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POISSON VOTING GAMES: PROPORTIONAL RULE

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ABSTRACT. We analyze strategic voting under pure proportional rule and two candidates, embedding the basic spatial model into the Poisson framework of population uncertainty. We prove that the Nash equilibrium exists and is unique. We show that it is characterized by a cutpoint in the policy space that is always located between the mean of the two candidates' positions and the median of the distribution of voters' types. We also show that, as the expected number of voters goes to infinity, the equilibrium converges to that of the complete information case.

KEY WORDS. Poisson games, strategic voting, proportional rule.

JEL CLASSIFICATION. C72, D72.

1. INTRODUCTION

Citizens often act strategically in their voting behavior. Their action influences the competing candidates' positioning choice, determines the electoral outcome and, consequently, the policy that is implemented according to the electoral rule in force. In particular, the outcome of an election held under proportional rule is a convex combination of the policies proposed by each candidate, with weights proportional to the number of votes that they obtain. Such an outcome function reflects the fact that every voted candidate participates to the decisional process to some extent. In this paper we analyze voters' strategic behavior under pure proportional rule when there are two candidates.

In a mass election a voter is typically unaware of the exact number of other voters in the population, but may have only some probabilistic information about it. To capture this kind of uncertainty we use the Poisson model introduced by Myerson (1998), in which the number of agents is not common knowledge but is a Poisson distributed random variable. As it seems reasonable to consider the size of the electorate as random from the viewpoint of an individual, voting settings were the immediate application of Poisson games (Myerson, 2000, 2002) and, thenceforth, the Poisson model has become the standard tool to model strategic voting in large elections (see, e.g., Krishna and Morgan, 2011, 2012; Bouton and Castanheira,

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2012; Bouton, 2013; Bouton and Gratton, 2015; Herrera et al., 2014). However, the convenient properties of Poisson games have been exploited to model also more general economic environments with a large number of actors (see Satterthwaite and Shneyerov, 2007; Makris, 2008, 2009; Ritzberger, 2009; Jehiel and Lamy, 2015 among others).

In this context of population uncertainty, we study the basic spatial model in which the positions of two candidates are exogenously given, and voters have to choose whom to vote for. Voters can be of different types, each type being characterized by a bliss point. The policy that is implemented after the voting stage is a weighted average of the candidates' positions, where weights are equal to the corresponding shares of votes. The further the implemented policy is from a type's bliss point, the worse off that type is.

In the context of complete information on the population's size, voters' strategic behavior in a pure proportional system with multiple candidates has been analyzed in De Sinopoli and Iannantuoni (2007). They show that, as the number of voters grows to infinity, in equilibrium basically voters split in two and only the two extremist candidates take votes. The policy outcome is precisely the "cutpoint outcome", that is, the outcome that is implemented when all the voters whose bliss points are on its left vote for the leftmost candidate, and all the voters whose bliss points are on its right vote for the rightmost one. This result has been applied to other electoral systems exhibiting positive degrees of power sharing in Meroni (2017), where the limit equilibrium outcome is characterized for the two-candidate case.[1]

With complete information, the equilibrium and the equilibrium outcome are unique at the limit, when there is a continuum of voters. In that case it is possible to talk about strategic voting–even if, in principle, nobody can affect the outcome–because the game with a continuum of players can be seen as the limit of a sequence of finite games. Thus, voters' strategic behavior is precisely identified by the limit of a sequence of equilibria of those games. Before the limit, when the number of players is finite and individuals' incentives to act strategically can be fully investigated, there is a plethora of equilibria and the equilibrium outcome is indeterminate.

The Poisson model allows to fully capture voters' strategic incentives, because the probability that an individual is decisive to the outcome is always positive. Very importantly, in the current framework such a model turns out to solve the indeterminacy of equilibrium entailed in the complete information case. In fact, we show that the equilibrium exists and is unique for every expected number of voters. It is

[1] Both in De Sinopoli and Iannantuoni (2007) and in Meroni (2017), voters' preferences are required to be single-peaked and representable by a continuously differentiable utility function. In this work we also require them to be symmetric with respect to the bliss point. (This assumption will be needed in the proof of Lemma 1.)

self-evident how such a uniqueness is essential for applications, as it allows to embed the simple voting game into more structured institutional frameworks of which that game represents just a stage.

Precisely, the equilibrium is characterized by a "cutpoint" type so that voters divide exactly in two, those to the left of the cutpoint voting for the leftist candidate and those to its right voting for the rightist one. This is due to the fact that, in the Poisson environment, if a type prefers the rightist candidate to the leftist one for a given behavior of the population then all the types located on her right will strictly do so. Furthermore, if a type is indifferent between the two candidates given a strategy of the others and this strategy changes so that the leftist candidate's vote share increases, then that type will strictly prefer the rightist candidate to the leftist one and, therefore, the indifferent type will move on her left. A consequence is that the best reply correspondence is a continuous decreasing function, hence existence and uniqueness of equilibrium follow. We analyze the characteristics of the equilibrium with respect to the parameters of the model and show that the cutpoint always lies between the average of the two candidates' positions and the median of the distribution of types.

We study the behavior of the equilibrium cutpoint as the expected number of voters varies. As this vanishes, the equilibrium converges to the mean of the two candidates' positions, that is, to the equilibrium under sincere voting. Indeed, the probability for an individual of being the only voter converges to one, and the type indifferent between the two candidates' platforms (i.e. the cutpoint type) is exactly the one located in their middle. On the other hand, as the expected number of voters grows to infinity, the equilibrium converges to the equilibrium of the model with a continuum of players studied in De Sinopoli and Iannantuoni (2007). In fact, for sufficiently large values of the expected number of players, the actual policy outcome remains sufficiently close to its expected value, and an individual vote can only marginally shift it. So basically all the types to the left of that value prefer to vote for the leftist candidate and all the types to its right prefer to vote for the rightist one. Furthermore, for sufficiently large values of the expected number of players, the expected policy outcome remains sufficiently close to the cutpoint outcome, so the equilibrium cutpoint converges to it.

Finally, we remove the hypothesis that candidates' platforms are exogenously given and explore their strategic decision of which position to take in the policy space in the case in which they are policy-motivated. Under such an assumption on candidates' preferences, the form of the outcome function implies that they potentially face a trade-off. The more moderate is the policy that they choose, the higher is their share of votes but the less extreme is the position to which such a weight is assigned. We show that, as the expected number of voters grows to infinity, candidates always choose extreme positions in equilibrium, in line with the result of the model with a continuum of strategic voters (Meroni, 2017). However, this does not hold for every value of that parameter. In fact, as the expected number of voters vanishes, candidates' positions may converge depending on the distribution of voters' types, in line with the result of the model with a continuum of sincere voters (Saporiti, 2014; Matakos et al., 2016).

