t}})$.
On the other hand, for $p\neq\bar{p}$ we have that $|\bar{t} - t_{i_p-1}| \to |\bar{t} - S_p| \geq \Delta S > 0$, thus $\frac{|\bar{t}-t_{i_p-1}|}{h} \to +\infty$, and $K_{i_p}\to 0$. So, for $p\neq\bar{p}$, $K_{i_p} f(c_p)\to 0$.
In other words, for sufficiently small Δ, $\sum_{i=1}^n K_i f\Big(\sum_{p=1}^{\Delta_i N} c_p\Big)$ only contains N_T non-zero terms, and all of them tend to 0 but one. Only the term for which $[t_{i-1}, t_i)$ contains $S_{\bar{p}} = \bar{t}$ has a non-zero limit, amounting to $K(0)f(c_{\bar{p}}) = K(0)f(\Delta J_{\bar{t}})$.

b) When $\bar{t}$ is not a jump time, we have that, for any given ω, each S_p is at positive distance from $\bar{t}$: we define $\underline{p}$ through

$$|\bar{t} - S_{\underline{p}}| \doteq \min_p |\bar{t} - S_p| > 0,$$

and again, for sufficiently small $\Delta = \Delta(\omega)$, we have $\sum_{p=1}^{N_T} K_{i_p} f(\Delta_{i_p} J) = \sum_{p=1}^{N_T} K_{i_p} f(c_p)$, which is a sum of N_T terms, where now all the terms K_{i_p} tend to 0, because, similarly as above, $t_{i_p-1}\to S_p$ but $|\bar{t} - S_p| \geq |\bar{t} - S_{\underline{p}}| > 0$, thus $\frac{|\bar{t}-t_{i_p-1}|}{h} \to +\infty$. However, since $f(\Delta J_{\bar{t}}) = 0$ we can also write $\sum_{i=1}^n K_i f(\Delta_i J) \to K(0)f(\Delta J_{\bar{t}})$. □

Proof of Theorem 1.

a) When $\bar{t}$ is a jump time. We show that a.s.
1a) $\sum_{i=1}^n K_i\Delta_i X \to K(0)\Delta X_{\bar{t}}$,
2a) $\sum_{i=1}^n K_i(\Delta_i X)^2 \to K(0)\Big(\Delta X_{\bar{t}}\Big)^2$,
which are sufficient to conclude.

As for 1a), using Lemma 1 for J, it remains to check that $\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds \overset{a.s.}{\to} 0$, which is almost immediate. In fact, we have

$$\left|\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds\right| \leq \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \lambda(\mathbb{R}) ds = \lambda(\mathbb{R}) h \cdot \frac{\sum_{i=1}^n K_i \Delta_i}{h}.$$

Since the second factor above tends a.s. to 1 we are done.

In order to show 2a) we write

$$\sum_{i=1}^n K_i (\Delta_i X)^2 = \sum_{i=1}^n K_i \Big(\sum_{p=1}^{\Delta_i N} c_p \Big)^2 + \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} a_s ds \Big)^2 - 2 \sum_{i=1}^n K_i \Big(\sum_{p=1}^{\Delta_i N} c_p \Big) \int_{t_{i-1}}^{t_i} a_s ds. \tag{12}$$

By Lemma 1 the first term tends to $K(0)(\Delta X_{\bar{t}})^2$. The second term in the rhs of (12) similarly as above tends to 0, because it is bounded from above by

$$\sum_{i=1}^n K_i (\lambda(\mathbb{R}) \Delta_i)^2 \leq C \Delta h \frac{\sum_{i=1}^n K_i \Delta_i}{h} \to 0.$$

The third term in (12) is a negligible mixed term. In fact, for small Δ it becomes

$$-2 \sum_{p=1}^{N_T} K_{i_p} c_p \int_{t_{i_p - 1}}^{i_p} a_s ds : \tag{13}$$

since on the fixed ω only finitely many jumps occurred, each with finite size, the random number $\bar{c} \doteq \max_{p=1,..,N_T} |c_p|$ is finite, further under Assumption **A1.1** the kernel K is bounded, then the latter sum is dominated in absolute value by

$$C \sum_{p=1}^{N_T} \Delta_{i_p} \lambda(\mathbb{R}) \leq C N_T \Delta \to 0.$$

Thus 2a) follows and a) is proved.

b) When $\bar{t}$ is not a jump time. Within

$$\sum_{i=1}^n K_i \Delta_i X = \sum_{p=1}^{N_T} K_{i_p} \Delta_{i_p} X - \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds,$$

as above, the second sum tends a.s. to 0, and now also the first one does, by Lemma 1. The same happens at the denominator of $T_{\bar{t}}^n$, thus we have a limit form $\frac{0}{0}$, and we look for the speed at which the two terms of the quotient tend to zero.

For that, note that, by virtue of the assumption that if $\mu(\omega, \mathbb{R}, \{s\}) \neq 0$ then $\int_{\mathbb{R}} \delta(\omega, x, s) \mu(dx, \{s\}) \neq 0$, for the fixed ω we have $|\underline{c}| \doteq \min_{p=1,..,N_T} |c_p| > 0$, and we can write $\sum_{i=1}^n K_i \Delta_i X$ as follows

$$\sum_{i=1}^n K_i \Delta_i X = \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|\delta(x,s)| > |\underline{c}|} \delta(x,s) \mu(dx, ds) - \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} a_s ds. \tag{14}$$

For a sufficiently small $\Delta = \Delta(\omega)$ the first sum contains the N_T vanishing terms $K_{i_p} c_p = K\Big(\frac{\bar{t} - t_{i_p - 1}}{h}\Big) c_p$, the leading of which, when $h \to 0$, by Assumption **A1.2** is the one having the smallest $\frac{\bar{t} - t_{i_p - 1}}{h}$. Since for all p we have $t_{i_p - 1} \to S_p$, the slowest term is $K\Big(\frac{\bar{t} - t_{i_{\underline{p}} - 1}}{h}\Big) |c_{\underline{p}}|$, being $|c_{\underline{p}}| > 0$. In other words, for the given ω the first sum in (14) tends to zero at speed $K\Big(\frac{\bar{t} - S_{\underline{p}}}{h}\Big)$.

Using Lemma 2, points 1) and 4),

$$\frac{1}{h}\sum_{i=1}^{n} K_i \int_{t_{i-1}}^{t_i} a_s ds = \int_0^T \frac{1}{h} K\Big(\frac{\bar{t}-s}{h}\Big) a_s ds + O_{a.s.}\left(\frac{\Delta}{h^2}\right) \to a_{\bar{t}}^{\star}(\omega),$$

thus if $a_{\bar{t}}^{\star}(\omega) \neq 0$ the last sum in (14) tends to 0 as $-ha_{\bar{t}}^{\star}$, which, by Assumption **A1.2**, dominates $K\Big(\frac{\bar{t}-S_{\underline{p}}}{h}\Big)$, so the numerator of $T_{\bar{t}}^n$ tends to zero as $-ha_{\bar{t}}^{\star}$.

As for the denominator of $T_{\bar{t}}^n$, from (12) analogously as above we find that the leading term of the first sum is $K\Big(\frac{\bar{t}-S_{\underline{p}}}{h}\Big)c_{\underline{p}}^2$; the third sum for small Δ is as in (13), thus it is bounded in absolute value by $C\sum_{p=1}^{N_T} K_{i_p}|c_p|\Delta_{i_p}$. The latter is in turn asymptotically dominated by $CK\Big(\frac{\bar{t}-S_{\underline{p}}}{h}\Big)|c_{\underline{p}}|\Delta << CK\Big(\frac{\bar{t}-S_{\underline{p}}}{h}\Big)$. This shows that the third sum is negligible with respect to the first one.

The second sum $\sum_{i=1}^n K_i\Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2$ in (12) is now shown to tend a.s. to 0 at speed $h\Delta \cdot (H'a^2)_{\bar{t}}^{\star}$. For that we proceed based on the following schedule:

1b) $\frac{1}{\Delta h}\sum_{i=1}^n K_i\Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 \simeq \frac{1}{\Delta h}\sum_{i=1}^n K_i a_{t_{i-1}}^2 \Delta_i^2$

2b) $\frac{1}{\Delta h}\sum_{i=1}^n K_i a_{t_{i-1}}^2 \Delta_i^2 \simeq \int_0^T \frac{1}{h} K_s H_s' a_s^2 ds$

3b) $\int_0^T \frac{1}{h} K_s H_s' a_s^2 ds \to (H'a^2)_{\bar{t}}^{\star}$,

which proves that the denominator of $T_{\bar{t}}^n$ tends to 0 as

$$\sqrt{K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right) + h\Delta(H'a^2)_{\bar{t}}^{\star}}. \tag{15}$$

However, from Assumption **A1.2** it will follow that the latter tends to 0 as $\sqrt{h\Delta \cdot (H'a^2)_{\bar{t}}^{\star}}$. Then note that

$$(H'a^2)_{\bar{t}}^{\star} = H_{\bar{t}-}' a_{\bar{t}-}^2 K_+ + H_{\bar{t}+}' a_{\bar{t}+}^2 K_- > 0,$$

because at least one between $a_{\bar{t}-}K_+$ and $a_{\bar{t}+}K_-$ is non zero, then at least one between $a_{\bar{t}-}^2 K_+$ and $a_{\bar{t}+}^2 K_-$ is strictly positive, and both $H_{\bar{t}+}', H_{\bar{t}-}'$ are strictly positive. Thus it will also follow that

$$T_{\bar{t}}^n \simeq \frac{-ha_{\bar{t}}^{\star}}{\sqrt{h\Delta(H'a^2)_{\bar{t}}^{\star}}} \simeq -\sqrt{\frac{h}{\Delta}}\frac{a_{\bar{t}}^{\star}}{\sqrt{H_{\bar{t}}'(a^2)_{\bar{t}}^{\star}}} \to \infty \cdot sgn\Big(-a_{\bar{t}}^{\star}\Big),$$

which will conclude the proof of b).

Let us now prove 2b), 3b) and then 1b). As for 2b), the difference of the terms at the two sides is

$$\int_0^T \frac{1}{h} K_s H_s' a_s^2 ds - \frac{1}{\Delta h}\sum_{i=1}^n K_i a_{t_{i-1}}^2 \Delta_i^2$$

$$= \frac{1}{h}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \Big[K_s - K_i\Big] H_s' a_s^2 ds + \frac{1}{h}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} K_i\Big[H_s' a_s^2 - a_{t_{i-1}}^2 \frac{\Delta_i}{\Delta}\Big] ds,$$

having subtracted and added $\int_{t_{i-1}}^{t_i} K_i H_s' a_s^2 ds$ for each i: since K is Lipschitz and H' and a are bounded, the first term of the rhs above is dominated by $\frac{C}{h}\sum_{i=1}^n \frac{\Delta_i^2}{h} \le C\frac{\Delta_{max}}{h^2} \to 0$. We thus remain with the second term, which is split as

$$\frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i H'_s\Big[a_s^2 - a_{t_{i-1}}^2\Big]ds + \frac{1}{h}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i\Big[H'_s - \frac{\Delta_i}{\Delta}\Big]a_{t_{i-1}}^2 ds, \tag{16}$$

where the second sum is

$$\frac{1}{h}\sum_{i\in I_H^{(n)}}\int_{t_{i-1}}^{t_i} K_i\Big[H'_s - \frac{\Delta_i}{\Delta}\Big]a_{t_{i-1}}^2 ds + \frac{1}{h}\sum_{i\notin I_H^{(n)}}\int_{t_{i-1}}^{t_i} K_i\Big[H'_s - \frac{\Delta_i}{\Delta}\Big]a_{t_{i-1}}^2 ds :$$

accounting for the boundedness of K, H', $\frac{\Delta_i}{\Delta}$ and a and for the fact that $\Delta_{max} \leq C\Delta$, the latter display is dominated in absolute value by

$$\frac{C}{h}m\Delta + \frac{C}{h}\sum_{i\notin I_H^{(n)}}\sup_{s\in[t_{i-1},t_i)}\Big|H'_s - \frac{\Delta_i}{\Delta}\Big|K_i\Delta_i,$$

$$\leq C\frac{\Delta}{h} + C\sup_{i\notin I_H^{(n)}}\sup_{s\in[t_{i-1},t_i)}\Big|H'_s - \frac{\Delta_i}{\Delta}\Big|\frac{\sum_{i=1}^n K_i\Delta_i}{h} \overset{a.s.}{\to} 0,$$

having used Lemma 2 part 2). We thus remain with only the first sum in (16), whose absolute value is dominated by

$$\frac{C}{h}\sum_{i=1}^{n}K_i\sup_{s\in[t_{i-1},t_i)}|a_s^2 - a_{t_{i-1}}^2|\Delta_i,$$

however note that

$$\sup_{s\in[t_{i-1},t_i)}|a_s^2 - a_{t_{i-1}}^2| = \sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}||a_s + a_{t_{i-1}}| \leq C\sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|$$

thus the last display is in turn dominated by

$$C\sup_{i=1,..,n}\sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|\cdot\frac{\sum_{i=1}^n K_i\Delta_i}{h} \overset{a.s.}{\to} 0,$$

which concludes the proof of 2b).

