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Some numerical aspects on a method for solving linear problems with complementarity constraints

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Abstract

A known method for solving linear problems with complementarity constraints is briefly recalled. The method decomposes the given problem in a sequence of parameterized problems and - by means of suitable cuts - allows to define an iterative procedure that leads to an optimal solution or to an approximation of it providing an estimate of the error. In this paper, for problems of different dimensions we have implemented some numerical experiments which show that in most cases the method converges linearly with respect to the dimension of the problem. Our results are also compared with those obtained by similar approaches where different kinds of cuts are considered.

Keywords Mathematical programs with complementarity constraints, duality, decomposition methods

AMS Classification 49M27, 65K05, 90C30, 90C33, 90C46

JEL Classification C61

1 Introduction

In the field of equilibrium models, mathematical programs with complementarity constraints (MPCC) form a class of important, but extremely difficult, problems. MPCC's constitute a subclass of the well-known mathematical programs with equilibrium constraints (MPEC), widely studied in recent years. Their relevance comes from many applications like urban traffic control, economy, problems arising from the electrical sector or from structural engineering [4, 5, 6, 27]. Their difficulty is due to the presence of the complementarity constraints, because the feasible region may not enjoy some fundamental properties: it may be not convex, even not connected and such that many of the standard constraint qualifications are violated at any feasible point. This last lack implies that the usual Karush-Kuhn-Tucker (KKT) conditions may not be fulfilled at an optimal solution, even in the linear case [15], denoted with LPCC. Specific constraint qualifications (CQs) for the MPCC were introduced to try to overcome this difficulty [8, 9, 15, 23, 24, 26, 31] and different notions of stationarity were also proposed [24]. Using these concepts, the behaviour of general nonlinear optimization algorithms for the MPCC was studied. For example, interior point methods [18, 22], penalty approaches [12, 21, 29, 30], relaxation methods [9, 13, 16, 25, 26], smoothing methods [1, 7, 28]. Also, many specific methods which combine the previous approaches for the MPCC were developed [11, 14, 19]; such methods have good convergence properties, but they are affected by the request of exact computation of KKT points [17].

In [20] the authors consider the linear case and propose a reformulation of the problem by means of a family of parameterized linear problems whose minimization leads to an optimal solution of LPCC. Exploiting the classic tools of the duality theory, an iterative method is outlined which explores the set of parameters, excluding at each step a subset of them, by means of a suitable cut. Indeed, the optimal values of the linear problems associated with such a subset are proved to be greater than or equal to the optimal value related to the current parameter. A similar approach for LPCC can be found in [10, 32] where different kinds of cuts are considered. In this work we define an algorithm which is implemented in an interactive way taking advantage of some devices that speed up the solving procedure, owing to the decomposition of the given problem in a sequence of parameterized problems.

The paper is organized as follows. In Section 2, we introduce the problem and describe the decomposition method of LPCC in a family of parameterized problems. In Section 3, the main results of [20] are resumed; in particular, we recall the sufficient optimality condition and how to determine lower and upper bounds of the optimum value. Using the results of the previous section, in Section 4, the iterative method is described and illustrated by means of an example. Finally, in Section 5, some numerical experiments involving problems with different dimensions are presented and commented.

2 A decomposition approach

We consider the following constrained minimization problem, whose objective function is linear and having a linear complementarity constraint besides affine ones:

$$(P) \qquad \begin{array}{l} f^0 := \min(\langle c, x\rangle + \langle d, y\rangle) \\ \text{s.t.} \quad (x, y) \in K := \{(x, y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0, \langle x, y\rangle = 0\}, \end{array}$$

where $A, B \in \mathbb{R}^{m\times n}$, $c, d \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

We will assume that the feasible set K is nonempty and a global minimum point of P exists; call it (x^0, y^0). Let us introduce the following penalized form for the gradient of the objective function of P:

$$\begin{array}{rcl} c(\alpha) & = & (c_j(\alpha_j) := c_j + \rho_j\alpha_j, j = 1, ..., n), \\ d(\alpha) & = & (d_j(\alpha_j) := d_j + \sigma_j(1-\alpha_j), j = 1, ..., n), \end{array}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \Delta := \{0,1\}^n$. For the sake of simplicity, we have used for the penalized form of the constants c_j and d_j (that are functions of α_j) the same name of the constants. For our purposes, we will assume that ρ_j and σ_j, $j = 1, ..., n$, are large enough positive constants; the meaning of this assumption will be clear inside the proof of Theorem 1.

The given problem P can be associated with a family $\{P(\alpha)\}_{\alpha\in\Delta}$ of subproblems

$$P(\alpha) \qquad \begin{array}{l} f^{\downarrow}(\alpha) := \min\ [f(x,y;\alpha) := \langle c(\alpha), x\rangle + \langle d(\alpha), y\rangle] \\ \text{s.t.} \quad (x,y) \in R := \{(x,y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0\}. \end{array}$$

Assumption 1. Let us suppose that, for large enough ρ_j and σ_j, $j = 1, ..., n$, the objective function of $P(\alpha)$ is bounded from below on R, $\forall \alpha \in \Delta$.

By Assumption 1, it follows that $\forall \alpha \in \Delta$ there exists a minimum point, say $(x(\alpha), y(\alpha))$, of $P(\alpha)$. A condition that guarantees Assumption 1 is that $\langle c, x\rangle + \langle d, y\rangle$ is bounded from below on R, which obviously implies that the objective function of P is bounded from below on K, which in turn yields that a global minimum point of P exists.

We have the following result.