The paper is organized as follows. The basic spatial model embedded in the Poisson framework is described in Section 2. In Section 3 we analyze the equilibrium of the model, while in Section 4 we study how it varies as the expected number of voters increases. We examine candidates' electoral competition in Section 5.

2. The model

Let the *policy space* be represented by the closed interval $\mathbb{X} = [0,1]$. Candidates L and R announce simultaneously their positions $x_L, x_R \in \mathbb{X}$, to which they are committed. Given these platforms, every voter chooses one of them. The resulting electoral outcome determines the policy that is implemented according to the proportional electoral rule. Let v be the share of votes gained by candidate L. The *policy outcome* is given by

$$x(v, x_L, x_R) = v x_L + (1-v) x_R,$$

i.e. it is a weighted average of the two candidates' positions where weights are given by the corresponding vote shares.

We assume that the number of voters is a random variable drawn from a Poisson distribution with mean n. The probability that the actual number of voters is k, hence, is equal to

$$p(k \mid n) = \frac{e^{-n} n^k}{k!}.$$

Each voter has a *type* that determines her preferences over the possible outcomes. We identify the set of types with $\Theta = [0,1]$, where $\theta_t \in \Theta$ represents the bliss point of type t voters, and we denote with θ the typical element of Θ. Each voter's type is independently drawn from Θ according to some continuous probability distribution F. That is, for every element $\theta \in \Theta$, the probability that a randomly sampled voter is of type $\theta_t \leq \theta$ is given by $F(\theta)$ and it is independent of the number and types of all the other voters. The decomposition property of the Poisson distribution implies that the number of voters with type smaller than or equal to θ is a Poisson random variable with parameter $nF(\theta)$.[2]

We denote each voter's action set with $C = \{L, R\}$. An *action profile* $z = (z_L, z_R) \in \mathbb{Z}_+^2$ is a vector that specifies for each action $c \in C$ the number of voters who have chosen that action, z_c. The *payoff* to a voter of type t who chooses action c when the

[2] For the properties of Poisson games see Myerson (1998).

realization of the rest of the population's behavior is z is a function of the distance between her bliss point and the induced policy. That is,

$$u_t(c,z) = f(|\theta_t - \hat{x}(c,z)|),$$

where $\hat{x}(c,z)$ is the outcome that the voter induces choosing c when $z = (z_L, z_R)$ is the action profile of the other voters, i.e.

$$\hat{x}(L,z) = \frac{z_L+1}{z_L+z_R+1}x_L + \frac{z_R}{z_L+z_R+1}x_R$$

and

$$\hat{x}(R,z) = \frac{z_L}{z_L+z_R+1}x_L + \frac{z_R+1}{z_L+z_R+1}x_R.$$

Note that $\hat{x}(R,z) > \hat{x}(L,z)$ for every z. The function $f : [0,1] \to \mathbb{R}$ is assumed to be bounded, continuous, strictly decreasing, and strictly concave in its argument.

When z is the entire population's realized behavior, let $x(z)$ denote the outcome that is implemented, i.e.

$$x(z) := x\left(\frac{z_L}{z_L+z_R} \mid x_L, x_R\right).$$

Moreover, let $v_t(x(z))$ be the corresponding payoff to a type t voter. Note that $v_t(x(z)) = u_t(L,z') = u_t(R,z'')$ where $z' = (z_L - 1, z_R)$ and $z'' = (z_L, z_R - 1)$. The function v_t is strictly concave because f is strictly concave. We also assume that v_t is differentiable in its argument.[3]

A *strategy function* (sometimes simply referred to as *strategy*) is a measurable function $\sigma : \Theta \to \Delta(C)$, which associates to each type a probability distribution on C, i.e. a mixed action. The mixed action employed by players of type t is denoted σ_t.[4] Strategy σ induces the *average behavior* $\tau(\sigma) \in \Delta(C)$ defined by

$$\tau_c(\sigma) = \int_\Theta \sigma_t(c)\,dF(\Theta)$$

for each $c \in C$. When voters vote according to σ, $\tau_c(\sigma)$ is the probability that a randomly sampled voter chooses action c. Note that $\tau_c(\sigma)$ is also the expected share of votes received by party c. The number of voters who choose c is a Poisson random variable with mean $n\tau_c(\sigma)$ and it is independent of the number of voters that choose

[3] This assumption will be used in the proof of Proposition 4. We make it for simplicity to directly apply the mean value theorem, but it can be eliminated at the cost of using a more cumbersome version of that theorem.

[4] Differently from Myerson (2000), we do not use distributional strategies but we employ the equivalent definition of strategies in terms of measurable functions (see Milgrom and Weber, 1985). We believe that the specification of the behavior of each single type is more appropriate in Poisson games, where agents are characterized only by their type and identities are unknown. Of course, all the strategy functions that differ only in a finite subset of Θ are outcome equivalent, as such a subset has measure zero being the function F continuous. Hence, a distributional strategy identifies a class of equivalence of our strategy functions.

the other action. bounded Under the population's average behavior $\tau \in \Delta(C)$, the probability of the action profile $z \in Z(C)$ is therefore given by

$$P(z \mid \tau) = e^{-n} n^{z_L + z_R} \frac{\tau_L{}^{z_L} \tau_R{}^{z_R}}{z_L! z_R!}.$$

From the perspective of any player of any type, the number of other players (not including herself) who choose action c is also a Poisson random variable with mean $n\tau_c$ due to the environmental equivalence property of Poisson games. Thus, the *expected payoff* to a type t voter who votes for c when the other voters vote according to τ is given by

$$U_t(c,\tau) = \sum_{z \in Z(C)} P(z \mid \tau) u_t(c,z).$$

Following standard terminology, we say that action c is a *pure best response* for type t voters against the population's behavior τ if $U_t(c,\tau) \geq U_t(c',\tau)$, $c' \neq c$. We denote the set of such actions $\text{PBR}_t(\tau)$. The set of type t voters' *best responses* against τ is $\text{BR}_t(\tau) = \Delta(\text{PBR}_t(\tau))$. Let $\text{BR}(\tau)$ be the collection of strategy functions such that $\sigma_t \in \text{BR}_t(\tau)$ for every t.

Definition 1. The strategy function σ is a *Nash equilibrium* if $\sigma \in \text{BR}(\tau(\sigma))$.

3. Equilibrium analysis

In this section we prove that the Nash equilibrium of the model exists and is unique. We show that in equilibrium voters split in two, those to the left of a cutpoint type voting for candidate L and those to its right voting for candidate R.

Formally, we say that strategy σ is a *cutpoint strategy* if there is a value $\theta \in \Theta$ such that $\sigma_t(L) = 1$ if $\theta_t < \theta$ and $\sigma_t(R) = 1$ if $\theta_t > \theta$. That is, all the voters to the left of θ vote for candidate L and all the voters to the right of θ vote for candidate R. We will usually refer to a cutpoint strategy with the correspondent cutpoint. Note that a given θ identifies an infinite number of strategy functions that differ only at the point $\theta_t = \theta$, which are therefore equivalent.