If in place of **A3** (i) we assume **A3** (ii), clearly the limit above is in probability. If instead in place of **A3** (i) we assume **A3** (iii) the first sum in (16) is dealt with as follows.

$$E\left[\frac{1}{h}\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} K_i H'_s\Big[a_s^2 - a_{t_{i-1}}^2\Big]ds\right|\right] \tag{17}$$

$$\leq C\frac{1}{h}\sum_{i=1}^{n}K_i\int_{t_{i-1}}^{t_i} E[|a_s - a_{t_{i-1}}|]ds \leq C\frac{1}{h}\sum_{i=1}^{n}K_i\Delta^{1+\rho} \to 0,$$

Thus again the convergence at 2b) takes place in probability.

3b) follows from Lemma 2, point 4).

1b) Writing, for each i, $\Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 = \Big(\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}}ds + a_{t_{i-1}}\Delta_i\Big)^2$ we obtain

$$\frac{1}{\Delta h}\sum_{i=1}^{n}K_i\Big(\int_{t_{i-1}}^{t_i} a_s ds\Big)^2 = \frac{1}{\Delta h}\sum_{i=1}^{n}K_i\Big(\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}}ds\Big)^2 \tag{18}$$

$$+\frac{2}{\Delta h}\sum_{i=1}^{n}K_i\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}}ds\cdot a_{t_{i-1}}\Delta_i + \frac{1}{\Delta h}\sum_{i=1}^{n}K_i a_{t_{i-1}}^2\Delta_i^2,$$

and, since by 2b) and 3b) $\frac{1}{\Delta h}\sum_{i=1}^n K_i a_{t_{i-1}}^2 \Delta_i^2 \to \left(H'a^2\right)^\star_{\bar t} \neq 0$, it is sufficient to show that the first two sums on the right hand side above tend to 0.

In both cases we use that

$$\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}} ds \leq \sqrt{\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i}\left(a_s - a_{t_{i-1}}\right)^2 ds}.$$

It follows that the first of the two sums is

$$\frac{1}{\Delta h}\sum_{i=1}^n K_i\Big(\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}} ds\Big)^2\Delta_i^2 \leq \frac{1}{\Delta h}\sum_{i=1}^n K_i\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i}(a_s - a_{t_{i-1}})^2 ds\Delta_i^2$$

$$\leq \frac{1}{\Delta h}\sum_{i=1}^n K_i \sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|^2\Delta_i^2 \leq C\sup_{i=1,..,n}\sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|^2\frac{\sum_{i=1}^n K_i\Delta_i}{h},$$

which, using Lemma 1, part 2), and Assumption **A3** (i), tends a.s. to 0.

The second sum at the rhs of (18) is

$$\frac{2}{\Delta h}\sum_{i=1}^n K_i\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} a_s - a_{t_{i-1}} ds\cdot a_{t_{i-1}}\Delta_i^2 \leq \frac{2}{\Delta h}\sum_{i=1}^n K_i\sqrt{\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i}\left(a_s - a_{t_{i-1}}\right)^2 ds}\cdot|a_{t_{i-1}}|\Delta_i^2$$

$$\leq \frac{C}{\Delta h}\sum_{i=1}^n K_i\sqrt{\sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|^2}\cdot\Delta_i^2 \leq C\sup_{i=1,..,n}\sup_{s\in[t_{i-1},t_i)}|a_s - a_{t_{i-1}}|\cdot\frac{\sum_{i=1}^n K_i\Delta_i}{h}\overset{a.s.}{\to} 0,$$

which concludes the proof of 1b).

If in place of **A3** (i) we assume **A3** (ii), clearly the last two limits above are in probability. If instead in place of **A3** (i) we assume **A3** (iii) then

$$E\left[\frac{1}{\Delta h}\sum_{i=1}^n K_i\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i}(a_s - a_{t_{i-1}})^2 ds\Delta_i^2\right], E\left[\frac{2}{\Delta h}\sum_{i=1}^n K_i\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i}|a_s - a_{t_{i-1}}|ds\cdot|a_{t_{i-1}}|\Delta_i^2\right]$$

tend to 0 because they turn out to be bounded exactly as in (17). □

Lemma 3. *Let $g:\mathbb{R}\to\mathbb{C}$ be a deterministic Lebesgue integrable function. Given a deterministic function φ defined on $\mathbb{R}_+$, assume that K satisfies* **IA1** *for φ. Then for fixed $\alpha > 0$, for any $s\in\mathbb{R}$, under (3) with $\frac{\Delta}{h^2}\to 0$, we have*

i) if K^α is Lipschitz and in $L^1(\mathbb{R})$ then

$$\sum_{i=1}^n\frac{K_i^\alpha}{h}\Delta_i\int_{|v|\leq\frac{K_i|s|}{\varphi(h)}} g(v)dv \to K_{(\alpha)}\int_{\mathbb{R}} g(v)dv \tag{19}$$

ii) if K is Lipschitz and $K\in L^1(\mathbb{R})$,

$$\sum_{i=1}^n\frac{K_i}{h}\Delta_i I_{\{\frac{|s|K_i}{\varphi(h)}>1\}}\to K_{(1)}.$$

iii) if $K^{\alpha/2}$ is Lipschitz and in $L^1(\mathbb{R})$, and $\Psi\in L^1(\mathbb{R})$ is a deterministic function then

$$\sum_{i=1}^n\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\int_{\mathbb{R}}\Psi(u)\int_{|v|\leq\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|} g(v)dvdu \to K_{(\alpha/2)}\cdot\int_{\mathbb{R}}\Psi(u)du\int_{\mathbb{R}} g(v)dv$$

Proof of Lemma 3. i) Since the difference of the two terms in (19) can be written as

$$\sum_{i=1}^{n}\frac{K_i^\alpha}{h}\Delta_i\left(\int_{|v|\le\frac{K_i|s|}{\varphi(h)}}g(v)dv-\int_{\mathbb{R}}g(v)dv\right)+\int_{\mathbb{R}}g(v)dv\left(\sum_{i=1}^{n}\frac{K_i^\alpha}{h}\Delta_i-K_{(\alpha)}\right),$$

it is sufficient to show that

$$\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\left(\int_{|v|\le\frac{K_i|s|}{\varphi(h)}}g(v)dv-\int_{\mathbb{R}}g(v)dv\right)\to 0, \tag{20}$$

because using then that, as in Lemma 2, 3), $\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\to K_{(\alpha)}$, the proof is concluded. The absolute value of the expression in (20) is dominated by

$$\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\int_{|v|>\frac{K_i|s|}{\varphi(h)}}|g(v)|dv.$$

We split $I\doteq\{1,2,...n\}=I'\cup I''$, where

$$I'=\{i\in I:|\bar t-t_{i-1}|\le\varepsilon_h\},\quad I''=\{i\in I:|\bar t-t_{i-1}|>\varepsilon_h\}.$$

For $i\in I'$ we have $K_i\ge K\left(\frac{\varepsilon_h}{h}\right)$, thus

$$\sum_{i\in I'}\frac{K_i^\alpha\Delta_i}{h}\int_{|v|>\frac{K_i|s|}{\varphi(h)}}|g(v)|dv\le\sum_{i\in I'}\frac{K_i^\alpha\Delta_i}{h}\int_{|v|>\frac{K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}}|g(v)|dv,$$

and the latter tends to 0, because the first factor is dominated by $\sum_{i=1}^{n}\frac{K_i^\alpha\Delta_i}{h}\to K_\alpha$, while the second factor is an integral of $|g|$ on a vanishing region.

On the other hand,

$$\sum_{i\in I''}\frac{K_i^\alpha\Delta_i}{h}\int_{|v|>\frac{K_i|s|}{\varphi(h)}}|g(v)|dv\le\sum_{i\in I''}\frac{K_i^\alpha\Delta_i}{h}\int_{\mathbb{R}}|g(v)|dv,$$

and usign Lemma 2, 1) we have

$$\sum_{i\in I''}\frac{K_i^\alpha\Delta_i}{h}\simeq\int_{r\in(0,T):|\bar t-r|>\varepsilon_h}\frac{K_r^\alpha}{h}dr=\int_{\frac{\bar t-T}{h}}^{-\frac{\varepsilon_h}{h}}K^\alpha(u)du+\int_{\frac{\varepsilon_h}{h}}^{\frac{\bar t}{h}}K^\alpha(u)du\to 0. \tag{21}$$

ii) We have that

$$\sum_{i=1}^{n}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K_i}{\varphi(h)}>1\right\}}-K_{(1)}\simeq\sum_{i=1}^{n}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K_i}{\varphi(h)}>1\right\}}-\sum_{i=1}^{n}\frac{K_i}{h}\Delta_i=\sum_{i=1}^{n}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K_i}{\varphi(h)}\le1\right\}},$$

and we show that the latter sum has limit 0. With I' and I'' as defined at point i), we immediately see that

$$\sum_{i\in I'}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K_i}{\varphi(h)}\le1\right\}}\to 0,$$

in fact if $|\bar t-t_{i-1}|\le\varepsilon_h$ then $K\left(\frac{|\bar t-t_{i-1}|}{h}\right)\ge K\left(\frac{\varepsilon_h}{h}\right)$, thus

$$\sum_{i\in I'}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K_i}{\varphi(h)}\le1\right\}}\le\sum_{i\in I'}\frac{K_i}{h}\Delta_iI_{\left\{\frac{|s|K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)}\le1\right\}}=I_{\left\{\frac{|s|K\left(\frac{\varepsilon_h}{h}\right)}{\varphi(h)}\le1\right\}}\sum_{i\in I'}\frac{K_i}{h}\Delta_i:$$

since the first factor tends to 0 and the second one is bounded, the latter product tends to 0.