Theorem 1. *If Assumption 1 holds, then*

$$f^0 = \min_{\alpha\in\Delta} f^{\downarrow}(\alpha) = \min_{\alpha\in\Delta} \min_{(x,y)\in R} f(x,y;\alpha). \tag{1}$$

Proof. Suppose that (x^0, y^0) is a minimum point of P, so that $f^0 = \langle c, x^0\rangle + \langle d, y^0\rangle$. Recalling that Δ is a finite set, let $i = 1$ and $\alpha^i \in \Delta := \{\alpha^1, \alpha^2, \ldots, \alpha^{2^n}\}$. From the definition of $c(\alpha)$ and $d(\alpha)$, we have

$$f(x,y;\alpha^i) = \langle c, x\rangle + \langle d, y\rangle + \sum_{j=1}^{n} \rho_j \alpha_j^i x_j + \sum_{j=1}^{n} \sigma_j (1-\alpha_j^i) y_j, \ \forall (x,y) \in R.$$

Let $vertR$ be the set of vertices of R. Consider $(\bar{x}, \bar{y}) \in vertR$; if $(\bar{x}, \bar{y})$ is such that

$$\sum_{j=1}^{n} \rho_j \alpha_j^i \bar{x}_j + \sum_{j=1}^{n} \sigma_j (1-\alpha_j^i) \bar{y}_j = 0 \tag{2}$$

(which happens independently on ρ and σ), then $\langle \bar{x}, \bar{y}\rangle = 0$ and $(\bar{x}, \bar{y})$ is a feasible solution of P. Therefore, $f^0 \leq f(\bar{x}, \bar{y}; \alpha^i)$; choose $\rho^i, \sigma^i \in \mathbb{R}^n$ arbitrarily, for example $\rho^i = \sigma^i = (1, \ldots, 1) \in \mathbb{R}^n$. Otherwise, if $(\bar{x}, \bar{y})$ is a vertex of R such that (2) is not fulfilled, we can choose $\rho^i = (\rho_1^i, \ldots, \rho_n^i)$ and $\sigma^i = (\sigma_1^i, \ldots, \sigma_n^i)$ such that $f^0 \leq f(\bar{x}, \bar{y}; \alpha^i)$. In fact,

$$f(\bar{x}, \bar{y}; \alpha^i) = \langle c, \bar{x}\rangle + \langle d, \bar{y}\rangle + \sum_{j=1}^{n} \rho_j^i \alpha_j^i \bar{x}_j + \sum_{j=1}^{n} \sigma_j^i (1-\alpha_j^i) \bar{y}_j$$

and there exists j such that $\rho_j^i \bar{x}_j > 0$ (or $\sigma_j^i \bar{y}_j > 0$). Now, if

$$\rho_j^i \geq \frac{\langle c, x^0 - \bar{x}\rangle + \langle d, y^0 - \bar{y}\rangle}{\bar{x}_j}$$

we get

$$\langle c, x^0\rangle + \langle d, y^0\rangle \le \langle c, \bar{x}\rangle + \langle d, \bar{y}\rangle + \rho^i_j \bar{x}_j \le f(\bar{x}, \bar{y}; \alpha^i),$$

and this means $f^0 \le f(\bar{x}, \bar{y}; \alpha^i)$. Analogously, in the case $\sigma^i_j \bar{y}_j > 0$.

Noticing that $vertR$ is a finite set, it is possible to find a couple (ρ^i, σ^i) such that

$$f^0 \le f(x, y; \alpha^i), \ \forall (x, y) \in vertR. \tag{3}$$

From Assumption 1 (by further increasing some components of ρ^i and σ^i, if necessary) a minimum point of $P(\alpha)$ exists and is attained in the set $vertR$. From (3), it follows $f^0 \le f^{\downarrow}(\alpha^i)$. Now, consider α^{i+1}; if $f^0 \le f(x, y; \alpha^{i+1})$, $\forall (x, y) \in vertR$, with $\rho = \rho^i$ and $\sigma = \sigma^i$, then set $\rho^{i+1} = \rho^i$ and $\sigma^{i+1} = \sigma^i$; in such a way, we obtain $f^0 \le f^{\downarrow}(\alpha^{i+1})$. Otherwise, choose ρ^{i+1} and σ^{i+1} by increasing the previous vectors $\rho = \rho^i$ and $\sigma = \sigma^i$, in order to get $f^0 \le f^{\downarrow}(\alpha^{i+1})$. Set $i = i + 1$ and repeat this procedure for all $i = 2, .., 2^n - 1$. It turns out that

$$f^0 \le \min_{\alpha \in \Delta} f^{\downarrow}(\alpha). \tag{4}$$

Moreover, starting again from (x^0, y^0) minimum point of P, let us define the following vector $\alpha^0 = \alpha(x^0, y^0)$:

$$\alpha^0 := \begin{cases} \alpha^0_j = 0, & \text{if} \quad x^0_j > 0 \\ \alpha^0_j = 1, & \text{if} \quad x^0_j = 0 \end{cases}$$

The above vector α^0 belongs to Δ and $f^0 = f(x^0, y^0; \alpha^0)$. Since $(x^0, y^0) \in R$ is a feasible solution of the problem $P(\alpha^0)$, then

$$f^0 = f(x^0, y^0; \alpha^0) \ge f^{\downarrow}(\alpha^0) \ge \min_{\alpha \in \Delta} f^{\downarrow}(\alpha). \tag{5}$$

Inequalities (4) and (5) imply (1) and this concludes the proof. □

Based on Theorem 1, we can propose a decomposition approach for solving problem P. In fact, Theorem 1 establishes that the minimum f^0 of problem P can be achieved by first determining $f^{\downarrow}(\alpha)$ $\forall \alpha \in \Delta$, and secondly by minimizing $f^{\downarrow}(\alpha)$ with respect to $\alpha \in \Delta$; in other words, equation (1) decomposes the problem P in a sequence of subproblems $P(\alpha)$.