We begin proving that, for any given aggregate behavior τ, the best response against τ is a cutpoint strategy and is unique, i.e. the best response correspondence is a function. In fact, the assumptions on the utility function f imply that, for a given average behavior of the population, if a type prefers to vote for candidate R than for candidate L then all the types on her right will (strictly) do so. Not only this holds for a given average behavior of the population, but it holds for every realization of such a behavior. Therefore, we prove first such a stronger result.

Lemma 1. *For every $z \in Z(C)$ and every $\theta_{t'} > \theta_t$, if type t prefers R to L given z then type t' strictly prefers R to L given z.*

Proof. Fix $z \in Z(C)$ and let $\Delta_t = u_t(R,z) - u_t(L,z)$ for every t. Recall that $\hat{x}(R,z) > \hat{x}(L,z)$ and consider $\theta_{t'} > \theta_t$. There are six possible ways in which the points $\hat{x}(R,z)$,

$\hat{x}(L,z)$, $\theta_{t'}$, and θ_t are ordered in the interval $[0,1]$. For each case, the strict decreasingness and the strict concavity of the utility function f imply the following. If $\hat{x}(L,z) < \hat{x}(R,z) \leq \theta_t < \theta_{t'}$ then $0 < \Delta_t < \Delta_{t'}$, while if $\theta_t < \theta_{t'} \leq \hat{x}(L,z) < \hat{x}(R,z)$ then $\Delta_t < \Delta_{t'} < 0$. If $\theta_t \leq \hat{x}(L,z) < \hat{x}(R,z) \leq \theta_{t'}$ then $\Delta_t < 0 < \Delta_{t'}$, while if $\hat{x}(L,z) \leq \theta_t < \theta_{t'} \leq \hat{x}(R,z)$ then either $\Delta_t < 0 < \Delta_{t'}$ or $0 < \Delta_t < \Delta_{t'}$ or $\Delta_t < \Delta_{t'} < 0$. Finally, if $\hat{x}(L,z) \leq \theta_t \leq \hat{x}(R,z) \leq \theta_{t'}$ then either $\Delta_t < 0 < \Delta_{t'}$ or $0 < \Delta_t < \Delta_{t'}$, while if $\theta_t \leq \hat{x}(L,z) \leq \theta_{t'} \leq \hat{x}(R,z)$ then either $\Delta_t < 0 < \Delta_{t'}$ or $\Delta_t < \Delta_{t'} < 0$. In all the cases, we have $\Delta_{t'} > \Delta_t$, that is, $u_{t'}(R,z) - u_{t'}(L,z) > u_t(R,z) - u_t(L,z)$. It follows that if $u_t(R,z) \geq u_t(L,z)$ then $u_{t'}(R,z) > u_{t'}(L,z)$. □

Recall that the environmental equivalence property of Poisson games implies that, for every aggregate behavior τ and action profile z, each type attaches the same probability $P(z \mid \tau)$ to z. It follows readily from Lemma 1 that, for every τ and every $\theta_{t'} > \theta_t$, if $U_t(R,\tau) \geq U_t(L,\tau)$ then $U_{t'}(R,\tau) > U_{t'}(L,\tau)$.

Lemma 2. *For every $\tau \in \Delta(C)$ and every $\theta_{t'} > \theta_t$, if type t prefers R to L given τ then type t' strictly prefers R to L given τ.*

A direct consequence of Lemma 2 is that $\mathrm{BR}(\tau)$ is a cutpoint strategy. In fact, for a given τ, there must be a cutpoint such that all the voters to its right will strictly prefer R over L, while all the voters to its left will strictly prefer L over R. The type whose bliss point is exactly the cutpoint will be indifferent between L and R. The previous lemmas imply also that there cannot be more than one indifferent type. So, for every τ, the cutpoint that characterizes $\mathrm{BR}(\tau)$ is unique.

Proposition 1. *For every τ, $\mathrm{BR}(\tau)$ is a cutpoint strategy and it is unique.*

It follows that every Nash equilibrium of the model is a cutpoint strategy and is therefore identified by a point in Θ. Existence and uniqueness of the equilibrium will be obtained exploiting this fact, which allows to consider the restriction of the best reply correspondence to cutpoint strategies. That is, we can consider the function $\mathrm{BR}(\theta) : [0,1] \to [0,1]$. In particular, we can show that such a function is continuous in θ, so a standard application of the Brouwer fixed point theorem implies existence. Moreover, we can show that the best reply function is decreasing in θ, which implies uniqueness.

Theorem 1. *There exists a unique Nash equilibrium.*

Proof. Given a cutpoint strategy θ, the probability that a randomly sampled voter votes for candidate L (resp. R) is given by $\tau_L(\theta) = F(\theta)$ (resp. $\tau_R(\theta) = 1 - F(\theta)$). Therefore, we have

$$U_t(c,\tau(\theta)) = \sum_{z \in Z(C)} e^{-n} n^{z_L + z_R} \frac{F(\theta)^{z_L}[1-F(\theta)]^{z_R}}{z_L! z_R!} u_t(c,z)$$

for every $\theta_t, \theta \in \Theta$ and $c \in C$.

By Lemma 1, BR(θ) is characterized by the point $\theta_t \in \Theta$ such that $U_t(R,\tau(\theta)) - U_t(L,\tau(\theta)) = 0$, that is,

$$\sum_{z \in Z(C)} e^{-n} n^{z_L+z_R} \frac{F(\theta)^{z_L}[1-F(\theta)]^{z_R}}{z_L! z_R!}[u_t(R,z) - u_t(L,z)] = 0. \quad (1)$$

It is easy to see that BR(θ) is continuous in θ since $F(\theta)$ is continuous in θ. By the Brouwer fixed point theorem, BR(θ) has at least one fixed point and, hence, an equilibrium exists.

Now, we show that the best response function is decreasing in θ. In fact we can prove that, given two strategies θ and θ' such that $\theta' > \theta$, the type who is indifferent between R and L given θ does strictly prefer R to L given θ'. That is, if $U_t(R,\theta) = U_t(L,\theta)$ then $U_t(R,\theta') > U_t(L,\theta')$. Then, by Lemma 1, the indifferent type against θ' will be on the left of type t, i.e. $BR(\theta') < BR(\theta)$.

We start proving that, for every t and every z and z' such that $z = (z_L, z_R)$ and $z' = (z_L+1, z_R-1)$,

$$u_t(R,z') - u_t(L,z') > u_t(R,z) - u_t(L,z). \quad (2)$$

Rearranging terms we have

$$u_t(R,z') + u_t(L,z) > u_t(L,z') + u_t(R,z),$$

which is equivalent to

$$v_t(x(z_L+1,z_R)) > \frac{1}{2}v_t(x(z_L+2,z_R-1)) + \frac{1}{2}v_t(x(z_L,z_R+1)) \quad (3)$$

as $u_t(R,z') = u_t(L,z)$. Since

$$x(z_L+1,z_R) = \frac{1}{2}x(z_L+2,z_R-1) + \frac{1}{2}x(z_L,z_R+1)$$

condition (3) and, therefore, condition (2) follow directly from the strict concavity of the function v_t.