Now we check that also

$$\sum_{i\in I''}\frac{K_i}{h}\Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\le 1\right\}}\to 0.$$

First of all note that

$$\sum_{i\in I''}\frac{K_i}{h}\Delta_i I_{\left\{\frac{|s|K_i}{\varphi(h)}\le 1\right\}}\le \sum_{i\in I''}\frac{K_i}{h}\Delta_i=\sum_{i=1}^{n}\frac{K_i}{h}\Delta_i I_{\left\{|\bar t-t_{i-1}|>\varepsilon_h\right\}},$$

then, with $b_r^{(n)}\doteq I_{\{|\bar t-r|>\varepsilon_h\}}$, as soon as we have verified $\sup_{i\in I''}\sup_{r\in[t_{i-1},t_i)}|b_r^{(n)}-b_{t_{i-1}}^{(n)}|\to 0$, we can apply Lemma 2 5) and 1) and conclude that

$$\sum_{i=1}^{n}\frac{K_i}{h}\Delta_i I_{\left\{|\bar t-t_{i-1}|>\varepsilon_h\right\}}=\sum_{i=1}^{n}\frac{K_i}{h}b_{t_{i-1}}^{(n)}\Delta_i\simeq\sum_{i=1}^{n}\frac{K_i}{h}\int_{t_{i-1}}^{t_i}b_r^{(n)}dr\simeq\int_0^T\frac{K_r}{h}b_r^{(n)}dr$$

$$=\int_0^T\frac{K_r}{h}I_{\left\{\frac{|\bar t-r|}{h}>\frac{\varepsilon_h}{h}\right\}}dr=\int_{\frac{\bar t-T}{h}}^{\frac{\bar t}{h}}K(u)I_{\left\{|u|>\frac{\varepsilon_h}{h}\right\}}du=\int_{\frac{\bar t-T}{h}}^{\frac{-\varepsilon_h}{h}}K(u)du+\int_{\frac{\varepsilon_h}{h}}^{\frac{\bar t}{h}}K(u)du\to 0.$$

So it remains to evaluate $\sup_{i\in I''}\sup_{r\in[t_{i-1},t_i)}|b_r^{(n)}-b_{t_{i-1}}^{(n)}|$, where, for $i\in I''$, we have $|\bar t-t_{i-1}|>\varepsilon_h$, thus

$$|b_r^{(n)}-b_{t_{i-1}}^{(n)}|=I_{|\bar t-r|\le\varepsilon_h,|\bar t-t_{i-1}|>\varepsilon_h}.$$

Now note that for any $r\in[t_{i-1},t_i)$ we have $\varepsilon_h<|\bar t-t_{i-1}|\le|\bar t-r|+|r-t_{i-1}|\le|\bar t-r|+\Delta$, thus for any $i\in I''$, for any $r\in[t_{i-1},t_i)$ we have

$$\frac{\varepsilon_h}{h}-\frac{\Delta}{h}<\frac{|\bar t-r|}{h}.$$

Since as $n\to\infty$ we have $\frac{\varepsilon_h}{h}\to\infty$ while $\frac{\Delta}{h}\to 0$, then $\frac{|\bar t-r|}{h}\to\infty$, and for sufficiently large n, uniformly on $i\in I''$, we have $\frac{\varepsilon_h}{h}<\frac{|\bar t-r|}{h}$, and thus $|\bar t-r|<\varepsilon_h$ cannot occur. That is, for sufficiently large n, for any $i\in I''$, for any $r\in[t_{i-1},t_i)$, $I_{|\bar t-r|\le\varepsilon_h,|\bar t-t_{i-1}|>\varepsilon_h}=0$, i.e. $\sup_{i\in I''}\sup_{r\in[t_{i-1},t_i)}I_{|\bar t-r|\le\varepsilon_h,|\bar t-t_{i-1}|>\varepsilon_h}\to 0$, and we are done.

As for iii), the proof is substantially the same as for i), we only point out some details. It is sufficient to prove that

$$\sum_{i=1}^{n}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\left(\int_{\mathbb{R}}\Psi(u)\int_{|v|\le\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|}g(v)dvdu-\int_{\mathbb{R}}\Psi(u)du\int_{\mathbb{R}}g(v)dv\right)$$
$$=\sum_{i=1}^{n}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\int_{\mathbb{R}}\Psi(u)\int_{|v|>\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|}g(v)dvdu\to 0,\tag{22}$$

because as in lemma 2, 3), we have $\sum_{i=1}^{n}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\to K_{(\alpha/2)}$. The sum in (22) is again split into the sum of the terms with $i\in I'$ and the sum of the ones with $i\in I''$: since for $i\in I'$ we have $\{|v|>\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|\}\subset\{|v|>\sqrt{\frac{2K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}}|u|\}$, the absolute value of the first sum is dominated by

$$\sum_{i\in I'}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\int_{\mathbb{R}}\Psi(u)\int_{|v|>\sqrt{\frac{2K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}}|u|}|g(v)|dvdu,$$

where for any u we have $\int_{|v|>\sqrt{\frac{2K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}}|u|}|g(v)|dv\to 0$ and $\Psi(u)\int_{|v|>\sqrt{\frac{2K\left(\frac{\varepsilon_h}{h}\right)|s|}{\varphi(h)}}|u|}|g(v)|dv\le C\Psi(u)\in L^1(\mathbb{R})$, where here $C=\int_{\mathbb{R}}|g(v)|dv$, thus by the dominated convergence theorem the sum over $i\in I'$ tends to 0.

On the other hand,

$$\sum_{i\in I''}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\int_{\mathbb{R}}\Psi(u)\int_{|v|>\sqrt{\frac{2K_i|s|}{\varphi(h)}}|u|}|g(v)|dvdu \le \sum_{i\in I''}\frac{K_i^{\frac{\alpha}{2}}}{h}\Delta_i\int_{\mathbb{R}}\Psi(u)du\int_{\mathbb{R}}|g(v)|dv,$$

where, as in (21), the first factor tends to 0. □

Lemma 4. *Assume that K satisfies* **IA1** *for φ_α in (9) and for $\varphi_\alpha^{(1)}(h) = h^{\frac{1}{\alpha}}$. Under* **IA2, IA3***, (3), $\Delta/h^2 \to 0$ and if K^α is Lipschitz and in $L^1(\mathbb{R})$ then, using the decomposition*

$$\Delta_i X = \Delta_i\tilde{J} + \Delta_i J^1, \quad \Delta_i\tilde{J} = \int_{t_{i-1}}^{t_i}\int_{|x|\le 1} xd\tilde{\mu}, \quad \Delta_i J^1 = \int_{t_{i-1}}^{t_i}\int_{|x|>1} xd\mu,$$

we have

$$\begin{cases} \text{if } \alpha\in(0,1) & \frac{\sum_{i=1}^n K_i\Delta_i\tilde{J}}{h} \xrightarrow{d} -a \text{ and } \frac{\sum_{i=1}^n K_i\int_{t_{i-1}}^{t_i}\int_{|x|\le 1} xd\mu}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha}, \\ \text{if } \alpha = 1 \text{ and } A_+\neq A_- & \frac{\sum_{i=1}^n K_i\Delta_i\tilde{J}}{h\log\frac{1}{h}} \xrightarrow{d} -(A_+ - A_-)K_{(1)}, \\ \text{if } \alpha\in(1,2) & \frac{\sum_{i=1}^n K_i\Delta_i\tilde{J}}{h^{\frac{1}{\alpha}}} \xrightarrow{d} Z_{1,\alpha}. \end{cases}$$

Proof. In each case, defined $Z_n \doteq \frac{\sum_{i=1}^n K_i\Delta_i\tilde{J}}{\varphi_\alpha(h)}$, we proceed by showing that the characteristic functions $E[e^{isZ_n}]$ converge to the characteristic function of the limit shown in the statement of the Lemma.

Since $\tilde{J}$ is a Lévy process,

$$E[e^{isZ_n}] = E\left[\prod_{j=1}^n e^{is\frac{K_j\Delta_i\tilde{J}}{\varphi_\alpha(h)}}\right] = \prod_{j=1}^n E\left[e^{is\frac{K_j\Delta_i\tilde{J}}{\varphi_\alpha(h)}}\right] = \prod_{j=1}^n e^{\Delta\int_{|x|\le 1} e^{is\frac{K_j}{\varphi_\alpha(h)}x}-1-is\frac{K_j}{\varphi_\alpha(h)}x\,\lambda(x)dx}$$

With $z \doteq s\frac{K_j}{\varphi_\alpha(h)}$, the integral at exponent is

$$A_+\int_{0<x\le 1}\left(e^{izx}-1-izx\right)x^{-1-\alpha}dx + A_-\int_{-1\le x<0}\left(e^{izx}-1-izx\right)|x|^{-1-\alpha}dx \tag{23}$$

$$= (A_+ + A_-)\int_0^1\frac{cos\Big(zx\Big)-1}{x^{1+\alpha}}dx + i(A_+ - A_-)\int_0^1\frac{sin\Big(zx\Big)-zx}{x^{1+\alpha}}dx.$$

By changing variable $v = |z|x$ that becomes

$$|z|^\alpha\left[(A_+ + A_-)\int_{0<v\le|z|}\frac{cos(v)-1}{v^{1+\alpha}}dv + i(A_+ - A_-)sgn(s)\int_{0<v\le|z|}\frac{sin(v)-v}{v^{1+\alpha}}dv\right],$$

so that

$$E[e^{isZ_n}] = e^{\sum_{j=1}^n\Delta\left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha\left[(A_+ + A_-)\int_{0<v\le\frac{|s|K_j}{\varphi_\alpha(h)}}\frac{cos(v)-1}{v^{1+\alpha}}dv + i(A_+ - A_-)sgn(s)\int_{0<v\le\frac{|s|K_j}{\varphi_\alpha(h)}}\frac{sin(v)-v}{v^{1+\alpha}}dv\right]}. \tag{24}$$

In each of the three cases $\alpha < 1, \alpha = 1, \alpha > 1$ the right speed is the $\varphi_\alpha(h)$ such that the exponent in the above expression converges to a finite quantity.

In the case $\alpha \in (0,1)$ we have $\varphi_\alpha(h) = h$, $\frac{cos(v)-1}{v^{1+\alpha}}, \frac{\sin(v)}{v^{1+\alpha}} \in L^1(R_+)$, while

$$\left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha sgn(s) \int_{0<v\leq\frac{|s|K_j}{h}} \frac{v}{v^{1+\alpha}} dv = \frac{sK_j}{h}\frac{1}{1-\alpha}.$$

It follows form (24) that $E[e^{isZ_n}]$ is give by

$$e^{\sum_{j=1}^n \Delta\left|\frac{sK_j}{h}\right|^\alpha (A_+ + A_-)\left[\int_{0<v\leq\frac{|s|K_j}{h}} \frac{cos(v)-1}{v^{1+\alpha}} dv + i\beta sgn(s)\int_{0<v\leq\frac{|s|K_j}{h}} \frac{sin(v)}{v^{1+\alpha}} dv\right] - i\sum_{j=1}^n \Delta\frac{sK_j}{h}\frac{A_+ - A_-}{1-\alpha}}$$

Recall that (from [19], Lemma 14.11)

$$\int_{\mathbb{R}_+} \frac{cos(v)-1}{v^{1+\alpha}} dv = \begin{cases} \Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right), & \alpha \in (0,1)\cup(1,2) \\ -\frac{\pi}{2}, & \alpha = 1, \end{cases} \tag{25}$$

$$\begin{cases} \int_0^{+\infty} \frac{\sin(v)}{v^{1+\alpha}} dv = -\Gamma(-\alpha)\sin\left(\frac{\pi\alpha}{2}\right), \text{ if } \alpha \in (0,1) \\ \\ \int_0^1 \frac{\sin(v)-v}{v^2} dv + \int_1^{+\infty} \frac{\sin(v)}{v^2} dv < +\infty \end{cases} \tag{26}$$

$$\int_0^{+\infty} \frac{e^{ir}-1-ir}{r^{1+\alpha}} dr = \Gamma(-\alpha)e^{-i\pi\frac{\alpha}{2}}, \quad \int_0^{+\infty} \frac{e^{-ir}-1+ir}{r^{1+\alpha}} dr = \Gamma(-\alpha)e^{i\pi\frac{\alpha}{2}}. \tag{27}$$

Thus, since the two integrals above are dominated by constants, $|s|^\alpha \sum_{j=1}^n \Delta\frac{K_j^\alpha}{h^\alpha} = |s|^\alpha \frac{\sum_{j=1}^n \Delta K_j^\alpha}{h} \cdot h^{1-\alpha} \to 0$, and since $a = \int_{|x|\leq 1} x\lambda(x)dx = \frac{A_+ - A_-}{1-\alpha}$, we have

$$E[e^{isZ_n}] \to e^{-is\frac{A_+ - A_-}{1-\alpha}} = e^{-isa},$$

where the limit is the characteristic function of the constant random variable $-a$.