In view of the definition of α, ρ_j and σ_j, $j = 1, ..., n$, and of problem $P(\alpha)$, by setting $\alpha_j = 1$ or $\alpha_j = 0$ one would expect $x_j = 0$ or $y_j = 0$, respectively, in an optimal solution of $P(\alpha)$; however, in general, an optimal solution of $P(\alpha)$ will not necessarily comply with such an expectation for a given $\alpha \in \Delta$. Anyway, if such an expectation does not occur for a given $\alpha \in \Delta$, then $f^{\downarrow}(\alpha) > f^0$, provided that we choose ρ and σ large enough. Consequently, we should work with the subset $\bar{\Delta}$ of Δ whose elements fulfill the following definition.

Definition 1. Let $\bar{\Delta}$ be the set of all $\alpha \in \Delta$ such that the system

$$Ax + By \ge b, \ x \ge 0, \ y \ge 0$$

is possible, when one sets $x_j = 0$ for $\alpha_j = 1$ and $y_j = 0$ for $\alpha_j = 0, j = 1, ..., n$.

Clearly, if a minimum point of P exists, then the set $\bar{\Delta}$ is nonempty.

Unfortunately, the cardinality of Δ, and also of $\bar{\Delta}$, is in general so large that the above outlined decomposition is not, by itself, of use. We aim at solving P through the penalized problems $P(\alpha)$'s, by running as less as possible on $\alpha \in \Delta$: at step k, having solved $P(\alpha^k)$, we try to determine α^{k+1}, such that

$$f(x(\alpha^k), y(\alpha^k), \alpha^k) > f(x(\alpha^{k+1}), y(\alpha^{k+1}), \alpha^{k+1}). \tag{6}$$

An initial effort in this direction is described in the first part of next section.

3 Optimality conditions and bounds for the optimum

Suppose that we have solved, at step k, the problem $P(\alpha^k)$; we try to determine α^{k+1}, such that (6) holds. To this aim, we introduce the dual problem of $P(\alpha)$ given by:

$$P^*(\alpha) \qquad \begin{array}{l} \max \langle \lambda, b\rangle \\ \text{s.t.} \quad \lambda \in R^*(\alpha) := \{\lambda \in \mathbb{R}^m : \ \lambda A \le c(\alpha), \ \lambda B \le d(\alpha), \ \lambda \ge 0\}. \end{array}$$

Theorem 2. *If $\alpha^{k+1} \in \Delta^k := \{\alpha \in \Delta : \bar\lambda A \le c(\alpha), \bar\lambda B \le d(\alpha)\}$, where $\bar\lambda$ is a maximum point of $P^*(\alpha^k)$, then*

$$f(x(\alpha^k), y(\alpha^k), \alpha^k) \le f(x(\alpha^{k+1}), y(\alpha^{k+1}), \alpha^{k+1}).$$

Proof. Observe that $\bar\lambda$ is a feasible solution of $P^*(\alpha^{k+1})$ so that the maximum of such a problem is greater than or equal to that of $P^*(\alpha^k)$:

$$\max_{\lambda \in R^*(\alpha^{k+1})} \langle \lambda, b\rangle \ge \langle \bar\lambda, b\rangle := \max_{\lambda \in R^*(\alpha^k)} \langle \lambda, b\rangle.$$

By the strong duality theorem, the minimum of $P(\alpha^{k+1})$ is greater than or equal to that of $P(\alpha^k)$. □

By the previous theorem, we infer that a necessary condition for (6) to hold is that

$$\alpha^{k+1} \notin \Delta^k := \{\alpha \in \Delta : \bar\lambda A \le c(\alpha), \bar\lambda B \le d(\alpha)\}. \tag{7}$$

In other words, having an optimal solution of $P(\alpha^k)$ for some $\alpha^k \in \Delta$, and, thus, a feasible solution to the given problem P, a necessary condition for obtaining a problem $P(\alpha^{k+1})$ with a minimum $f^{\downarrow}(\alpha^{k+1}) < f^{\downarrow}(\alpha^k)$ (namely, a feasible solution of P "better" than the current one), is that α^{k+1} belongs to the set $\Delta \setminus \Delta^k$.

Theorem 2 provides a criterion for eliminating the subset Δ^k from the subsequent iterations; observe that Δ^k cannot be empty as it contains at least α^k.

Let us introduce the following sets of indexes

$$I_x(\bar\lambda) := \left\{ j = 1, \dots, n : \frac{\langle \bar\lambda, A^j\rangle - c_j}{\rho_j} > 0 \right\} = \left\{ j = 1, \dots, n : \langle \bar\lambda, A^j\rangle - c_j > 0 \right\} \tag{8}$$

$$I_y(\bar\lambda) := \left\{ j = 1, \dots, n : \frac{\langle \bar\lambda, B^j\rangle - d_j}{\sigma_j} > 0 \right\} = \left\{ j = 1, \dots, n : \langle \bar\lambda, B^j\rangle - d_j > 0 \right\} \tag{9}$$

where, both in (8) and in (9), the second equality is true because $\rho_j > 0$ and $\sigma_j > 0$, $j = 1, \dots, n$.