Let $\theta' > \theta$ and $\Delta_t(z) = u_t(R,z) - u_t(L,z)$. Moreover, let

$$B(z_L \mid m, F(\theta)) = \binom{m}{z_L} F(\theta)^{z_L}[1-F(\theta)]^{m-z_L}$$

be the probability that the number of voters choosing L is equal to z_L given that the population's size is m and given the cutpoint strategy θ. Condition (1) is equivalent to

$$\sum_{m=0}^{\infty} \frac{e^{-n}n^m}{m!} \sum_{z_L=0}^{m} B(z_L \mid m, F(\theta))\Delta_t(z_L, m-z_L) = 0. \quad (4)$$

Note that the binomial distribution with parameters m and $F(\theta')$ first order stochastically dominates the one with parameters m and $F(\theta)$, i.e.

$$\sum_{z_L=0}^{\bar{z}} \binom{m}{z_L} F(\theta')^{z_L}[1-F(\theta')]^{m-z_L} < \sum_{z_L=0}^{\bar{z}} \binom{m}{z_L} F(\theta)^{z_L}[1-F(\theta)]^{m-z_L}$$

for every $\bar{z} \in \mathbb{Z}_+$. Fix m and let for simplicity $B(z_L \mid m, F(\theta)) = B(z_L \mid \theta)$ and $\Delta_t(z_L, m - z_L) = \Delta_t(z_L)$. We have

$$\begin{aligned}
&\sum_{z_L=0}^{m} B(z_L \mid \theta)\Delta_t(z_L) < \\
&B(0 \mid \theta')\Delta_t(0) + [B(1 \mid \theta) + B(0 \mid \theta) + B(0 \mid \theta')]\Delta_t(1) + \sum_{z_L=2}^{m} B(z_L \mid \theta)\Delta_t(z_L) < \\
&B(0 \mid \theta')\Delta_t(0) + B(1 \mid \theta')\Delta_t(1) + [B(2 \mid \theta) + B(0 \mid \theta) - B(0 \mid \theta') + B(1 \mid \theta) - B(1 \mid \theta')]\Delta_t(2) + \\
&+ \sum_{z_L=3}^{m} B(z_L \mid \theta)\Delta_t(z_L) < \cdots < \\
&\sum_{z_L=0}^{m-1} B(z_L \mid \theta')\Delta_t(z_L) + \left[B(z_L \mid \theta) + \sum_{z_L=0}^{m-1} B(z_L \mid \theta) - \sum_{z_L=0}^{m-1} B(z_L \mid \theta') \right] \Delta_t(m) = \\
&\sum_{z_L=0}^{m} B(z_L \mid \theta')\Delta_t(z_L) + \left[\sum_{z_L=0}^{m} B(z_L \mid \theta) - \sum_{z_L=0}^{m} B(z_L \mid \theta') \right] \Delta_t(m) = \\
&\sum_{z_L=0}^{m} B(z_L \mid \theta')\Delta_t(z_L),
\end{aligned}$$

where the first inequality derives from $B(0 \mid \theta) - B(0 \mid \theta') > 0$ and $\Delta_t(1) > \Delta_t(0)$, the second one from $B(0 \mid \theta) - B(0 \mid \theta') + B(1 \mid \theta) - B(1 \mid \theta') > 0$ and $\Delta_t(2) > \Delta_t(1)$, and similarly for every term of the summation until the last inequality, which derives from $\sum_{z_L=0}^{m-1} B(z_L \mid \theta) - \sum_{z_L=0}^{m-1} B(z_L \mid \theta') > 0$ and $\Delta_t(m) > \Delta_t(m-1)$.

We can conclude that

$$U_t(R, \theta') - U_t(L, \theta') > U_t(R, \theta) - U_t(L, \theta)$$

and, hence, if $U_t(R, \theta) = U_t(L, \theta)$ then $U_t(R, \theta') > U_t(L, \theta')$. That is, the type who is indifferent between L and R given θ will strictly prefer R to L given $\theta' > \theta$, so $BR(\theta') < BR(\theta)$. It follows that the best response function is decreasing, and therefore the Nash equilibrium is unique. □

We can now show some characteristics of the Nash equilibrium relatively to the parameters of the model. In particular, we prove that the cutpoint that characterizes it is always located between the median of the distribution of types and the mean of the two parties' positions. To this end, let θ^* be the equilibrium cutpoint, let $\bar{\theta} = \frac{x_L + x_R}{2}$ and $F(\theta_m) = \frac{1}{2}$.

Proposition 2. *If $\theta_m > \bar{\theta}$ then $\bar{\theta} < \theta^* < \theta_m$, while if $\theta_m < \bar{\theta}$ then $\theta_m < \theta^* < \bar{\theta}$.*

Proof. Take the cutpoint strategy θ_m. Clearly, $\tau_L(\theta_m) = \tau_R(\theta_m) = \frac{1}{2}$. Therefore any two symmetric realizations, i.e. any two realizations z and z' such that $z = (z_1, z_2)$ and $z' = (z_2, z_1)$, have the same probability given θ_m.

Consider the type t whose bliss point is $\bar{\theta}$. For every realization z such that $z_L = z_R$ we have $u_t(L, z) = u_t(R, z)$, while for every symmetric realizations z and z' we have $u_t(L, z) - u_t(R, z) = u_t(R, z') - u_t(L, z')$. It follows that type t is indifferent between choosing L and choosing R given θ_m, that is, $BR(\theta_m) = \bar{\theta}$. Since the best reply function is continuous and decreasing, both when $\theta_m > \bar{\theta}$ and when $\theta_m < \bar{\theta}$ its fixed point is between these two values. Of course, if $\theta_m = \bar{\theta}$ then $\theta^* = \theta_m = \bar{\theta}$. □

The result in Proposition 2 holds for every value of the expected number of voters n. In the next section we analyze the behavior of the equilibrium cutpoint as n grows to infinity.

4. Large electorate

If we fix x_L, x_R, and F, the Nash equilibrium θ^* is a function of the expected number of players n. In this section we study the behavior of this function $\theta^*(n)$ as the electorate becomes large. We show that, as n goes to infinity, the equilibrium converges to the equilibrium of the model with a continuum of voters studied in De Sinopoli and Iannantuoni (2007).

For the sake of completeness, we analyze first the other limit case, that is, the behavior of the equilibrium as the expected number of voters vanishes. The next proposition establishes that, as n goes to zero, the equilibrium converges to the mean of the two candidates' positions, i.e. voters vote sincerely. In this case, indeed, the probability for every voter of being the only player in the game converges to one, and the type that is located in the middle of the two proposed platforms is the one indifferent between them.