If we do not compensate the small jumps and only consider $Y_n \doteq \frac{\sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|x|\leq 1} x d\mu}{h^{1/\alpha}}$, we only have

$$E[e^{isY_n}] = e^{\sum_{j=1}^n \Delta\left|\frac{sK_j}{h^{1/\alpha}}\right|^\alpha \left[(A_+ + A_-)\int_{0<v\leq\frac{|s|K_j}{h^{1/\alpha}}} \frac{cos(v)-1}{v^{1+\alpha}} dv + i(A_+ - A_-)sgn(s)\int_{0<v\leq\frac{|s|K_j}{h^{1/\alpha}}} \frac{sin(v)}{v^{1+\alpha}} dv\right]}, \tag{28}$$

and by Lemma 3 i) we have

$$\sum_{j=1}^n \frac{K_j^\alpha}{h}\Delta \int_{0<v\leq\frac{|s|K_j}{h^{\frac{1}{\alpha}}}} \frac{cos(v)-1}{v^{1+\alpha}} dv \to K_{(\alpha)}\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right),$$

$$\sum_{j=1}^n \frac{K_j^\alpha}{h}\Delta \int_{0<v\leq\frac{|s|K_j}{h^{\frac{1}{\alpha}}}} \frac{sin(v)}{v^{1+\alpha}} dv \to -K_{(\alpha)}\Gamma(-\alpha)\sin\left(\frac{\pi\alpha}{2}\right).$$

Thus

$$E[e^{isY_n}] \to e^{|s|^\alpha K_{(\alpha)}\Gamma(-\alpha)\left((A_+ + A_-)\cos\left(\frac{\pi\alpha}{2}\right) - i\, sgn(s)(A_+ - A_-)\sin\left(\frac{\pi\alpha}{2}\right)\right)} = E[e^{isZ_{1,\alpha}}],$$

having used notation (8).

If $\alpha = 1$, with $\varphi_\alpha(h) = h \log \frac{1}{h}$ and $z_j = \frac{sK_j}{h\log\frac{1}{h}}$, from (24) we have

$$E[e^{isZ_n}] = e^{\sum_{j=1}^n \Delta|z_j|\left[(A_+ + A_-)\int_0^{|z_j|} \frac{cos(v)-1}{v^2}dv + i(A_+ - A_-)sgn(z_j)\int_0^{|z_j|} \frac{sin(v)-v}{v^2}dv\right]} \tag{29}$$

The exponent above is

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\left[|s|(A_+ + A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}} \frac{cos(v)-1}{v^2}dv + is(A_+ - A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}} \frac{sin(v)-v}{v^2}dv\right],$$

which is shown to tend to $-is(A_+ - A_-)$: the first integrand $\frac{cos(v)-1}{v^2}I_{v>0}$ is in $L^1(\mathbb{R})$, thus, applying Lemma 3 i) we obtain that

$$|s|\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}(A_+ + A_-)\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}} \frac{cos(v)-1}{v^2}dv \to 0.$$

The second integral is written as

$$\int_0^{|z_j|} \frac{\sin(v)-v}{v^2}dv I_{|z_j|\le 1} + \left[\int_0^1 \frac{\sin(v)-v}{v^2}dv + \int_1^{|z_j|}\frac{\sin(v)}{v^2}dv - \log(|z_j|)\right] I_{|z_j|>1}, \tag{30}$$

where $\frac{sin(v)-v}{v^2} \in L^1((0,1))$, and $\frac{sin(v)}{v^2}I_{v\in(1,+\infty)} \in L^1(\mathbb{R})$. Note that if $s = 0$ we directly find that $E[e^{isZ_n}] = 1$, we thus only concentrate on a fixed $s \neq 0$. We have that

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\left(\int_0^{\frac{|s|K_j}{h\log\frac{1}{h}}}\left|\frac{sin(v)-v}{v^2}\right|dv I_{|z_j|\le 1} + \int_0^1 \left|\frac{sin(v)-v}{v^2}\right|dv\right) \le$$

$$\frac{\sum_{j=1}^n \Delta K_j}{h} 2\int_0^1 \left|\frac{sin(v)-v}{v^2}\right|dv\frac{1}{\log\frac{1}{h}} \le \frac{\sum_{j=1}^n \Delta K_j}{h}\frac{C}{\log\frac{1}{h}} \to 0,$$

and

$$\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\int_1^{\frac{|s|K_j}{h\log\frac{1}{h}}} \frac{sin(v)}{v^2}dv I_{|z_j|>1} \le \frac{C}{\log\frac{1}{h}}\frac{\sum_{j=1}^n \Delta K_j}{h} \to 0.$$

Finally, recalling that K is bounded (by **IA1**),

$$-is(A_+ - A_-)\frac{\sum_{j=1}^n \Delta K_j}{h\log\frac{1}{h}}\log\left(\frac{|s|K_j}{h\log\frac{1}{h}}\right) I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}} \to -is(A_+ - A_-)K_{(1)},$$

since within

$$\frac{\sum_{j=1}^n K_j\Delta}{h\log\frac{1}{h}}\left[\log(|s|) + \log(K_j) + \log(\frac{1}{h}) - \log\left(\log\frac{1}{h}\right)\right] I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}}$$

the first two terms are bounded in absolute value by

$$\frac{1}{\log\frac{1}{h}}\left[\frac{\sum_{j=1}^n |K_j\log(K_j)|\Delta}{h} + \frac{C\sum_{j=1}^n K_j\Delta}{h}\right] \to 0,$$

the third term converges by Lemma 3 i):

$$\sum_{j=1}^n \frac{K_j}{h}\Delta I_{\left\{\frac{|s|K_j}{h\log\frac{1}{h}}>1\right\}} \to K_{(1)};$$

and the fourth one

$$\sum_{j=1}^{n} \frac{K_j}{h} \Delta I_{\left\{\frac{|s|K_j}{h \log \frac{1}{h}} > 1\right\}} \frac{\log\left(\log \frac{1}{h}\right)}{\log \frac{1}{h}} \to 0.$$

Thus the statement is proved.

If $\alpha \in (1,2)$ we can directly use the relations in (27) In fact, from (23), where $z_j = s\frac{K_j}{\varphi_\alpha(h)} = s\frac{K_j}{h^{1/\alpha}}$, we change variable $v = |z_j|x$ in the first integral, while in the second one we firstly change in $y = -x$, then in $v = |z_j|y$, and we reach

$$|z_j|^\alpha \left[A_+ \int_0^{|z_j|} \frac{e^{iv \cdot sgn(z_j)} - 1 - iv \cdot sgn(z_j)}{v^{1+\alpha}} dv + A_- \int_0^{|z_j|} \frac{e^{-iv \cdot sgn(z_j)} - 1 + iv \cdot sgn(z_j)}{v^{1+\alpha}} dv \right] \tag{31}$$

With $g(v) = \frac{e^{iv}-1-iv}{v^{1+\alpha}} I_{v>0} \in L^1(\mathbb{R})$, and $\bar{g}$ its complex conjugate, the above equals

$$|z_j|^\alpha \left(A_+ \int_0^{|z_j|} g(v) I_{z_j>0} + \bar{g}(v) I_{z_j<0}\, dv + A_- \int_0^{|z_j|} \bar{g}(v) I_{z_j>0} + g(v) I_{z_j<0}\, dv \right)$$

thus

$$E[e^{isZ_n}] = e^{\sum_{j=1}^n \Delta \left|\frac{sK_j}{\varphi_\alpha(h)}\right|^\alpha \left[I_{z_j>0} \int_0^{|z_j|} A_+ g(v) + A_- \bar{g}(v)\ dv + I_{z_j<0} \int_0^{|z_j|} A_+ \bar{g}(v) + A_- g(v)\ dv \right]}.$$

With $\varphi_\alpha(h) = h^{\frac{1}{\alpha}}$, by Lemma 3 i), the exponent

$$\frac{|s|^\alpha \sum_{j=1}^n \Delta K_j^\alpha}{h} \left[I_{s>0} \int_0^{|z_j|} A_+ g(v) + A_- \bar{g}(v) dv + I_{s<0} \int_0^{|z_j|} A_+ \bar{g}(v) + A_- g(v) dv \right]$$

tends to

$$|s|^\alpha K_{(\alpha)} \Gamma(-\alpha) \left(I_{s>0} \left(A_+ e^{-i\pi\frac{\alpha}{2}} + A_- e^{i\pi\frac{\alpha}{2}} \right) + I_{s<0} \left(A_+ e^{i\pi\frac{\alpha}{2}} + A_- e^{-i\pi\frac{\alpha}{2}} \right) \right).$$

By developing and simplifying, the above expression becomes

$$-|s|^\alpha K_{(\alpha)} c \left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right) sign(s) \right),$$

where $c = -\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)(A_+ + A_-)$, $\beta = \frac{A_+ - A_-}{A_+ + A_-}$, and the statement is proved. $\square$

Lemma 5. *Assume that K satisfies* **IA1** *for ψ_α, then* **IA2, IA3**, *(3), $\frac{\Delta}{h^2} \to 0$ and that $K^{\alpha/2}$ is Lipschitz and in $L^1(\mathbb{R})$. In the case $\alpha = 1$ assume also $\sqrt{K}\log(K)$ bounded and $\frac{\Delta \log^2 \frac{1}{h}}{h^2} \to 0$. Then*

$$\begin{cases} \textit{if } \alpha \in (0,1) & \frac{\sum_{i=1}^n K_i (\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu)^2}{h^{\frac{2}{\alpha}}} \xrightarrow{d} Z_{2,\alpha}, \\ \textit{if } \alpha = 1 & \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^2} \xrightarrow{d} Z_{2,\alpha}, \\ \textit{if } \alpha \in (1,2) & \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}} \xrightarrow{d} Z_{2,\alpha}, \end{cases}.$$

Proof. Defined now $\tilde V_n \doteq \frac{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}{\psi_\alpha(h)}$, since $\tilde V_n \geq 0$, we proceed by showing that the Laplace transforms $E[e^{-s\tilde V_n}]$ converge to the Laplace transform of the limit shown in the statement of this Lemma (see [6], theorem 6.6.3 for the properties of the Laplace transforms limit). Since $\tilde J$ is a Lévy process, with $s \geq 0$,

$$E[e^{-s\tilde V_n}] = \prod_{j=1}^n E\left[e^{-s\frac{K_j(\tilde J_\Delta)^2}{\psi_\alpha(h)}}\right] = \prod_{j=1}^n \int_{\mathbb{R}} e^{-\lambda_j x^2}\tilde p(x)dx,$$

where we defined $\lambda_j = \lambda_j^{(\alpha)} \doteq \frac{sK_j}{\psi_\alpha(h)}$, and $\tilde p(x) = \tilde p_\alpha(x)$ is the, not explicitly known, density of the law of $\tilde J_\Delta$. In order to deal with $\int_{\mathbb{R}} e^{-\lambda_j x^2}\tilde p(x)dx$ we interpret $e^{-\lambda_j x^2}$ as the characteristic function $\mathcal{C}\phi \doteq E[e^{ixW}]$ of a Gaussian random variable W, with mean 0, variance $\sigma_j^2 \doteq 2\lambda_j$ and density ϕ, and we use that

$$\int (\mathcal{C}\phi)(x)\tilde p(x)dx = \int \phi(x)(\mathcal{C}\tilde p)(x)dx,$$

$(\mathcal{C}\tilde p)(x)$ being $E[e^{ix\tilde J_\Delta}]$. The latter equality holds true since

$$\int E[e^{ixW}]\tilde p(x)dx = \int\int e^{ixz}\phi(z)dz\ \tilde p(x)dx = \int\phi(z)\int e^{ixz}\tilde p(x)dx\ dz = \int\phi(z)E[e^{iz\tilde J_\Delta}]dz.$$

So we write

$$\int_{\mathbb{R}} e^{-\lambda_j x^2}\tilde p(x)dx = \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2\sigma_j^2}}}{\sigma\sqrt{2\pi}} \cdot e^{\Delta\int_{|r|\leq 1} e^{ixr}-1-ixr\ \lambda(dr)}dx.$$

With $u \doteq \frac{x}{\sigma_j}$ we reach

$$\int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta\int_{|r|\leq 1} e^{i\sigma_j ur}-1-i\sigma_j ur\ \lambda(dr)}du. \tag{32}$$

Case $\alpha \in (0,1)$**.** No compensation of the small jumps is required, we thus consider the special case with null compensator, $V_n \doteq \frac{\sum_{i=1}^n K_i(\Delta_i J)^2}{\psi_\alpha(h)}$, and we only deal with $\int_{\mathbb{R}} e^{-\lambda_j x^2}p(x)dx$, where $p(x) = p_\alpha(x)$ is the density of the law of $\int_0^\Delta\int_{|x|\leq 1} xd\mu$, and

$$\int_{\mathbb{R}} e^{-\lambda_j x^2}p(x)dx = \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta\int_{|r|\leq 1} e^{i\sigma_j ur}-1\ \lambda(dr)}du. \tag{33}$$