The following results are proved in [20]. We assume that $\alpha^k \in \Delta$ and $\bar\lambda$ is a maximum point of $P^*(\alpha^k)$.

Theorem 3. (**sufficient optimality condition**) If

$$I_x(\bar\lambda) \cup I_y(\bar\lambda) = \varnothing, \tag{10}$$

then an optimal solution $(\bar x^k, \bar y^k)$ of the problem $P(\alpha^k)$ is an optimal solution of problem P.

If the above sufficient condition is not satisfied, that is $I_x(\bar\lambda) \cup I_y(\bar\lambda) \ne \varnothing$, the following result establishes a condition equivalent to the necessary condition (7).

Theorem 4. *Suppose that* $I_x(\bar\lambda) \cup I_y(\bar\lambda) \neq \varnothing$ *and* $\alpha \in \Delta$. *Then*

$$\alpha \notin \{\alpha \in \Delta : \bar\lambda A \leq c(\alpha), \bar\lambda B \leq d(\alpha)\} \tag{11}$$

iff

$$\sum_{i \in I_x(\bar\lambda)} \alpha_i - \sum_{j \in I_y(\bar\lambda)} \alpha_j \leq |I_x(\bar\lambda)| - 1. \tag{12}$$

Define the relaxed problem of P obtained by dropping the complementarity constraints, i.e.

$$(RP) \qquad \begin{cases} \min(\langle c, x\rangle + \langle d, y\rangle) \quad \text{s.t.} \\ Ax + By \geq b, \quad x \geq 0, \quad y \geq 0 \end{cases}$$

and denote by $\bar f$ the optimal value of RP (possibly $-\infty$).

The next result deepens the meaning of the sufficient optimality condition given by Theorem 3.

Proposition 1. *Let* $\alpha \in \Delta$ *and* $\bar\lambda$ *be an optimal solution of* $P^*(\alpha)$. *If* $I_x(\bar\lambda) \cup I_y(\bar\lambda) = \varnothing$, *then* $f^0 = \bar f$.

Proof. The assumption $I_x(\bar\lambda) \cup I_y(\bar\lambda) = \varnothing$ is equivalent to say that $\bar\lambda A \leq c$, $\bar\lambda B \leq d$. Therefore, since $\bar\lambda \geq 0$, we have that $\bar\lambda$ is a feasible solution for the dual of RP: if $\bar f = -\infty$, we achieve a contradiction. Suppose that $\bar f > -\infty$; then, by strong duality $f^{\downarrow}(\alpha) = \langle \bar\lambda, b\rangle$ and, by weak duality,

$$f^{\downarrow}(\alpha) = \langle \bar\lambda, b\rangle \leq \bar f,$$

which implies $f^0 := \min_{\alpha \in \Delta} f^{\downarrow}(\alpha) \leq \bar f$. On the other hand, $f^0 \geq \bar f$ and this completes the proof. □

The sufficient optimality condition (10) given in Theorem 3 is a very restrictive condition. Indeed, it directly implies that the minimum value of the relaxed problem RP is equal to the one of P, as proved by Proposition 1.

Therefore, we propose an alternative iterative approach that leads, not only to a different sufficient optimality condition, but mainly to the possibility to evaluate the difference between the current value of the objective function of P and its minimum value, i.e., $f^{\downarrow}(\alpha^k) - f^0$. We will achieve this purpose by defining a finite sequence of lower and upper bounds of the minimum of P.

We restrict our attention to the case where it is possible to find vectors of upper bounds, say $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$, for x and y respectively, in such a way that the optimal value of problem P does not change. Therefore, alternatively to $\{P(\alpha)\}_{\alpha \in \Delta}$, we can associate with the given problem P the following family $\{Q(\alpha)\}_{\alpha \in \Delta}$ of subproblems:

$$Q(\alpha) \qquad \begin{array}{l} \min(\langle c, x\rangle + \langle d, y\rangle) \\ \text{s.t. } Ax + By \geq b, \\ \quad\quad 0 \leq x_j \leq X_j(1 - \alpha_j), \quad j = 1, \ldots, n \\ \quad\quad 0 \leq y_j \leq Y_j \alpha_j, \quad j = 1, \ldots, n. \end{array}$$

We assume that $Q(\alpha)$ admits a solution, $\forall \alpha \in \Delta$.

Denote by $S(\alpha)$ the feasible set of $Q(\alpha)$. The dual of $Q(\alpha)$, say $Q^*(\alpha)$, is given by:

$$Q^*(\alpha) \qquad \begin{array}{l} \max\left(\langle \lambda, b\rangle - \sum_{j=1}^n \mu_j X_j(1 - \alpha_j) - \sum_{j=1}^n \nu_j Y_j \alpha_j\right) \\ \text{s.t. } \lambda A - \mu \leq c \\ \quad\quad \lambda B - \nu \leq d \\ \quad\quad \lambda \geq 0, \mu \geq 0, \nu \geq 0. \end{array}$$

An optimal basic solution of $Q^*(\alpha)$ can be immediately derived from an optimal basic solution of $P^*(\alpha)$, as proved by the following proposition [20].

Proposition 2. *If $\bar{\lambda}$ is an optimal basic solution of $P^*(\alpha)$, then the vector $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$, where*

$$\bar{\mu}_j = max\{0, \langle \bar{\lambda}, A^j \rangle - c_j\}, \quad j = 1, \dots, n, \tag{13a}$$
$$\bar{\nu}_j = max\{0, \langle \bar{\lambda}, B^j \rangle - d_j\}, \quad j = 1, \dots, n \tag{13b}$$

is an optimal basic solution of $Q^*(\alpha)$.