Proposition 3. *The limit point of $\theta^*(n)$ as n goes to zero is $\bar{\theta}$.*

Proof. Given n and the corresponding equilibrium $\theta^*(n)$, the equilibrium condition of the indifferent type t such that $\theta_t = \theta^*(n)$ can be written as

$$U_t(R,\theta^*(n)) - U_t(L,\theta^*(n)) =$$
$$P((0,0) \mid \tau(\theta^*(n)))[u_t(R,(0,0)) - u_t(L,(0,0))] +$$
$$\sum_{\substack{z \in Z(C) \\ z \neq (0,0)}} P(z \mid \tau(\theta^*(n)))[u_t(R,z) - u_t(L,z)] = 0.$$

Suppose that the limit point of $\theta^*(n)$ as n goes to zero is different form $\bar{\theta}$. This implies that, for n sufficiently small, there exists a $\delta > 0$ such that $|\theta^*(n) - \delta| > \bar{\theta}$. Assume without loss of generality that $\theta^*(n) > \bar{\theta} + \delta$. For $\theta_t = \theta^*(n)$, we have $u_t(R,(0,0)) - u_t(L,(0,0)) = v_t(x_R) - v_t(x_L) = K$ for some $K > 0$. Since

$$\lim_{n \to 0} P((0,0) \mid \tau) = \lim_{n \to 0} e^{-n} = 1$$

for every $\tau \in \Delta(C)$, we have that, for $\theta_t = \theta^*(n)$,

$$\lim_{n \to 0} [U_t(R,\theta^*(n)) - U_t(L,\theta^*(n))] = K > 0;$$

that is, at the limit type t is not indifferent between L and R. The desired contradiction follows. □

Let the *cutpoint outcome* $\hat{\theta}$ be the unique policy outcome implemented when all the types to its left vote for candidate L and all the types to its right vote for candidate R, i.e., the unique solution to

$$\hat{\theta} = F(\hat{\theta})x_L + [1 - F(\hat{\theta})]x_R.$$

De Sinopoli and Iannantuoni (2007) show that $\hat{\theta}$ is the unique equilibrium of the model with complete information in which there is a continuum of voters, when this is seen as the limit of a sequence of finite games. We are now going to prove that $\hat{\theta}$ is the limit point of the equilibrium of our Poisson model as the expected number of voters goes to infinity.

First, it is useful to present some results that follow from the properties of the Poisson distribution.

The Chernoff bound gives exponentially decreasing bounds for the tail probabilities of a Poisson random variable.[5] It directly implies that, in a Poisson game, the probability of the population realizations that are smaller than a given value becomes smaller and smaller as n increases.

Lemma 3. *For every $\delta > 0$ and $\bar{m} > 0$ there exists a value $\bar{n} \in \mathbb{R}$ such that, for every $n \geq \bar{n}$, $\sum_{m=0}^{\bar{m}} p(m \mid n) < \delta$.*

The second result regards the shares of votes of the two candidates. Given the expected number of players n and the population behavior τ, the vote share of candidate L is a random variable that takes value $\frac{z_L}{z_L+z_R}$ with probability $P(z \mid \tau)$ (similarly for the vote share of candidate R). If we fix the realization of the population m, such a random variable takes value $\frac{z_L}{m}$ with probability $\binom{m}{z_L}\tau^{z_L}(1-\tau)^{m-z_L}$. Let us denote it $s(m)$. Note that

$$s(m) = \sum_{i=1}^{m} \frac{\omega_i}{m},$$

where ω_i is a player's vote for candidate L given τ, i.e. the random variable that takes value 1 with probability τ_L and 0 with probability $1-\tau_L$. It is easy to see that the expected value of $s(m)$ is τ_L. The next result follows directly from the weak law of large numbers, which states that the sample mean converges in probability to the expected value.

Lemma 4. *For every $\delta > 0$ and $\epsilon > 0$ there exists a value $\bar{m} \in \mathbb{R}$ such that, for every $m \geq \bar{m}$, $Pr(|s(m) - \tau_L| > \epsilon) < \delta$.*

We can use Lemmas 3 and 4 to prove that the sample mean converges in probability to the expected value also when the size of the population is random. That is, when also the sample size is a (Poisson) random variable. Even if we will not explicitly use this result, it is important per se as it gives a clear intuition of the

[5] In particular, given a Poisson random variable with parameter n, $p(k \leq h \mid n) \leq e^{-n}(en)^h/h^h$ for $h < n$ and $p(k \geq h \mid n) \leq e^{-n}(en)^h/h^h$ for $h > n$.

fact that the actual policy outcome, which depends on the actual vote shares of the candidates, remains sufficiently close to its expected value for n large enough. This fact is at the basis of the result presented in Proposition 4.

Lemma 5. *For every $\delta > 0$ and $\epsilon > 0$ there exists a value $\bar{n} \in \mathbb{R}$ such that, for every $n \geq \bar{n}$,*

$$\sum_{m=0}^{\infty} p(m \mid n)[Pr(|s(m) - \tau(L)| > \epsilon)] < \delta.$$

Proof. Given $\delta > 0$, let $\delta_1 = \delta_2 = \delta/2$. By Lemma 4, there exists a value $\bar{m}$ such that, for every $m \geq \bar{m}$, $\Pr(|s(m) - \tau(L)| > \epsilon) < \delta_1$. Given $\bar{m}$, then, there exists by Lemma 3 a value $\bar{n}$ such that, for every $n \geq \bar{n}$, $\sum_{m=0}^{\bar{m}} p(m \mid n) < \delta_2$. Therefore, for $n \geq \bar{n}$ we have

$$\sum_{m=0}^{\infty} p(m \mid n)[\Pr(|s(m) - \tau(L)| > \epsilon)] =$$

$$\sum_{m=0}^{\bar{m}} p(m \mid n)[\Pr(|s(m) - \tau(L)| > \epsilon)] + \sum_{m=\bar{m}+1}^{\infty} p(m \mid n)[\Pr(|s(m) - \tau(L)| > \epsilon)] <$$

$$\sum_{m=0}^{\bar{m}} p(m \mid n) + \left[1 - \sum_{m=0}^{\bar{m}} p(m \mid n)\right] \delta_1 < \delta_2 + \delta_1 = \delta.$$

□

Given the average behavior $\tau \in \Delta(C)$, let the *expected outcome* be

$$X(\tau) = \tau_L x_L + (1 - \tau_L) x_R.$$

We now show that, for n sufficiently large, in equilibrium all the types to the left of the expected outcome vote for candidate L and all the types to its right vote for candidate R, except for a neighborhood that shrinks as the expected number of voters increases. Let $\sigma^*(n)$ be the cutpoint strategy corresponding to the Nash equilibrium $\theta^*(n)$, and let $\tau^*(n) = \tau(\sigma^*(n))$.