Similarly as when from (23) we obtained (24) and then (28), with $z = \frac{sK_j}{h^{1/\alpha}}$ there replaced by $\sigma_j u = \sqrt{2\lambda_j}\cdot u$ here, we have

$$\begin{aligned}\int_{|r|\leq 1} e^{i\sigma_j ur} - 1\ \lambda(dr) &= \sigma_j^\alpha|u|^\alpha(A_+ + A_-)\int_0^{\sigma_j|u|}\left[\frac{cos(v)-1}{v^{1+\alpha}} + i\beta sgn(u)\frac{\sin(v)}{v^{1+\alpha}}\right]dv \\ &\doteq \sigma_j^\alpha|u|^\alpha\int_0^{\sigma_j|u|} f(v)dv \doteq \sigma_j^\alpha|u|^\alpha g_j(u),\end{aligned} \tag{34}$$

then we are left with

$$E[e^{-sV_n}] = \prod_{j=1}^n \int_{\mathbb{R}} e^{-\lambda_j x^2}p(x)dx = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta\sigma_j^\alpha|u|^\alpha g_j(u)}du.$$

By developing $e^y = \sum_{k=0}^{+\infty}\frac{y^k}{k!}$, we obtain $\prod_{j=1}^n\left(1+\theta_j^{(n)}\right) \doteq$

$$\prod_{j=1}^n\left(\int_{\mathbb{R}}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}du + \int_{\mathbb{R}}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot\Delta\sigma_j^\alpha|u|^\alpha g_j(u)du + \sum_{k\geq 2}\int_{\mathbb{R}}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot\frac{\Delta^k\left(\sigma_j^\alpha|u|^\alpha g_j(u)\right)^k}{k!}du\right) \tag{35}$$

We are now going to show that

$$\begin{array}{ll} (1) & \forall j = 1,..,n, \theta_j^{(n)} \to 0 \text{ and } \max_{j=1,..,n} |\theta_j^{(n)}| \to 0 \\ (2) & \sum_{j=1}^n |\theta_j^{(n)}| \leq M < \infty \\ (3) & \sum_{j=1}^n \theta_j^{(n)} \to \theta, \end{array}$$

where M does not depend on n, and $\theta \doteq s^{\frac{\alpha}{2}} 2^\alpha K_{(\alpha/2)}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\pi}}\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) < 0$. That allows to conclude ([6], Lemma p.199) that

$$E[e^{-sV_n}] = \prod_{j=1}^n \left(1 + \theta_j^{(n)}\right) \to e^\theta,$$

which is the Laplace transform of the law of the $Z_{2,\alpha}$ in the notations, and the stated result follows.

Let us now evaluate the numbers $\theta_j^{(n)}$. Denoted

$$\theta_{j,1}^{(n)} \doteq \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha |u|^\alpha g_j(u) du = 2\int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha u^\alpha (A_+ + A_-) \int_0^{\sigma_j u} \frac{\cos(v) - 1}{v^{1+\alpha}} dv du, \tag{36}$$

we preliminarily show that

$$\begin{array}{ll} (4) & \sum_{j=1}^n \theta_{j,1}^{(n)} \to \theta \\ (5) & \sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| \to 0. \end{array}$$

Note that the function $\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}|u|^{\alpha k}$ is in $L^1(\mathbb{R})$ for any integer k, with

$$\int_{\mathbb{R}_+} e^{-\frac{u^2}{2}} |u|^{\alpha k} du = 2^{\frac{\alpha k - 1}{2}} \Gamma\left(\frac{\alpha k + 1}{2}\right). \tag{37}$$

As for (4), using the notation in (34), Lemma 3 iii), (37) and (25) and with $\sigma_j = \sqrt{2\lambda_j} = \sqrt{2\frac{sK_j}{\psi_\alpha(h)}}$ we have

$$\begin{aligned} \sum_{j=1}^n \theta_{j,1}^{(n)} &= \sum_{j=1}^n \Delta\sigma_j^\alpha \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha g_j(u) du = s^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}}\Delta}{h} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha \int_0^{\sigma_j |u|} f(v) dv du \\ &= s^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}}\Delta}{h} 2(A_+ + A_-) \int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha \int_0^{\sigma_j |u|} \frac{\cos(v) - 1}{v^{1+\alpha}} dv du \to \theta. \end{aligned} \tag{38}$$

As for (5), since for all $j = 1,..,n$, $|g_j(u)| \leq C\int_{\mathbb{R}_+} \frac{|cos(v)-1|}{v^{1+\alpha}} + \frac{|\sin(v)|}{v^{1+\alpha}} dv < \infty$, $g_j(u)$ is bounded uniformly in j and u, thus we have that $\sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}|$ is dominated by

$$\sum_{j=1}^n \sum_{k\geq 2} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \frac{\Delta^k \left(C\sigma_j^\alpha |u|^\alpha\right)^k}{k!} du = \sum_{j=1}^n \sum_{k\geq 2} C^k \left(\frac{\Delta}{h}\right)^k \frac{2^{\frac{\alpha k}{2}} K_j^{\frac{\alpha k}{2}}}{k!} 2^{\frac{\alpha k - 1}{2}} \Gamma\left(\frac{\alpha k + 1}{2}\right) : \tag{39}$$

since the kernel K is bounded, the above is dominated by

$$\left(\frac{\Delta}{h}\right)^2 n \sum_{k\geq 2} \left(\frac{\Delta}{h}\right)^{k-2} C^k \frac{2^{\alpha k - \frac{1}{2}}}{k!} \Gamma\left(\frac{\alpha k + 1}{2}\right) \tag{40}$$

and since for large n we have $\Delta/h < 1$, the series is absolutely convergent (quotient criterion), and (40) is $O\left(\frac{\Delta}{h^2}\right)$, thus it tends to 0, and (5) is verified.

It follows that, since $\theta_{j,1}^{(n)} = \Delta\sigma_j^\alpha \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} |u|^\alpha g_j(u) du$, where $\sigma_j^\alpha \leq C \frac{\sqrt{K(0)^\alpha}}{h}$ and $g_j(u)$ is uniformly bounded, thus $|\theta_{j,1}^{(n)}| \leq C\Delta/h$ uniformly in j, and

$$\max_{j=1,..,n} |\theta_j^{(n)}| \leq \max_{j=1,..,n} |\theta_j^{(n)} - \theta_{j,1}^{(n)}| + \max_{j=1,..,n} |\theta_{j,1}^{(n)}| \leq \sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| + C\frac{\Delta}{h} = O\left(\frac{\Delta}{h^2}\right) \to 0,$$

which solves (1).

As for (2), using again Lemma 3 iii), we have

$$\sum_{j=1}^n |\theta_{j,1}^{(n)}| \leq \sum_{j=1}^n \sigma_j^\alpha \Delta \int_{\mathbb{R}} \Psi(u)|g_j(u)|du \leq C \sum_{j=1}^n \frac{K_j^{\frac{\alpha}{2}}}{h} \Delta \int_{\mathbb{R}} \Psi(u) \int_0^{|u|\sqrt{\frac{2|s|K_j}{h^{2/\alpha}}}} |f(v)|dv\, du \to C,$$

thus using also that (39) is $O(\Delta/h^2)$ we reach

$$\sum_{j=1}^n |\theta_j^{(n)}| \leq \sum_{j=1}^n |\theta_j^{(n)} - \theta_{j,1}^{(n)}| + \sum_{j=1}^n |\theta_{j,1}^{(n)}| \leq C\frac{\Delta}{h^2} + C \leq M.$$

Finally (3) follows directly from (4) and (5).

Case $\alpha \in [1, 2)$**.** From (32), the integral in $\lambda(dr)$ is given by

$$\int_0^1 (A_+ + A_-)\frac{cos(\sigma_j ur) - 1}{r^{1+\alpha}} + i(A_+ - A_-)\frac{\sin(\sigma_j ur) - \sigma_j ur}{r^{1+\alpha}}\ dr$$

$$= \sigma_j^\alpha |u|^\alpha \int_0^{\sigma_j|u|} (A_+ + A_-)\frac{cos(v) - 1}{v^{1+\alpha}} + i(A_+ - A_-)sgn(u)\frac{\sin(v) - v}{v^{1+\alpha}}\ dv \doteq \sigma_j^\alpha |u|^\alpha \tilde{g}_j(u).$$

Thus

$$E[e^{-s\tilde{V}_n}] = \prod_{j=1}^n \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot e^{\Delta\sigma_j^\alpha|u|^\alpha\tilde{g}_j(u)}du$$

$$= \prod_{j=1}^n \left(1 + \sum_{k\geq1} \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \frac{\left(\Delta\sigma_j^\alpha|u|^\alpha\tilde{g}_j(u)\right)^k}{k!} du\right) \doteq \prod_{j=1}^n \left(1 + \tilde{\theta}_j^{(n)}\right). \tag{41}$$

Again, we show that $\tilde{\theta}_{j,1}^{(n)} \doteq \int_{\mathbb{R}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha|u|^\alpha\tilde{g}_j(u)du$ turns out to be the leading term of $\tilde{\theta}_j^{(n)}$, and that the conditions (1) to (5) above are satisfied also for $\tilde{\theta}_j^{(n)}$, which allows to conclude the proof.

Note that for any $\alpha \in [1, 2)$

$$\tilde{\theta}_{j,1}^{(n)} = 2\int_{\mathbb{R}_+} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \cdot \Delta\sigma_j^\alpha u^\alpha \int_0^{\sigma_j u} (A_+ + A_-)\frac{cos(v) - 1}{v^{1+\alpha}} dvdu,$$

has the same expression of $\theta_{j,1}^{(n)}$ at (36), thus $\sum_{j=1}^n \tilde{\theta}_{j,1}^{(n)}$ coincides exactly with the right hand side of (38). By Lemma 3 iii), using (37) and the relations in (25) we obtain that for $\alpha = 1$ then $\sum_{j=1}^n \tilde{\theta}_{j,1}^{(n)} \to \tilde{\theta} \doteq -s^{\frac{\alpha}{2}} 2^{\alpha-1}\sqrt{\pi}K_{(\alpha/2)}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)$, while for $\alpha \in (1, 2)$ then $\sum_{j=1}^n \tilde{\theta}_{j,1}^{(n)} \to \theta$, and a condition of type (4) is satisfied in any case.

As for (5), we need to bound differently $|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}|$ in the two cases $\alpha = 1$, $\alpha \in (1, 2)$.