Let us observe that the feasible set of $Q^*(\alpha)$ does not depend on α; call this set $S^*, \forall \alpha \in \Delta$, and notice that $S^* \neq \varnothing$ by Proposition 2. Denote by $V := vertS^*$, the set of all vertices of S^*, or, equivalently, of all basic solutions of $Q^*(\alpha)$. Then, we have the following result.

Theorem 5. *The minimum f^0 of problem P equals the minimum of the problem:*

$$\begin{array}{ll} \min\limits_{\alpha, f} & f \\ \text{s.t.} & \alpha \in \Delta \\ & f \geq \big(\langle \lambda^h, b \rangle - \sum_{j=1}^n \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^n \nu_j^h Y_j \alpha_j\big), \ (\lambda^h, \mu^h, \nu^h) \in V. \end{array} \tag{14}$$

Proof. The following relations are readily seen to hold:

$$\begin{aligned} f^0 &= \min_{\alpha \in \Delta} \min_{(x,y) \in S(\alpha)} (\langle c, x \rangle + \langle d, y \rangle) \\ &= \min_{\alpha \in \Delta} \max_{(\lambda, \mu, \nu) \in S^*} \big(\langle \lambda, b \rangle - \sum_{j=1}^n \mu_j X_j (1 - \alpha_j) - \sum_{j=1}^n \nu_j Y_j \alpha_j\big) \\ &= \min_{\alpha \in \Delta} \max_{(\lambda^h, \mu^h, \nu^h) \in V} \big(\langle \lambda^h, b \rangle - \sum_{j=1}^n \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^n \nu_j^h Y_j \alpha_j\big). \end{aligned} \tag{15}$$

The last equality and the introduction of the scalar variable f prove the thesis of the theorem. □

Remark 1. Observe that if $\bar{V}$ is any subset of V, then by (15) we have:

$$f^0 \geq \min_{\alpha \in \Delta} \max_{(\lambda^h, \mu^h, \nu^h) \in \bar{V}} \big(\langle \lambda^h, b \rangle - \sum_{j=1}^n \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^n \nu_j^h Y_j \alpha_j\big). \tag{16}$$

Therefore, the minimum in (16) is a lower bound t of f^0. At every α met in the solution process, an optimal basic solution of $Q(\alpha)$ and, hence, an optimal basic solution of $Q^*(\alpha)$ is available. Let V_k be the set of the basic solutions of $Q^*(\alpha^k)$ considered until step k. Thus, V_k which is initially empty, gains a new element. Every time this happens, problem (16) may be solved to find a new lower bound on f^0, say it t_k. Moreover, the minimum $f^{\downarrow}(\alpha^k)$ of problem $P(\alpha^k)$ for every $\alpha^k \in \Delta$, is obviously an upper bound on the optimal value f^0; let T_k be the minimum of all the previously found upper bounds. Obviously, at k-th step, the equality $T_k = t_k$ is a sufficient condition for optimality; moreover, $|T_k - t_k|$ is un upper bound of the difference between the current value of the objective function of P and its minimum. We can decide to stop the iterative procedure if such a difference is small enough, in a sense that can be specified case by case according to the meaning of the given complementarity problem.

4 The iterative method

The analysis developed in the previous sections allows us to define an iterative method for the minimization of the problem P.

General Algorithm

Step 0) (initialization).
Consider RP, the relaxed problem of P, and let $\bar{f}$ be the optimal value of RP (possibly $-\infty$). If $\bar{f} > -\infty$, let $(\bar{x}, \bar{y})$ be an optimal solution of RP. If $\langle \bar{x}, \bar{y} \rangle = 0$, then $(\bar{x}, \bar{y})$ is an optimal solution of P too; hence $\rightarrow$ **STOP**. Otherwise (i.e., if $\bar{f} = -\infty$ or $\langle \bar{x}, \bar{y} \rangle > 0$), set $k = 0$, $\Delta_0 = \Delta$, $T_{-1} = +\infty$, $t_{-1} = \bar{f}$. Go to Step 1.

Step 1) (solution of problem $P(\alpha^k)$).
Choose $\alpha^k \in \Delta_k$ and solve $P(\alpha^k)$. Let $(x(\alpha^k), y(\alpha^k))$ be an optimal solution of $P(\alpha^k)$ with optimal value $f^{\downarrow}(\alpha^k)$. If $f^{\downarrow}(\alpha^k) < T_{k-1}$ then $T_k = f^{\downarrow}(\alpha^k)$; otherwise $T_k = T_{k-1}$. Go to Step 2.

Step 2) (computation of a cut on the set Δ).
Let $\bar{\lambda}$ be an optimal solution of $P^*(\alpha^k)$. If $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$, then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. Otherwise, by means of inequality (13), determine the set $\{\alpha \in \Delta : \bar{\lambda}A \leq c(\alpha), \bar{\lambda}B \leq d(\alpha)\}$, that gives the vectors α to be rejected in the sequel. Let[1] $\Delta_{k+1} = \Delta_k \setminus \{\alpha \in \Delta : \bar{\lambda}A \leq c(\alpha), \bar{\lambda}B \leq d(\alpha)\}$. If $\Delta_{k+1} = \varnothing$, then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. (Remark that $\Delta_{k+1} = \varnothing$ is implied by $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$). Otherwise, go to Step 3.