Proposition 4. *For every $\delta > 0$ there exists a value $\bar{n} \in \mathbb{R}$ such that, for every $n \geq \bar{n}$, if $\theta_t \leq X(\tau^*(n)) - \delta$ then $\sigma_t^*(n)(L) = 1$, while if $\theta_t \geq X(\tau^*(n)) + \delta$ then $\sigma_t^*(n)(R) = 1$.*

Proof. Consider a type t such that $\theta_t \leq X(\tau^*(n)) - \delta$, and suppose that $U_t(R, \theta^*(n)) \geq U_t(L, \theta^*(n))$. That is,

$$\sum_{m=0}^{\infty} p(m \mid n) \sum_{z_L=0}^{m} B(z_L \mid m, \tau^*)[v_t(\hat{x}(R, z_L, m - z_L)) - v_t(\hat{x}(L, z_L, m - z_L)] \geq 0, \quad (5)$$

where $B(z_L \mid m, \tau^*) = \binom{m}{z_L} {z_L}^{\tau_L^*} (m - z_L)^{1-\tau_L^*}$ (as in the proof of Theorem 1). Note that

$$\hat{x}(L, z_L, m - z_L) = \frac{m}{m+1} x(z_L, m - z_L) + \frac{1}{m+1} x_L$$

and

$$\hat{x}(R, z_L, m - z_L) = \frac{m}{m+1} x(z_L, m - z_L) + \frac{1}{m+1} x_R.$$

Hence, we can rewrite condition (5) as

$$\sum_{m=0}^{\infty} p(m \mid n)\cdot \tag{6}$$

$$\sum_{z_L=0}^{m} Pr\left(s(m)=\frac{z_L}{m}\mid \tau^*\right)\left[v_t(\hat{x}(R,z_L,m-z_L))-v_t\left(\hat{x}(R,z_L,m-z_L)-\frac{x_R-x_L}{m+1}\right)\right]\geq 0,$$

where we have also substituted $B(z_L \mid m,\tau^*)$ with the equivalent term $Pr\left(s(m)=\frac{z_L}{m}\mid \tau^*\right)$, i.e. the probability that the vote share of candidate L is z_L/m.

Since the function v_t is continuous on $[0,1]$ and differentiable on $(0,1)$, the mean value theorem implies that for each m and z_L there exists a value

$$\bar{x}\in\left[\hat{x}(R,z_L,m-z_L)-\frac{x_R-x_L}{m+1},\hat{x}(R,z_L,m-z_L)\right]$$

such that

$$v_t(\hat{x}(R,z_L,m-z_L))-v_t\left(\hat{x}(R,z_L,m-z_L)-\frac{x_R-x_L}{m+1}\right)=\frac{x_R-x_L}{m+1}\cdot\frac{\partial v_t(x)}{\partial x}\bigg|_{x=\bar{x}}.$$

We can use this fact in condition (6), adopting bounds of $\partial v_t(x)/\partial x|_{x=\bar{x}}$. In fact, as the utility function f is strictly concave, its slope is the greatest (in absolute value) when its argument is the maximum, so

$$\max_{\theta_t\in\Theta}\left(\max_{x\in[0,1]}\frac{\partial v_t(x)}{\partial x}\right)=\left|f'(1)\right|.$$

Also, the slope of f is the lower (in absolute value) the smaller is its argument, so

$$\max_{x>\theta_t+\eta}\frac{\partial v_t(x)}{\partial x}=f'(\eta)<0$$

for every $\eta>0$. Let $\left|f'(1)\right|=G(1)$ and $\left|f'(\eta)\right|=G(\eta)$. Note that, for every m, $G(1)\frac{x_R-x_L}{m+1}$ is an upper bound of the gain that any player can get choosing R instead of L for any given outcome induced by the opponents. On the other hand, $G(\eta)\frac{x_R-x_L}{m+1}$ is a lower bound of the loss that any player incurs choosing R instead of L when the induced outcome is on the right of her bliss point plus η.

Let $\eta<\delta$ and, for every m, let $\varepsilon_m=\max\left\{0,\delta-\eta-\frac{1}{m+1}\right\}$. Suppose that the vote share of candidate L induced by the m other voters is no further than $\varepsilon_m>0$ from its expected value $\tau_L^*(n)$, that is, $s(m)\in\left[\tau_L^*(n)-\varepsilon_m,\tau_L^*(n)+\varepsilon_m\right]$. Then the outcome that a voter induces voting for L (and, consequently, for R) is on the right of $\theta_t+\eta$. To see this, consider $s(m)=\tau_L^*(n)+\varepsilon_m$ and, with abuse of terminology, denote the outcome that the voter induces choosing L with $\hat{x}\left(L,\tau_L^*(n)+\varepsilon_m\right)$.[6] We have

$$\hat{x}\left(L,\tau_L^*(n)+\varepsilon_m\right)=$$
$$\frac{m}{m+1}\left\{\left[\tau_L^*(n)+\varepsilon_m\right]x_L+\left[1-\tau_L^*(n)-\varepsilon_m\right]x_R\right\}+\frac{1}{m+1}x_L=$$
$$\frac{m}{m+1}X(\tau^*(n))-\frac{m}{m+1}\varepsilon_m(x_R-x_L)+\frac{1}{m+1}x_L>$$

[6] We take the largest vote share of candidate L in the interval under consideration, so that the induced outcome is the leftmost possible.

$$
\begin{aligned}
&X(\tau^*(n)) - \frac{1}{m+1} - \frac{m}{m+1}\varepsilon_m > \\
&X(\tau^*(n)) - \frac{1}{m+1} - \varepsilon_m = \\
&X(\tau^*(n)) - \delta + \eta \geq \\
&\theta_t + \eta,
\end{aligned}
$$

where the first inequality derives from $x_R - x_L \leq 1$, $x_L \geq 0$, and $X(\tau^*(n)) < 1$, while the last one from the assumption that $\theta_t \leq X(\tau^*(n)) - \delta$. Thus, the left hand side of condition (6) is smaller than

$$
\begin{aligned}
\sum_{m=0}^{\infty} p(m \mid n) \Big\{ & Pr\big(|s(m) - \tau_L^*(n)| > \varepsilon_m\big) \frac{1}{m+1} G(1) + \\
& - Pr\big(|s(m) - \tau_L^*(n)| \leq \varepsilon_m\big) \frac{1}{m+1} G(\eta) \Big\}. \qquad (7)
\end{aligned}
$$

Now, let

$$\bar{\delta} = \frac{1}{k} \cdot \frac{G(\eta)}{G(1) + G(\eta)} > 0$$

with $k > 1$. By Lemma 4, given $\bar{\delta}$ and ε_m there exists a value $\bar{m}$ such that, for $m > \bar{m}$, $Pr\left(|s(m) - \tau_L^*(n)| > \varepsilon_m\right) < \bar{\delta}$.[7] Thus, the expression in (7) is smaller than

$$\sum_{m=0}^{\bar{m}} p(m \mid n) \frac{1}{m+1} G(1) + \sum_{m=\bar{m}+1}^{\infty} p(m \mid n) \frac{1}{m+1} \left[\bar{\delta} G(1) - (1-\bar{\delta}) G(\eta)\right]. \qquad (8)$$