If $\alpha = 1$, splitting as in (30), we write

$$\tilde{g}_j(u) = (A_+ + A_-)\int_0^{\sigma_j|u|}\frac{cos(v)-1}{v^2}dv + i(A_+ - A_-)sgn(u)\int_0^{\sigma_j|u|}\frac{\sin(v)-v}{v^2}dv I_{\sigma_j|u|\leq 1}$$

$$+i(A_+ - A_-)sgn(u)\left[\int_0^1 \frac{\sin(v)-v}{v^2}dv + \int_1^{\sigma_j|u|}\frac{\sin(v)}{v^2}dv - \log(\sigma_j|u|)\right]I_{\sigma_j|u|>1},$$

where $\log(\sigma_j|u|) = \frac{1}{2}\log(2s) + \frac{1}{2}\log(K_j) + \log\left(\frac{1}{h}\right) + \log(|u|)$, thus

$$\tilde{g}_j(u) \doteq \ell_j(u) - i(A_+ - A_-)sgn(u)\left[\tfrac{1}{2}\log(K_j) + \log\left(\tfrac{1}{h}\right) + \log(|u|)\right]I_{\sigma_j|u|>1},$$

where $\ell_j(u)$ is uniformly bounded in j and u. Using that $|u\log(|u|)| \leq |u|^2 I_{|u|>1} + \frac{1}{e}I_{0<|u|<1}$, then for any triplet of positive quantities A_1, A_2, A_3 with $A = A_1 + A_2 + A_3$, we have

$$|u|^k\left[A + |\log|u||\right]^k \leq |u|^k 2^k\left[A^k + |\log|u||^k\right] = 2^k\Big(|u|^k A^k + (|u\log|u||)^k\Big) \leq$$

$$2^k\Big(|u|^k A^k + (u^2+C)^k\Big) \leq 8^k\Big(|u|^k(A_1^k + A_2^k + A_3^k) + u^{2k} + C^k\Big).$$

Thus

$$\left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right| \leq \sum_{k\geq 2}\frac{\Delta^k}{h^k}C^k\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}|u|^k\left[C + |\log(K_j)| + \log\left(\frac{1}{h}\right) + |\log|u||\right]^k du$$

$$\leq \sum_{k\geq 2}\frac{\Delta^k}{h^k}C^k\frac{K_j^{\frac{k}{2}}}{k!}\cdot 2\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\left[u^k C^k + u^k|\log(K_j)|^k + u^k\log^k\left(\frac{1}{h}\right) + u^{2k} + C^k\right]du:$$

similarly as above,

$$\sum_{k\geq 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = \frac{\Delta^2}{h^2}\sum_{k\geq 2}C^k\frac{\Delta^{k-2}}{h^{k-2}}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\frac{\Delta^2}{h^2}\right),$$

$$\sum_{k\geq 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^{2k}du = O\left(\frac{\Delta^2}{h^2}\right),\quad \sum_{k\geq 2}C^k\frac{\Delta^k}{h^k}\frac{K_j^{\frac{k}{2}}}{k!} = O\left(\frac{\Delta^2}{h^2}\right);$$

since $\sqrt{K}|\log(K)|$ is bounded, also

$$\sum_{k\geq 2}\frac{\Delta^k}{h^k}C^k\frac{\left(K_j^{\frac{1}{2}}|\log(K_j)|\right)^k}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\frac{\Delta^2}{h^2}\right).$$

Finally,

$$\sum_{k\geq 2}\left(\frac{\Delta\log\left(\frac{1}{h}\right)}{h}\right)^k C^k\frac{K_j^{\frac{k}{2}}}{k!}\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}u^k du = O\left(\left(\frac{\Delta\log\left(\frac{1}{h}\right)}{h}\right)^2\right),$$

thus $\sum_{j=1}^n\left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right| = O\left(\frac{\Delta\log^2\left(\frac{1}{h}\right)}{h^2}\right) \to 0$, and (5) for $\tilde{\theta}_j^{(n)}$ is proved.

Thus (1), (2) and (3) for $\tilde{\theta}_j^{(n)}$ follow analogously as for $\theta_j^{(n)}$.

If $\alpha \in (1,2)$**,** due to (27), $\tilde{g}_j(u)$ is uniformly bounded in j and u, thus $\sum_{j=1}^n\left|\tilde{\theta}_j^{(n)} - \tilde{\theta}_{j,1}^{(n)}\right|$ is dealt exactly as in (39), thus it is $O\left(\frac{\Delta}{h^2}\right) \to 0$, and (5) is done. From (4) and (5) then the properties (1) to (3) again follow as above, and now the proof of the Lemma is complete. □

Lemma 6. *Under* **A1**, **IA2**, **IA3** *and (3): if* $\alpha \in (1,2)$ *and* $\frac{\Delta}{h^2} \to 0$ *then*

$$\left(\frac{\left(\sum_{i=1}^n K_i \Delta_i X\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i X)^2}{h^{\frac{2}{\alpha}}} \right) \xrightarrow{d} (Z_{1,\alpha}^2, Z_{2,\alpha}).$$

Remark. Note that under **A1** K is bounded and then also K^2 is Lipschitz and in $L^1(\mathbb{R})$.

Proof of Lemma 6. We proceed through the following steps. Recalling the decomposition $\Delta_i X = \Delta_i \tilde{J} + \Delta_i J^1$:

1) due to the negligibility of the contribution of J^1 we show that a.s.

$$\frac{\left(\sum_{i=1}^n K_i \Delta_i X\right)^2}{h^{\frac{2}{\alpha}}} \simeq \frac{\left(\sum_{i=1}^n K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}}, \quad \frac{\sum_{i=1}^n K_i \left(\Delta_i X\right)^2}{h^{\frac{2}{\alpha}}} \simeq \frac{\sum_{i=1}^n K_i \left(\Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}}. \tag{42}$$

After that, it is sufficient to prove the convergence in distribution of

$$\left(\frac{\left(\sum_{i=1}^n K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}} \right).$$

2) We develop

$$\frac{\left(\sum_{i=1}^n K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}} = \frac{\sum_{i=1}^n \left(K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}} + \frac{\sum_{i,j=1..n: i \neq j} K_i K_j \Delta_i \tilde{J} \Delta_j \tilde{J}}{h^{\frac{2}{\alpha}}}$$

and we show that $\frac{\sum_{i \neq j} K_i K_j \Delta_i \tilde{J} \Delta_j \tilde{J}}{h^{\frac{2}{\alpha}}} \xrightarrow{L^1} 0$, so the stated limit in distribution is the same as for

$$\left(\frac{\sum_{i=1}^n \left(K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}}, \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}} \right)$$

3) For $s_1, s_2 > 0$ we show that

$$\mathcal{L}_n(s_1, s_2) \doteq E\left[e^{-s_1 \frac{\sum_{i=1}^n \left(K_i \Delta_i \tilde{J}\right)^2}{h^{\frac{2}{\alpha}}} - s_2 \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{\frac{2}{\alpha}}}} \right] \to E\left[e^{-s_1 Z_{1,\alpha}^2 - s_2 Z_{2,\alpha}} \right] \doteq \mathcal{L}(s_1, s_2), \tag{43}$$

which concludes the proof.

Let us start by 1). For the first result it is sufficient to show that a.s.

$$\frac{\sum_{i=1}^n K_i \Delta_i X}{h^{\frac{1}{\alpha}}} \simeq \frac{\sum_{i=1}^n K_i \Delta_i \tilde{J}}{h^{\frac{1}{\alpha}}}.$$

The difference of the two above terms is $\frac{\sum_{i=1}^n K_i \Delta_i J^1}{h^{\frac{1}{\alpha}}}$: recalling that the probability that $\Delta J^1_{\bar{t}} \neq 0$ is zero, for the convergence in distribution we can focus on those ω where there is no jump at $\bar{t}$. For any fixed ω such that $\Delta J^1_{\bar{t}} = 0$, using the notation at the proof of Lemma 1, part b), $\bar{t} - S_{\underline{p}}$ is a fixed quantity, and

$$\frac{\sum_{i=1}^n K_i \Delta_i J^1}{h^{\frac{1}{\alpha}}} \simeq \frac{K\left(\frac{\bar{t} - S_{\underline{p}}}{h}\right)}{h^{\frac{1}{\alpha}}} :$$

by assumption $K\left(\frac{\bar{t}-S_p}{h}\right)=o(h)$, and since $\alpha>1$ then $h=o(h^{\frac{1}{\alpha}})$, thus the above display tends a.s. to 0.

As for the second result in (42),

$$\frac{\sum_{i=1}^n K_i\left[\left(\Delta_i X\right)^2-\left(\Delta_i \tilde{J}\right)^2\right]}{h^{\frac{2}{\alpha}}}=\frac{\sum_{i=1}^n K_i\left[\left(\Delta_i J^1\right)^2+2\Delta_i\tilde{J}\Delta_i J^1\right]}{h^{\frac{2}{\alpha}}}:$$

the first term $\sum_{i=1}^n K_i\left(\Delta_i J^1\right)^2/h^{\frac{2}{\alpha}}$ has the same limit as

$$\frac{K\left(\frac{\bar{t}-S_p}{h}\right)}{h^{\frac{2}{\alpha}}}=\frac{o(\Delta h)}{h^{\frac{2}{\alpha}}}=\frac{o(\Delta)}{h^{\frac{2}{\alpha}-1}}=o\left(\frac{\Delta}{h}\right)\to 0.$$

While the square of the second term $2\sum_{i=1}^n K_i\Delta_i\tilde{J}\Delta_i J^1/h^{\frac{2}{\alpha}}$ by the Schwartz inequality is dominated by

$$\frac{C\sum_{i=1}^n K_i\left(\Delta_i\tilde{J}\right)^2}{h^{\frac{2}{\alpha}}}\frac{\sum_{i=1}^n K_i\left(\Delta_i J^1\right)^2}{h^{\frac{2}{\alpha}}},$$

where the first factor converges in distribution by Lemma 5 and the second one tends to 0. Thus also the second term tends to 0.

As for 2), let us evaluate $E\left[\left|\frac{\sum_{i,j:i\neq j}K_iK_j\Delta_i\tilde{J}\Delta_j\tilde{J}}{h^{\frac{2}{\alpha}}}\right|\right]$: since $\Delta_i\tilde{J}$ and $\Delta_j\tilde{J}$ are independent, such a quantity is dominated by

$$\frac{\sum_{i,j:i\neq j}K_iK_jE\left[\left|\Delta_i\tilde{J}\right|\right]E\left[\left|\Delta_j\tilde{J}\right|\right]}{h^{\frac{2}{\alpha}}}\leq C\frac{\sum_{i,j:i\neq j}K_iK_j\Delta^2}{h^{\frac{2}{\alpha}}},$$

having used the estimate (2.1.36) with $p=1$ in [14] for the last inequality. Now, by Lemma 2, part 6),

$$\sum_{i=1}^n\sum_{j<i}K_i^2K_j^2\Delta^2\simeq\int_0^T K_u^2\int_0^u K_s^2dsdu=h\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}}K^2(v)\int_0^{\bar{t}-vh}K_s^2dsdv$$

$$=h^2\int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}}K^2(v)\int_v^{\frac{\bar{t}}{h}}K^2(w)dwdv\leq h^2K_{(2)}^2,$$

thus

$$C\frac{\sum_{i,j:i\neq j}K_iK_j\Delta^2}{h^{\frac{2}{\alpha}}}=C\frac{\sum_{i,j:i\neq j}K_iK_j\Delta^2}{h^2}\frac{1}{h^{\frac{2}{\alpha}-2}}\to 0$$

and 2) is done.

As for 3), we have

$$\mathcal{L}_n(s_1,s_2)=E\left[e^{-\sum_{i=1}^n\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}(\Delta_i\tilde{J})^2}\right]=\prod_{i=1}^n E\left[e^{-u_i(\Delta_i\tilde{J})^2}\right],$$

having set $u_i\doteq\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}>0$. The i-th term in above display is then the same as in (32), where now $u_i=\frac{s_1K_i^2+s_2K_i}{h^{2/\alpha}}$ is in place of $\lambda_j=\frac{sK_j}{h^{2/\alpha}}$, and thus $(\sigma_i)^\alpha=(2u_i)^{\frac{\alpha}{2}}=\frac{2^{\frac{\alpha}{2}}(s_1K_i^2+s_2K_i)^{\frac{\alpha}{2}}}{h}$ is now in place of $(\sigma_j)^\alpha=(2\lambda_j)^{\frac{\alpha}{2}}=\frac{(2sK_j)^{\frac{\alpha}{2}}}{h}$. Thus, similarly to (41), the above display is $\prod_{j=1}^n\left(1+\tilde{\theta}_i^{(n)}\right)$, where now

$$\sum_{i=1}^n\tilde{\theta}_{i,1}^{(n)}=\sum_{i=1}^n 2\Delta\sigma_i^\alpha\int_{\mathbb{R}_+}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\cdot u^\alpha\int_0^{\sigma_i u}(A_++A_-)\frac{cos(v)-1}{v^{1+\alpha}}dvdu.$$