Step 3) (computation of a lower bound).
Determine an optimal solution $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ of the problem $Q^*(\alpha^k)$ and let t the minimum in (16) obtained by adding the (basic) solution $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ to the set $\bar{V}$. If $t > t_{k-1}$, then $t_k = t$; otherwise $t_k = t_{k-1}$. If $T_k = t_k$ then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. Otherwise, set $k = k + 1$ and go to Step 1.

Remark 2. The following observations are worth noting:

(a) In the solution of problem $P(\alpha)$, we need to choose the components of the vectors $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ to penalize the costs $c(\alpha)$ and $d(\alpha)$. To obtain such a penalization, the components of ρ and σ must be of order of magnitude greater than that of the elements of c and d. In the subsequent Example 1, any components of ρ and σ will be set equal to 10^{10}.

(b) In the formulation of problem $Q(\alpha)$ and hence of its dual $Q^*(\alpha)$, we need to choose the values of the upper bounds $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ of which we have assumed the existence. The selection of such vectors is a crucial aspect of the method. Even if it is valid only for the particular case, a suggestion for this choice is given in Example 1 at Step 3 of Iteration 1.

(c) If, at a certain iteration k, we can prove that $f^{\downarrow}(\alpha) \geq f^{\downarrow}(\alpha^k)$ $\forall \alpha \in \Delta_{k+1}$, then $f^{\downarrow}(\alpha^k)$ is the optimal value of P and the current solution $(x(\alpha^k), y(\alpha^k))$ is an optimal solution. Suppose for example that $|I_x(\bar{\lambda})| + |I_y(\bar{\lambda})| = 1$; from the inequality (12) it follows that there is a unique index j such that any α to be considered in the sequel has $\alpha_j = 0$ if

[1] Recall that $\{\alpha \in \Delta : \bar{\lambda}A \leq c(\alpha), \bar{\lambda}B \leq d(\alpha)\}$ is the set Δ^k defined in (7).

$|I_x(\bar\lambda)| = 1$, and $\alpha_j = 1$ if $|I_y(\bar\lambda)| = 1$. If, by solving RP with the additional condition $x_j = 0$ when $\alpha_j = 1$ or $y_j = 0$ when $\alpha_j = 0$, we obtain a minimum greater than or equal to $f^{\downarrow}(\alpha^k)$, the current solution is an optimal one. Otherwise, such a minimum is a lower bound of f^0 and it will replace the current lower bound, computed at Step 3, if it is better.

(d) Recall that in the proposed decomposition method we should work with the subset $\bar\Delta$ of Δ, introduced in Definition 1. If a vector $\alpha \notin \bar\Delta$, the optimal value of the corresponding $P(\alpha)$ is of the same magnitude of ρ_j's and σ_j's. In this case, we skip Step 1 and we generate a new α. We refer to this case as a *null step*.

Example 1. Let us apply the iterative method to the following problem P:

$$\min(2x_1 + 2x_2 + x_3 + 2x_4 + 2y_1 + 2y_2 + 2y_3 + 2y_4)$$

$$\begin{cases} x_1 + x_4 + y_1 + y_2 + y_3 \geq 20 \\ x_1 + x_2 + y_1 + y_3 \geq 14 \\ x_2 + x_3 + y_1 \geq 10 \\ x_2 + y_3 + y_4 \geq 10 \\ x_1 + x_3 + y_4 \geq 5 \\ \langle x, y\rangle = 0 \\ x \geq 0, y \geq 0 \end{cases} \tag{17}$$

For the solution of some of the steps, the numerical software MATLAB has been used.

Step 0) The solution of the relaxed problem RP is $\bar x = \left(\frac{5}{2}, 0, \frac{5}{2}, 0\right)$, $\bar y = \left(\frac{15}{2}, 0, 10, 0\right)$, with optimal value $\bar f = \frac{85}{2}$. As $\langle \bar x, \bar y\rangle > 0$ and hence the complementarity condition is not satisfied, set $k = 0$, $\Delta_0 = \Delta = \{0,1\}^4$, $T_{-1} = +\infty$, $t_{-1} = \bar f = \frac{85}{2}$. Go to Step 1.

Iteration 1 with $k = 0$
Step 1) Let's take a first $\alpha^0 = (0,0,0,0)$. Let's solve $P(\alpha^0)$. We get

$$x(\alpha^0) = (5, 10, 0, 15)\ ,\ y(\alpha^0) = (0,0,0,0), \quad \text{with } f^{\downarrow}(\alpha^0) = 60.$$

As $f^{\downarrow}(\alpha^0) < T_{k-1}$ then $T_0 = 60$. Go to Step 2.
Step 2) The optimal solution of $P^*(\alpha^0)$ is $\bar\lambda = (2, 0, 0, 2, 0)$. The sets of indexes defined in (8) and (9) are $I_x(\bar\lambda) = \varnothing$ and $I_y(\bar\lambda) = \{3\}$. Since $I_x(\bar\lambda) \cup I_y(\bar\lambda) \neq \varnothing$, according to Theorem 4 the subsequent α's to be considered must satisfy the inequality $\alpha_3 \geq 1$ (see inequality (12)). Therefore, we can disregard all the α's with $\alpha_3 = 0$, namely

$$\alpha^0 = (0,0,0,0)\ ;\ (0,0,0,1)\ ;\ (0,1,0,0)\ ;\ (0,1,0,1)\ ;$$

$$(1,0,0,0)\ ;\ (1,0,0,1)\ ;\ (1,1,0,0)\ ;\ (1,1,0,1).$$

Go to Step 3.
Step 3) Let us consider $\{Q(\alpha)\}_{\alpha\in\Delta}$ where the upper bounds are

$$X = \left(20, 10, 20, \tfrac{31}{2}\right) \qquad \text{and} \qquad Y = \left(20, \tfrac{31}{2}, 25, 10\right).$$