Substituting the value of $\bar{\delta}$ we obtain

$$\bar{\delta} G(1) - (1 - \bar{\delta}) G(\eta) = \frac{1-k}{k} G(\eta) < 0,$$

so we can rewrite (8) as

$$
\begin{aligned}
&\sum_{m=0}^{\bar{m}} p(m \mid n) \frac{1}{m+1} G(1) + \sum_{m=\bar{m}+1}^{\infty} p(m \mid n) \frac{1}{m+1} \frac{1-k}{k} G(\eta) = \\
&\sum_{m=0}^{\bar{m}} \left[p(m \mid n) \frac{1}{m+1} G(1) - p(m + \bar{m} + 1 \mid n) \frac{1}{m + \bar{m} + 2} \frac{k-1}{k} G(\eta) \right] + \\
&+ \sum_{m=2\bar{m}+2}^{\infty} p(m \mid n) \frac{1}{m+1} \frac{1-k}{k} G(\eta). \qquad (9)
\end{aligned}
$$

We can show that, for n sufficiently large, every term in the square brackets is negative. That is, for every $m \in [0, \bar{m}]$, we have

$$\frac{p(m \mid n)}{p(m + \bar{m} + 1 \mid n)} \cdot \frac{m + \bar{m} + 2}{m+1} \leq \frac{k-1}{k} \cdot \frac{G(\eta)}{G(1)}.$$

[7] Differently from Lemma 4, the value of ε_m depends on m. However, as m increases, ε_m becomes higher so $Pr(|s(m) - \tau_L^*(n)| > \varepsilon_m)$ becomes lower. That is, if $Pr(|s(\bar{m}) - \tau_L^*(n)| > \varepsilon_{\bar{m}}) < \bar{\delta}$ then, for every $m > \bar{m}$, $Pr(|s(m) - \tau_L^*(n)| > \varepsilon_{\bar{m}}) < \bar{\delta}$ so $Pr(|s(m) - \tau_L^*(n)| > \varepsilon_m) < \bar{\delta}$.

This is due to the fact that, as n increases, every realization m of the Poisson random variable becomes relatively less likely than the higher realization $m+\bar{m}+1$. More precisely, we have

$$\operatorname*{argmax}_{m\in[0,\bar{m}]} \frac{p(m\mid n)}{p(m+\bar{m}+1\mid n)}\cdot\frac{m+\bar{m}+2}{m+1} = \operatorname*{argmax}_{m\in[0,\bar{m}]} \frac{(m+\bar{m}+2)!}{(m+1)!}\cdot\frac{1}{n^{\bar{m}+1}} = \bar{m}.$$

Moreover, if

$$n \geq \bar{n}_L = \left[\frac{(2\bar{m}+2)!}{(\bar{m}+1)!}\cdot\frac{k}{k-1}\cdot\frac{G(1)}{G(\eta)}\right]^{\frac{1}{\bar{m}+1}}$$

then

$$\frac{(2\bar{m}+2)!}{(\bar{m}+1)!}\cdot\frac{1}{n^{\bar{m}+1}} \leq \frac{k-1}{k}\cdot\frac{G(\eta)}{G(1)}.$$

We can conclude that, for $n \geq \bar{n}_L$, the whole expression in (9) is negative. Thus, it must be $U_t(L,\theta^*(n)) > U_t(R,\theta^*(n))$ for every $\theta_t \leq X(\tau^*(n))-\delta$.

An analogous argument implies that there exists a value $\bar{n}_R$ such that, for $n \geq \bar{n}_R$, $U_t(R,\theta^*(n)) > U_t(L,\theta^*(n))$ for every $\theta_t \geq X(\tau^*(n))+\delta$. Setting $\bar{n} = \max\{\bar{n}_L,\bar{n}_R\}$, we obtain the desired result. □

We can finally prove that the limit point of the equilibrium $\theta^*(n)$ as n goes to infinity is the cutpoint outcome $\hat{\theta}$. The reason is that, as n becomes large, the expected outcome remains close to the cutpoint outcome. By the previous proposition, hence, if n is sufficiently large then in equilibrium all the types to the left of $\hat{\theta}$ vote for candidate L and all the types to its right vote for candidate R, except for a neighborhood that shrinks as the expected number of voters increases.

Theorem 2. *The limit point of $\theta^*(n)$ as n goes to infinity is $\hat{\theta}$.*

Proof. We will show that, for every $\delta > 0$, there exists a value $\bar{n} \in \mathbb{R}$ such that, for every $n \geq \bar{n}$, if $\theta_t \leq \hat{\theta}-\delta$ then $\sigma_t^*(n)(L) = 1$, while if $\theta_t \geq \hat{\theta}+\delta$ then $\sigma_t^*(n)(R) = 1$.

Similarly to the proof of Corollary 4 in De Sinopoli and Iannantuoni (2007), let $\delta_1 = \delta/2$ and suppose $X(\tau^*(n)) < \hat{\theta}-\delta_1$. Proposition 4 implies that there exists a value n_1 such that, for $n \geq n_1$, all the types to the right of $X(\tau^*(n))+\delta_1$ (and, therefore, of $\hat{\theta}$) vote for candidate R in equilibrium. It follows that the expected outcome must be on the right of $\hat{\theta}$, i.e. $X(\tau^*(n)) \geq \hat{\theta}$. A similar argument contradicts $X(\tau^*(n)) > \hat{\theta}+\delta_1$. Therefore, for n sufficiently large, it must be $\hat{\theta}-\delta_1 \leq X(\tau^*(n)) \leq \hat{\theta}+\delta_1$, that is, $\hat{\theta}-\delta \leq X(\tau^*(n))-\delta_1$ and $\hat{\theta}+\delta \geq X(\tau^*(n))+\delta_1$. The result then follows from Proposition 4. □

5. Policy-motivated candidates

So far, we have considered the two candidates' platforms x_L and x_R as exogenous. In this section we remove this hypothesis and explore the electoral competition that happens before the voting stage, that is, candidates' strategic decision of which position to take in the policy space. To this end, we assume that candidates L and R

are purely policy-motivated with single-peaked preferences characterized by ideal policies θ_L and θ_R respectively, where $\theta_L < \theta_R$, and they choose positions $x_L, x_R \in \mathbb{X}$.