Since

$$\sum_{i=1}^{n} \Delta\sigma_i^{\alpha} \simeq 2^{\frac{\alpha}{2}} \int_0^T [s_1 K_r^2 + s_2 K_r]^{\frac{\alpha}{2}} \frac{dr}{h} = 2^{\frac{\alpha}{2}} \int_{\frac{\bar{t}-T}{h}}^{\frac{\bar{t}}{h}} [s_1 K^2(u) + s_2 K(u)]^{\frac{\alpha}{2}} du$$

tends to $2^{\frac{\alpha}{2}} \int_{\mathbb{R}} [s_1 K^2(u) + s_2 K(u)]^{\frac{\alpha}{2}} du$ then, similarly as for Lemma 3, part iii), we have

$$\sum_{i=1}^{n} \tilde{\theta}_{i,1}^{(n)} \to 2 \cdot 2^{\frac{\alpha}{2}} \int_{\mathbb{R}} [s_1 K^2(u) + s_2 K(u)]^{\frac{\alpha}{2}} du \cdot \frac{2^{\frac{\alpha-1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{\alpha+1}{2}\right) \cdot (A_+ + A_-)\Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)$$

and, similarly as in Lemma 5,

$$\prod_{j=1}^{n} \left(1 + \tilde{\theta}_i^{(n)}\right) \simeq \prod_{j=1}^{n} \left(1 + \tilde{\theta}_{i,1}^{(n)}\right) \to e^{\underline{\theta}} \doteq \mathcal{L}_\infty(s_1, s_2),$$

where

$$\underline{\theta} \doteq \frac{2^{\alpha}}{\sqrt{\pi}} (A_+ + A_-) \Gamma\left(\frac{\alpha+1}{2}\right) \Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \int_{\mathbb{R}} [s_1 K^2(u) + s_2 K(u)]^{\frac{\alpha}{2}} du.$$

The function $\mathcal{L}_\infty$ is the Laplace transform of a probability law (bacause $\mathcal{L}_\infty(0,0) = 1$ and the function is continuous at (0,0)), and we see that it is the one of a proper joint law having marginals $Z_{1,\alpha}^2$ and $Z_{2,\alpha}$. In fact, with $s_2 = 0$ we have

$$e^{\frac{2^{\alpha}}{\sqrt{\pi}}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int_{\mathbb{R}}[s_1 K^2(u)]^{\frac{\alpha}{2}} du} = \mathcal{L}_\infty(s_1, 0) = \lim_n \mathcal{L}_n(s_1, 0)$$

$$= \lim_n E\left[e^{-s_1 \frac{\sum_{i=1}^n (K_i \Delta_i \tilde{J})^2}{h^{2/\alpha}}}\right]:$$

$\frac{\sum_{i=1}^n (K_i \Delta_i \tilde{J})^2}{h^{2/\alpha}} \overset{d}{\simeq} \left(\frac{\sum_{i=1}^n K_i \Delta_i \tilde{J}}{h^{1/\alpha}}\right)^2$, as we saw above at 2), and, by Lemma 4, the latter term converges in distribution to $Z_{1,\alpha}^2$.

On the other hand, with $s_1 = 0$ we have

$$e^{\frac{2^{\alpha}}{\sqrt{\pi}}(A_+ + A_-)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int_{\mathbb{R}}[s_2 K(u)]^{\frac{\alpha}{2}} du} = \mathcal{L}_\infty(0, s_2) = \lim_n \mathcal{L}_n(0, s_2)$$

$$= \lim_n E\left[e^{-s_2 \frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{2/\alpha}}}\right]$$

and, by Lemma 5, $\frac{\sum_{i=1}^n K_i (\Delta_i \tilde{J})^2}{h^{2/\alpha}} \overset{d}{\to} Z_{2,\alpha}$. Thus $\mathcal{L}_\infty$ describes a specific joint law of $\left(Z_{1,\alpha}^2, Z_{2,\alpha}\right)$. □

Remark. The joint law of $\left(Z_{1,\alpha}^2, Z_{2,\alpha}\right)$ has a Laplace transform of type $e^{-C \int_{\mathbb{R}} [s_1 K^2(u) + s_2 K(u)]^{\frac{\alpha}{2}} du}$ with positive C: no linear part in s_1, s_2 is present, thus there are no drift terms. The law could resemble a bidimensional $\alpha/2$-stable, however this is not the case, because it is concentrated on a parabola (if $x_2 = K(u)$ then $x_1 = x_2^2$) rather than on the unit sphere.

Proof of theorem 2.

a) Since X is a càdlàg process, for fixed $\varepsilon \in (0,1)$ we have a.s. $\nu(\omega, (\varepsilon, 1] \times [0,T]) < \infty$, i.e. the jumps occurring on $[0,T]$ with size larger than ε in absolute value are only finitely many. Define now N_T^{ε} the a.s. finite number of jumps of X with size absolute value $|\Delta X_p| > \varepsilon$, and S_p^{ε} the times of such jumps, $p = 1, .., N_T^{\varepsilon}$.

For any n, for any $p = 1,..,N^\varepsilon$ we call $I_p = I_p^\varepsilon$ the unique interval $(t_{i-1}, t_i] = (t_{i-1}^\varepsilon, t_i^\varepsilon]$ containing S_p^ε, and we rename its extremes $t_{i_p-1} = t_{i_p^\varepsilon-1}, t_{i_p} = t_{i_p^\varepsilon}$. For any $\varepsilon > 0$ we split

$$X_t = \tilde{J}_t^\varepsilon - C_t^\varepsilon + J_t^{1,\varepsilon},$$

where

$$\tilde{J}_t^\varepsilon \doteq \int_0^t \int_{|\delta(x,s)|\le\varepsilon} \delta(x,s) d\tilde{\mu},\ C_t^\varepsilon \doteq \int_0^t \int_{|\delta(x,s)|\in(\varepsilon,1]} \delta(x,s)\lambda(x)dxds,\ J_t^{1,\varepsilon} \doteq \int_0^t \int_{|\delta(x,s)|>\varepsilon} \delta(x,s)d\mu,$$

and we proceed through the following steps.

1) For any fixed $\varepsilon \in (0,1)$, $J^{1,\varepsilon}$ is a FA jump process with piece-wise constant paths, so that, by Lemma 1 we have that, as $n \to \infty$, $F^n(J^{1,\varepsilon}) \stackrel{a.s.}{\to} F(J^{1,\varepsilon})$ with both $f(x) = x$ and $f(x) = x^2$, where $F(J^{1,\varepsilon})$ is finite a.s..

2) Note that as $\varepsilon \to 0$ then, for both $f(x) = x$ and $f(x) = x^2$,

$$F(J^{1,\varepsilon}) = K(0)f(\Delta J_{\bar{t}}^{1,\varepsilon}) \stackrel{a.s.}{\to} F(X) = K(0)f(\Delta X_{\bar{t}}).$$

3) Now we check that

$$\forall\eta > 0,\ \lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\Big(\Big\{|F^n(X) - F^n(J^{1,\varepsilon})| > \eta\Big\}\Big) = 0. \tag{44}$$

The three properties allow to conclude (10) by Proposition 2.2.1 in [14].

We define

$$a_s(\varepsilon) \doteq \int_{|\delta(s,x)|\in(\varepsilon,1]} \delta(s,x)\lambda(x)dx, \quad \sigma_s^2(\varepsilon) \doteq \int_{|\delta(s,x)|\le\varepsilon} \delta^2(s,x)\lambda(x)dx.$$

Note that $a(0)$ is the process a that we defined in Section 3, and that it has finite values only if X has finite variation jumps ($\alpha < 1$). For proving part a), without loss of generality, through a localization procedure, we can assume that for any fixed $\varepsilon > 0$ the processes $a_s(\varepsilon)$ and $\sigma_s^2(\varepsilon)$ are bounded in absolute value by constants A^ε and Σ^ε respectively, depending on ε.

Case $f(x) = x$**:**

$$P\Big(\Big\{|F^n(X) - F^n(J^{1,\varepsilon})| > \eta\Big\}\Big) \le P\Big(\Big\{|\sum_{i=1}^n K_i\Delta_i\tilde{J}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) + P\Big(\Big\{|\sum_{i=1}^n K_i\Delta_i C^\varepsilon| > \frac{\eta}{2}\Big\}\Big):$$

the first probability is bounded by

$$\frac{||\sum_{i=1}^n K_i\Delta_i\tilde{J}^\varepsilon||_{L^2}}{\eta/2} = \frac{\sqrt{\sum_{i=1}^n K_i^2 E[(\Delta_i\tilde{J}^\varepsilon)^2]}}{\eta/2}$$

$$= \frac{\sqrt{\sum_{i=1}^n K_i^2 E[\int_{t_{i-1}}^{t_i}\int_{|\delta|\le\varepsilon} \delta^2\lambda(x)dxds]}}{\eta/2} \le \frac{\sqrt{\Sigma^\varepsilon \cdot \sum_{i=1}^n K_i^2\Delta_i}}{\eta/2},$$

having used for the first equality that $K_i\Delta_i\tilde{J}^\varepsilon$ are martingale increments. Since under **A1** we have $K^2 \in L^1(\mathbb{R})$ then for fixed ε, as $n \to \infty$, $\Sigma^\varepsilon\sum_{i=1}^n K_i^2\Delta_i \simeq \Sigma^\varepsilon h \to 0$, then $\limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i\Delta_i\tilde{J}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) = 0$ for all $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i\Delta_i\tilde{J}^\varepsilon| > \frac{\eta}{2}\Big\}\Big) = 0.$$

As for $\sum_{i=1}^n K_i \Delta_i C^\varepsilon$, we have

$$\Big|\sum_{i=1}^n K_i \Delta_i C^\varepsilon\Big| \le A^\varepsilon \sum_{i=1}^n K_i \Delta_i,$$

which does not depend on ω and, for fixed ε, tends a.s. to 0, as $n \to \infty$, so again

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\Big(\Big\{|\sum_{i=1}^n K_i\Delta_i C^\varepsilon| > \frac{\eta}{2}\Big\}\Big) \le \lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\Big(\Big\{A^\varepsilon \sum_{i=1}^n K_i \Delta_i > \frac{\eta}{2}\Big\}\Big) = \lim_{\varepsilon\to 0} 0 = 0$$

For the case $f(x) = x^2$ we reason similarly. In fact

$$F^n(X) - F^n(J^{1,\varepsilon}) = \sum_{i=1}^n K_i \left(\Delta_i \tilde{J}^\varepsilon\right)^2 + \sum_{i=1}^n K_i \left(\Delta_i C^\varepsilon\right)^2$$

$$+2\sum_{i=1}^n K_i\Big(\Delta_i \tilde{J}^\varepsilon \Delta_i J^{1,\varepsilon} - \Delta_i \tilde{J}^\varepsilon \Delta_i C^\varepsilon - \Delta_i J^{1,\varepsilon}\Delta_i C^\varepsilon\Big), \tag{45}$$

and we show that for fixed ε each term tends to 0 in probability as $n \to \infty$: $\sum_{i=1}^n K_i \left(\Delta_i \tilde{J}^\varepsilon\right)^2$ tends to 0 in probability because its L^1-norm tends to 0;

$$\sum_{i=1}^n K_i \left(\Delta_i C^\varepsilon\right)^2 \le (A^\varepsilon)^2 \sum_{i=1}^n K_i \Delta_i^2 \le (A^\varepsilon)^2 \Delta_{max} \sum_{i=1}^n K_i \Delta_i \overset{a.s.}{\to} 0.$$

Finally, the double products are all dealt with using the Schwartz inequality, and shown to be negligible:

$$\Big|\sum_{i=1}^n K_i \Delta_i Z \Delta_i Y\Big| = \Big|\sum_{i=1}^n \sqrt{K_i}\Delta_i Z \sqrt{K_i}\Delta_i Y\Big| \le \sqrt{\sum_{i=1}^n K_i (\Delta_i Z)^2}\sqrt{\sum_{i=1}^n K_i(\Delta_i Y)^2},$$

and for each one of the three double products in (45) at least one of the square roots on the right hand side above tends to 0 in probability, while $\sum_{i=1}^n K_i(\Delta_i J^{1,\varepsilon})^2 = F^n(J^{1,\varepsilon})$ converges to the finite quantity $F(J^{1,\varepsilon}) = K(0)(\Delta J^{1,\varepsilon}_{\bar t})^2$.