The values of the upper bounds can be determined by combining the inequality

$$2x_1 + 2x_2 + x_3 + 2x_4 + 2y_1 + 2y_2 + 2y_3 + 2y_4 \leq 60,$$

given by the objective function less than or equal to its current value, with the inequalities coming from the constraints. For example, by considering the first constraint together with the above inequality, we get

$$40 + 2x_2 + x_3 + 2y_4 \leq 2(x_1 + x_4 + y_1 + y_2 + y_3) + 2x_2 + x_3 + 2y_4 \leq 60,$$

that gives the bounds $x_2 \leq 10$, $x_3 \leq 20$ and $y_4 \leq 10$. Similar bounds for the other variables may be obtained by means either of other constraints, taken singularly, or linear combinations of them. The optimal solution of $Q^*(\alpha^0)$, obtained from (13), is $(\bar{\lambda}; \bar{\mu}; \bar{\nu}) = (2, 0, 0, 2, 0; 0, 0, 0, 0; 0, 0, 2, 0)$. Therefore, problem (16) is equivalent to

$$t = \min_{\alpha, f} f \quad \text{s.t.} \quad \alpha \in \Delta \ , \ f \geq 60 - 2Y_3\alpha_3$$

with solution $t = 10$. Since $t = 10 < t_{-1}$, set $t_0 = t_{-1} = \frac{85}{2}$. As $T_0 \neq t_0$, let us continue in Step 3. $\Delta_1 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta \ : \ \alpha_3 \geq 1\}$. As $\Delta_1 \neq \varnothing$, set $k = 1$ and go to Step 1.

Iteration 2 with $k = 1$
Step 1) We have to select $\alpha^1 \in \Delta_1$. Let us choose $\alpha^1 = (0, 0, 1, 0) \in \Delta_1$. The optimal solution of $P(\alpha^1)$ is

$$x(\alpha^1) = (5, 10, 0, 15) \ , \ y(\alpha^1) = (0, 0, 0, 0), \quad \text{with } f^{\downarrow}(\alpha^1) = 60.$$

As $f^{\downarrow}(\alpha^1) = T_0$ then $T_1 = T_0 = 60$. Go to Step 2.
Step 2) By solving $P^*(\alpha^1)$ we get $\bar{\lambda} = (2, 0, 2, 0, 0)$. The sets of indexes defined in (8) and (9) are $I_x(\bar{\lambda}) = \{3\}$ and $I_y(\bar{\lambda}) = \{1\}$; $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. The subsequent vectors to be considered belong to the set $\Delta_2 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta \ : \ \alpha_3 \geq 1, \alpha_3 - \alpha_1 \ \leq 0\}$. Therefore, in the following analysis we can disregard the following α's

$$\alpha^1 = (0, 0, 1, 0) \ ; \ (0, 0, 1, 1) \ ; \ (0, 1, 1, 0) \ ; \ (0, 1, 1, 1).$$

Go to Step 3.
Step 3) The optimal solution of $Q^*(\alpha^1)$ is $(\bar{\lambda}; \bar{\mu}; \bar{\nu}) = (2, 0, 2, 0, 0; 0, 0, 1, 0; 2, 0, 0, 0)$. Therefore, problem (16) becomes

$$t = \min_{\alpha, f} f \quad \text{s.t.} \quad \alpha \in \Delta \ , \ f \geq 60 - 2Y_3\alpha_3 \ ; \ f \geq 60 - X_3(1 - \alpha_3) - 2Y_1\alpha_1$$

with solution $t = 20$; observe that this value of t ($t = 20$) improves the previous one ($t = 10$). Since $t = 20 < t_0 = \frac{85}{2}$, set $t_1 = t_0$. As $T_1 \neq t_1$, let us continue in Step 3. As $\Delta_2 \neq \varnothing$, set $k = 2$ and go to Step 1.

Iteration 3 with $k = 2$
Step 1) We have to choose $\alpha^2 \in \Delta_2$; let us take $\alpha^2 = (1, 0, 1, 0)$. The solution of $P(\alpha^2)$ produces a null step (see item (d) of Remark 2). Let's choose a different element in Δ_2: $\alpha^2 = (1, 0, 1, 1)$. The optimal solution of $P(\alpha^2)$ is

$$x(\alpha^2) = (0, 0, 0, 0) \ , \ y(\alpha^2) = (10, 0, 10, 5), \quad \text{with } f^{\downarrow}(\alpha^2) = 50.$$

As $f^{\downarrow}(\alpha^2) < T_1$ then $T_2 = 50$. Go to Step 2.
Step 2) By solving $P^*(\alpha^2)$ we get $\bar{\lambda} = (2, 0, 0, 0, 2)$. The sets of indexes defined in (8) and (9) are $I_x(\bar{\lambda}) = \{1, 3\}$ and $I_y(\bar{\lambda}) = \varnothing$; $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. The set Δ_3 is given by the solutions of the following system

$$\begin{cases} \alpha \in \Delta \\ \alpha_3 \geq 1 \\ \alpha_3 - \alpha_1 \leq 0 \\ \alpha_1 + \alpha_3 \leq 1 \end{cases}$$

As Δ_3 is empty, the current solution $(x(\alpha^2), y(\alpha^2))$ is an optimal solution of P. $\rightarrow$ **STOP**.