In the complete information case with a continuum of voters, Meroni (2017) proves that two policy-motivated candidates choose extreme positions in equilibrium. It follows from Theorem 2 that the same result is true in the Poisson model when the expected number of voters is sufficiently large. Precisely, let $\theta^*_{x_L,x_R}(n)$ be the equilibrium of the voting game with expected number of voters n when candidate L's proposed platform is x_L and candidate R's proposed platform is x_R. The following proposition establishes an analogous result to Proposition 1 in Meroni (2017).

Proposition 5. *If F is strictly increasing, there exists a value $\bar{n} \in \mathbb{R}$ such that, for every $n \geq \bar{n}$, the following holds:*

- *if $\theta_L < \theta^*_{0,1}(n) < \theta_R$, then the unique equilibrium is $(x_L, x_R) = (0,1)$;*
- *if $\theta^*_{0,1}(n) \leq \theta_L < \theta_R$, then the unique equilibrium is $(x_L, x_R) = (\tilde{x}_L, 1)$ with $\tilde{x}_L$ such that $\theta^*_{\tilde{x}_L,1}(n) = \theta_L$;*
- *if $\theta_L < \theta_R \leq \theta^*_{0,1}(n)$, then the unique equilibrium is $(x_L, x_R) = (0, \tilde{x}_R)$ with $\tilde{x}_R$ such that $\theta^*_{0,\tilde{x}_R}(n) = \theta_R$.*

Proof. First, note that the assumption that $\theta_L < \theta_R$ and the form of the outcome function imply that in every equilibrium $x_L < x_R$. Moreover, for every $x_L < x_R$, the cutpoint outcome $\hat{\theta}$ is strictly increasing in x_L and x_R. In fact, we have

$$\frac{\partial \hat{\theta}}{\partial x_L} = \frac{F(\hat{\theta})}{1 + F'(\hat{\theta})(x_R - x_L)} \qquad \text{and} \qquad \frac{\partial \hat{\theta}}{\partial x_R} = \frac{1 - F(\hat{\theta})}{1 + F'(\hat{\theta})(x_R - x_L)}.$$

Both derivatives are strictly positive when $\hat{\theta} \notin \{0,1\}$ since F is strictly increasing, and they are continuous since F is continuous. Theorem 2 implies that, for every $\varepsilon > 0$, there exists a value $\bar{n}$ such that $\hat{\theta} - \varepsilon \leq \theta^*_{x_L,x_R}(n) \leq \hat{\theta} + \varepsilon$ for every $n \geq \bar{n}$. It follows that, for n sufficiently large,

$$\frac{\partial \theta^*_{x_L,x_R}(n)}{\partial x_L} > 0 \qquad \text{and} \qquad \frac{\partial \theta^*_{x_L,x_R}(n)}{\partial x_R} > 0.$$

It is clear that in an equilibrium (x_L, x_R) such that $\theta_L < \theta^*_{x_L,x_R}(n) < \theta_R$ each candidate would prefer the outcome to be closer to her bliss point. By the above result, the unique equilibrium of this kind is necessarily $(0,1)$. On the other hand, if $\theta^*_{0,1}(n) \leq \theta_L < \theta_R$ then $(\tilde{x}_L, 1)$ such that $\theta^*_{\tilde{x}_L,1}(n) = \theta_L$ is the unique Nash equilibrium. In fact, candidate L's best reply against $x_R = 1$ is unique. Moreover, if there existed another equilibrium $(\bar{x}_L, \bar{x}_R)$ with $\bar{x}_R \neq 1$ it would be $\theta^*_{\bar{x}_L,\bar{x}_R} \geq \theta_R$, so $\bar{x}_L = 0$ would be candidate L's unique best reply. But then we would have $\theta^*_{0,\bar{x}_R} \geq \theta_R$, which contradicts $\theta^*_{0,\bar{x}_R} < \theta^*_{0,1} \leq \theta_L < \theta_R$. An analogous and symmetric argument applies to prove the last case. □

The above proposition states that, when n is sufficiently large, policy-motivated candidates choose extreme positions in equilibrium. This holds for every strictly increasing distribution function F. It is possible to see that such a result does not extend to every expected number of voters for every function F with the same property. In fact, by Proposition 3, if n is sufficiently small then the equilibrium of the voting game is close to the average of the two candidates' positions. This value, differently from the cutpoint outcome, does not depend on the distribution function F. As a consequence, in this case candidates' positions can converge for some specifications of that function, as informally illustrated in the following example.

Example 1. Consider two candidates with ideal policies $\theta_L = 0$ and $\theta_R = 1$, and suppose that F is the uniform distribution function. For n sufficiently large, Proposition 5 implies that in equilibrium candidates choose positions $x_L = 0$ and $x_R = 1$. As n goes to zero, given the candidates' choices x_L and x_R, the equilibrium cutpoint of the voting stage approaches the point $\frac{x_L+x_R}{2}$ and the induced outcome approaches

$$\theta^*_{x_L,x_R}(0) = \left(\frac{x_L + x_R}{2}\right)x_L + \left(1 - \frac{x_L + x_R}{2}\right)x_R.$$

Note that

$$\frac{\partial\theta^*_{x_L,x_R}(0)}{\partial x_L} = x_L \qquad \text{and} \qquad \frac{\partial\theta^*_{x_L,x_R}(0)}{\partial x_R} = 1 - x_R,$$

which are strictly positive for $x_L > 0$ and $x_R < 1$ respectively. By the same argument as in the proof of Proposition 5, it follows that in equilibrium candidates choose positions $x_L = 0$ and $x_R = 1$ also as n vanishes.

Suppose now that F is strictly increasing but highly concentrated in a neighborhood of $\theta = \frac{1}{3}$. Again, for n sufficiently large, Proposition 5 implies that in equilibrium candidates choose $x_L = 0$ and $x_R = 1$. However, if candidates choose these positions when n is close to zero, the equilibrium of the voting game remains close to $\bar{\theta} = \frac{1}{2}$. Given the assumption on F, in this case candidate L takes almost all the votes. So candidate R has now the incentive to choose a more moderate position, and both candidates' equilibrium choices will move towards the point $\frac{1}{3}$.

6. Conclusions

We have proposed a model of strategic voting under pure proportional rule and two candidates, with uncertainty about the size of the electorate. The equilibrium of the model exists and is unique for every possible expected number of voters. Moreover, it is characterized by a cutpoint type; that is, in equilibrium all the voters whose bliss points are on the left of the cutpoint vote for the leftist candidate, and all the voters whose bliss points are on her right vote for the rightist one. The cutpoint is always located between the mean of the two candidates' positions and the median of the distribution of types. As the expected number of voters grows to infinity, the equilibrium converges to the one of the complete information case.

The uniqueness result is extremely important for applications. In fact, it allows to embed the model into a more complex economical framework, of which the voting game is just a stage. In particular, we have considered the electoral competition between the two candidates that happens before the voting stage in the case in which they are policy-motivated. As the expected number of voters goes to infinity, candidates choose extreme positions in equilibrium, while their choices may converge for vanishing values of that parameter depending on the distribution of voters' types.

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