It follows that, for fixed $\varepsilon > 0$, $F^n(X) - F^n(J^{1,\varepsilon}) \overset{P}{\to} 0$ as $n \to \infty$, thus again

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\Big(\Big\{|F^n(X) - F^n(J^{1,\varepsilon})| > \eta\Big\}\Big) = \lim_{\varepsilon\to 0} 0 = 0.$$

b) We concentrate on the set $\{\Delta X_{\bar t} = 0\}$, having probability 1. On that set both the numerator and the denominator of $T^n_{\bar t}$ tend to 0 in probability: using Lemmas 4 and 5 we reach the following speeds, as explained below:

$$\sum_{i=1}^n K_i \Delta_i X \overset{d}{\simeq} \begin{cases} -ah, & \text{if } \alpha \in (0,1) \\ -(A_+ - A_-)K_{(1)} \cdot h \log \frac{1}{h}, & \text{if } \alpha = 1 \text{ and } A_+ \ne A_- \\ h^{\frac{1}{\alpha}} Z_{1,\alpha}, & \text{if } \alpha \in (1,2) \end{cases}; \tag{46}$$

$$\sum_{i=1}^n K_i (\Delta_i X)^2 \overset{d}{\simeq} \begin{cases} h^{\frac{2}{\alpha}} Z_{2,\alpha} + o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}), & \text{if } \alpha \in (0,1) \\ h^2 Z_{2,\alpha}, & \text{if } \alpha = 1 \\ h^{\frac{2}{\alpha}} Z_{2,\alpha}, & \text{if } \alpha \in (1,2), \end{cases} \tag{47}$$

where for $\alpha < 1$ we have $a = (A_+ - A_-)/(1-\alpha)$. It follows that for $\alpha \in (0,1)$ and $a \ne 0$ then

$$T_n \overset{d}{\simeq} \frac{-a}{\sqrt{h^{\frac{2}{\alpha}-2} Z_{2,\alpha} + O\left(\frac{\Delta}{h}\right) + o_P\left(\sqrt{\frac{\Delta}{h}}\right)}} \to -sgn(a)\cdot\infty,$$

for $\alpha = 1$ then

$$T_n \overset{d}{\simeq} -\frac{(A_+ - A_-)K_{(1)}}{\sqrt{Z_{2,\alpha}}} \log \frac{1}{h} \overset{a.s.}{\to} -sgn(A_+ - A_-) \cdot \infty$$

while for $\alpha \in (1, 2)$ numerator and denominator of $T_{\bar{t}}^n$ have the same speed, and by Lemma 6 the theorem is proved.

To obtain (46) from Lemma 4, we simply note that a.s. the speed of $\sum_{i=1}^n K_i \Delta_i J^1$ is $K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)$, where $S_{\underline{p}}$ is the time of the jump of J^1 closest to $\bar{t}$ (see the proof of Theorem 1 after (14)). Since $K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right) = o(h\Delta)$ by assumption **A1.2**, $K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)$ is negligible, for any α, with respect to $\varphi_\alpha(h)$.

To obtain (47) from Lemmas 4 and 5 we first note that, similarly as above, $\sum_{i=1}^n K_i(\Delta_i J^1)^2$ tends to zero still at speed $K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right) = o(h\Delta)$. Then

· for $\alpha \in (0, 1)$ the squared denominator of $T_{\bar{t}}^n$ is

$$\sum_{i=1}^n K_i(\Delta_i X)^2 = \sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu\Big)^2 + \sum_{i=1}^n K_i(\Delta_i \cdot a)^2 + \sum_{i=1}^n K_i(\Delta_i J^1)^2$$

$$-2\Delta a \sum_{i=1}^n K_i \int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu - 2\Delta a \sum_{i=1}^n K_i \Delta_i J^1 + 2\sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu\Big) \Delta_i J^1 :$$

within the last term $\sum_{i=1}^n \sqrt{K_i}\Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu\Big) \sqrt{K_i} \Delta_i J^1$ is dominated by $\sqrt{\sum_{i=1}^n K_i \Big(\int_{t_{i-1}}^{t_i} \int_{|x|\le 1} x d\mu\Big)^2} \sqrt{\sum_{i=1}^n K_i(\Delta_i J^1)^2} = O_P\Big(h^{\frac{1}{\alpha}} \sqrt{K\Big(\frac{\bar{t}-S_{\underline{p}}}{h}\Big)}\Big) = o_P(h\sqrt{h\Delta})$, thus the above display is asymptotically equivalent to

$$h^{\frac{2}{\alpha}} Z_{2,\alpha} + O_P\left(\Delta h + K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right) + \Delta h^{\frac{1}{\alpha}}\right) + o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}) = h^{\frac{2}{\alpha}} Z_{2,\alpha} + o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}).$$

· for $\alpha = 1$ we instead split $\Delta_i X$ into $\Delta_i \tilde{J}$ and $\Delta_i J^1$ and, using again the Schwartz inequality, the mixed term within the squared denominator of $T_{\bar{t}}^n$ is shown to be dominated by

$$2\sqrt{\sum_{i=1}^n K_i\Big(\Delta_i \tilde{J}\Big)^2} \sqrt{\sum_{i=1}^n K_i(\Delta_i J^1)^2} = O_P\left(h\sqrt{K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)}\right) = o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}).$$

Thus

$$\sum_{i=1}^n K_i\Big(\Delta_i X\Big)^2 \overset{d}{\simeq} h^2 Z_{2,\alpha} + O_P\left(K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)\right) + o_P(h^{\frac{3}{2}}\Delta^{\frac{1}{2}}) \overset{d}{\simeq} h^2 Z_{2,\alpha}.$$

· for $\alpha \in (1, 2)$ we again split $\Delta_i X$ into $\Delta_i \tilde{J}$ and $\Delta_i J^1$ and use the Schwartz inequality:

$$\sum_{i=1}^n K_i\Big(\Delta_i X\Big)^2 \overset{d}{\simeq} h^{\frac{2}{\alpha}} Z_{2,\alpha} + O_P\left(K\left(\frac{\bar{t}-S_{\underline{p}}}{h}\right)\right) + o_P(h^{\frac{1}{\alpha}}\sqrt{\Delta h}) \overset{d}{\simeq} h^{\frac{2}{\alpha}} Z_{2,\alpha}.$$

□

Proof of Corollary 1. Let us split $Y = Y^1 + \tilde{J}$, where $Y_t^1 \doteq Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t^1$, then

$$T_{\bar{t}}^n = \frac{\sum_{i=1}^n K_i \Delta_i Y}{\sqrt{\sum_{i=1}^n K_i (\Delta_i Y)^2}} = \frac{\sum_{i=1}^n K_i \Delta_i Y^1 + \sum_{i=1}^n K_i \Delta_i \tilde{J}}{\sqrt{\sum_{i=1}^n K_i(\Delta_i Y^1)^2 + \sum_{i=1}^n K_i(\Delta_i \tilde{J})^2 + 2\sum_{i=1}^n K_i \Delta_i Y^1 \Delta_i \tilde{J}}}$$

with $S_n \doteq \sum_{i=1}^n K_i(\Delta_i Y^1)^2$, the above equals

$$\frac{\frac{\sum_{i=1}^n K_i\Delta_i Y^1}{\sqrt{S_n}} + \frac{\sum_{i=1}^n K_i\Delta_i \tilde J}{\sqrt{S_n}}}{\sqrt{1+\frac{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}{S_n} + 2\frac{\sum_{i=1}^n K_i\Delta_i Y^1\Delta_i \tilde J}{S_n}}},$$

and we show that the last display tends to $\mathcal{N}(0,1)$ in distribution.

In fact first of all note that with probability 1 there is no jump at $\bar t$, and when $\Delta X_{\bar t}=0$ the leading term of S_n is $\sum_{i=1}^n K_i(\int_{t_{i-1}}^{t_i} b_s ds + \int_{t_{i-1}}^{t_i}\sigma_s dW_s)^2 \sim h\sigma^\star_{\bar t}$ ([18], thm 2.7) because $\sum_{i=1}^n K_i(\Delta_i J^1)^2 \sim K(\frac{\bar t - S_p}{h}) = o(\Delta h)$. Thus, with probability 1, $S_n \sim h$.

Then, the first quotient of the above numerator tends in distribution to a standard Gaussian r.v. because Y^1 has finite variation jumps, so the result in [5] applies. We now show that all the other terms tend to 0. If $\alpha \in (1,2)$, by Lemma 4, $\sum_{i=1}^n K_i\Delta_i \tilde J$ tends to 0 at speed $h^{1/\alpha} << h^{1/2}$, thus the second quotient at numerator tends to 0; the second term at denominator

$$\frac{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}{S_n} \sim \frac{h^{\frac{2}{\alpha}}}{h} \to 0$$

and the third one

$$\frac{\sum_{i=1}^n K_i\Delta_i Y^1\Delta_i \tilde J}{S_n} \le \frac{\sqrt{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}\sqrt{S_n}}{S_n} \sim \frac{h^{\frac{1}{\alpha}}}{\sqrt h} \to 0.$$

If instead $\alpha = 1$, the second quotient at numerator is

$$\frac{\sum_{i=1}^n K_i\Delta_i \tilde J}{\sqrt{S_n}} \sim \frac{h\log\frac{1}{h}}{\sqrt h} \to 0,$$

the second term at denominator

$$\frac{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}{S_n} \sim \frac{h^2}{h} \to 0$$

and the third one

$$\frac{\sum_{i=1}^n K_i\Delta_i Y^1\Delta_i \tilde J}{S_n} \le \frac{\sqrt{\sum_{i=1}^n K_i(\Delta_i \tilde J)^2}\sqrt{S_n}}{S_n} \sim \frac{h}{\sqrt h} \to 0.$$

□

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References

[1] Ait-Sahalia, Y., Jacod, J. (2009), Estimating the degree of activity of jumps in high frequency data, Ann. Statist. 37(5A), 2202-2244.

[2] Ait-Sahalia, Y., Jacod, J. (2010). Is Brownian motion necessary to model high-frequency data?, The Annals of Statistics, 38(5), 3093-3128.

[3] Barndorff-Nielsen, O. (1998) Processes of Normal Inverse Gaussian type, Finance and Stochastics, 2, 41-68.

[4] Carr, P., Geman, H., Madan D., Yor, M. (2002) The Fine Structure of Asset Returns: An Empirical Investigation, The Journal of Business, 75(2), 305-332

[5] Christensen, K., Oomen, R., Renò, R. (2020) The drift burst hypothesis, forthcoming on Journal of Econometrics

[6] Chung, K.L. (1974) A course in probability theory, Academic Press, Inc.

[7] Cont, R., Mancini, C. (2011) Nonparametric tests for pathwise properties of semimartingales, Bernoulli, 17(2), 781-813

[8] Cont, R., Tankov, P. (2004) Financial modelling with jump processes, Chapman & Hall/CRC press

[9] Eberlein, E., Keller, U. (1995) Hyperbolic Distributions in Finance, Bernoulli 1, 281-299.

[10] Gallant, A.R., Tauchen, G. (2018) Exact Bayesian moment based inference for the distribution of the small-time movements of an Ito semimartingale. Journal of Econometrics 205(1): 140-155.

[11] Geman, H. (2002) Pure jump Lévy processes for asset price modelling. Journal of Banking & Finance 26(7), 1297-1316.

[12] Geman, H., Madan, D., Yor, M. (2001) Times Changes for Lévy Processes, Mathematical Finance, 11, 79-96

[13] Hounyo, U., Varneskov, R. T. (2020) Inference for local distributions at high sampling frequencies: A bootstrap approach. Journal of Econometrics 215, 1-34

[14] Jacod, J., Protter, P. E. (2012) Discretization of Processes, Springer-Verlag, Berlin, 2nd edn.

[15] Jacod, J., Todorov, V. (2009) Testing for common arrivals of jumps for discretely observed multidimensional processes, The Annals of Statistics 37 (4), 1792-1838

[16] Jing, B.Y., Kong, X.B., Liu, Z. (2012) Modeling high-frequency financial data by pure jump processes. The Annals of Statistics 40(2), 759-784.

[17] Kolokolov, A., Renò, R. (2021) Jumps or staleness?, ssrn.com

[18] Mancini, C., Mattiussi, V., Renò, R. (2015) Spot volatility estimation using delta sequences, Finance and Stochastics, 19(2), 261-293.

[19] Sato, K. (1999) Lévy processes and infinitely divisible distributions, Cambridge studies in advanced mathematics 68, Cambridge University Press, ISBN 0 521 553024

[20] Todorov, V., Tauchen, G. (2014) Limit theorems for the empirical distribution function of scaled increments of Ito semimartingales at high frequencies, Annals of Applied Probability, 24(5), 1850-1888.