5 Numerical experiments

By using MATLAB we have developed some numerical tests. The decomposition method requires to process the set of binary vectors $\alpha \in \Delta$; therefore, a crucial aspect is the choice of $\alpha^k \in \Delta_k$ in Step 1. Since Δ_k is given by a system of inequalities, each one defined by (12) (see Theorem 4), to choose a new $\alpha^k \in \Delta_k$ we solve such a system by using the MATLAB routine INTLINPROG, where the objective function is a constant or is arbitrary.

We have tested randomly generated problems of different dimensions (m is the number of linear constraints and $2n$ is the number of total variables). In order to generate the test problems we used the MATLAB function RAND for the random construction of matrices A, B and vectors c, d, b. With the aim of avoiding very dense matrices we converted to zero a fixed proportion of elements. The results of the tests with $n = 50, 100, 200$ and $m = n/2$ are shown in the following Tables 1-2-3 which report in the $\bar{f}$ column the optimal value of the relaxed problem RP, in the $fval$ column the optimal value of the complementarity problem P, in the *niter* column the number of iterations the method took to terminate (a maximum number of 350 iterations has been set), in the *null* column the number of iterations that do not generate a cut (null steps) and in the *time* column the total computational time.

Tables 1-2-3 may be compared with Tables 6.1-6.2-6.3 of [10] and Tables 5-6 of [32]; concerning Tables 5-6 of [32], we have noticed that the computational time is roughly of the same order of ours, while in Tables 6.1-6.2-6.3 of [10] the computational time is not reported. Particular attention should be paid to the number of iterations corresponding to the number of the choices of α^k for different sizes of the problem: our tests show that in general there is not an exponential growth w.r.t. the number of iterations, as the cutting technique on the α's seems to be efficient. The algorithm stops when the set Δ_k turns out to be empty; for two problems (Problem n.1 in Table 2 and Problem n.8 in Table 3) the maximum number of iterations is reached. We observe that the number of iterations requested by our method is in general comparable with the number of LPs considered in Tables 6.1-6.2-6.3 of [10] and with the number of nodes reported in Tables 5-6 of [32].

#	$\bar{f}$	$fval$	$niter$	$null$	$time(sec)$
1	7.8556	7.9671	7	0	2.176
2	14.6305	14.6889	14	0	2.440
3	1.6133	1.7718	16	0	2.441
4	0.4049	0.4605	10	0	2.085
5	5.6777	6.7316	28	0	2.913
6	8.3240	8.4158	8	0	2.042
7	7.2269	7.2621	7	0	1.924
8	4.4460	4.5095	11	0	2.016
9	15.0657	15.1771	97	0	6.148
10	4.3320	4.3695	28	0	2.747

Table 1. Test problems with $m = 25, n = 50$

#	$\bar{f}$	$fval$	$niter$	$null$	$time(sec)$
1	3.2411	4.5590	350	0	–
2	11.4017	12.2509	110	0	6.961
3	9.2597	9.4326	11	0	2.129
4	6.4122	6.9625	116	0	8.489
5	8.7454	9.3954	22	0	2.901
6	5.2693	5.5325	11	0	2.320
7	6.6466	6.8519	17	0	2.366
8	10.8341	11.5061	17	0	2.391
9	7.0492	7.4918	20	0	2.552
10	1.4972	1.6253	16	0	2.423

Table 2. Test problems with $m = 50, n = 100$

#	$\bar{f}$	$fval$	$niter$	$null$	$time(sec)$
1	−0.5737	−0.2342	22	0	5.148
2	9.4322	9.8799	130	0	12.476
3	10.5890	10.6160	26	0	5.106
4	5.4462	5.7384	8	0	3.005
5	5.8129	6.7195	142	0	20.624
6	3.0496	3.3473	20	0	3.887
7	1.4058	3.1474	12	0	3.570
8	4.9035	6.8678	350	0	–
9	8.7765	8.8197	12	0	2.525
10	8.0579	8.6748	28	0	4.813

Table 3. Test problems with $m = 100, n = 200$

We point out that for problems where $m \approx n$ the convergence is not always reached. In Table 4 we report the results of some experiments with $m = 40$ and $n = 50$. In half of the cases we have convergence, while in two tests (5-9) of the other half, where the maximum number of iterations is reached, we obtain a good approximation of the optimal value. For the remaining three cases (2-4-7) the algorithm is not successful; we remark that in such cases there is a significant number of null steps.

#	$\bar{f}$	$fval$	$niter$	$null$	$time(sec)$
1	7.0898	7.1985	11	0	3.770
2	29.1028	40.4113	350	189	–
3	13.9111	14.6377	144	0	9.023
4	25.4643	29.5511	350	202	–
5	22.5266	22.7318	350	174	–
6	5.5505	5.6451	17	0	4.704
7	33.8596	41.6825	350	176	–
8	14.9497	15.2073	44	0	4.064
9	10.9119	11.4158	350	147	–
10	17.3990	18.1171	31	0	3.316

Table 4. Test problems with $m = 40, n = 50$

6 Conclusions

We have recalled a decomposition method for a linear problem with complementarity constraints in a sequence of parameterized problems. The method allows to define an algorithm that leads to an optimal solution or to an approximation of it providing an estimate of the error. For problems of different dimensions we have implemented some numerical experiments which show that in most of the cases the method converges linearly w.r.t. the dimension n of the problem. A completely unified MATLAB code fully implementing the whole algorithm is still in progress. This is a possible further development, together with some more numerical testing.

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