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Abstract

We investigate the relationship between the notion of progressive taxation and inequality reduction under a general version of the concept of inequality equivalence. We consider a two parameter formalization of the concept of inequality equivalence that both includes as special cases the intermediate inequality equivalence and the unit-consistent inequality equivalence. Both criteria could range from relative to absolute inequality views as the parameters in the formulation change. For the unit-consistent inequality equivalence the condition of non-decreasing average tax rate is necessary and sufficient to guarantee the post-tax inequality reduction for all the inequality views in between the relative and the absolute.

Keywords: Progressive taxation, Inequality equivalence, Intermediate measures, Unit consistency.

JEL classification: D31, D63, H23.

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1 Introduction

Since the seminal works of Jakobsson (1976), Fellman (1976) and Kakwani (1977) [henceforth JFK] the notion of progressivity of an income taxation scheme has been associated with the feature of *non-decreasing average tax rate.* For any fixed pre-tax income distribution, if the labour supply is not affected by taxation, the application of a progressive tax system will reduce the inequality in the post-tax income distribution as measured using the Lorenz curve. This result holds irrespective of the revenue collected and of the pre-tax distribution of income. It however depends on the notion of inequality considered and in particular on the fact that the Lorenz criterion implicitly incorporates the view that inequality is unaffected if all incomes are proportionally scaled. This view on how inequality changes across distributions of different total income has been questioned since the work of Kolm (1976a,b). Alternative notions of inequality have been suggested in the literature, mostly formalizing views that range between the *relative perception* based on comparisons of income ratios across individuals and the *absolute perception* focussing on absolute income differentials.

The generalization of the results of JFK by Pfingsten (1988) that incorporates an intermediate parametric definition of inequality clarifies that the condition of non-decreasing average tax rate is only linked to the relative inequality view and that as the inequality perceptions move towards the absolute view this condition becomes gradually weaker approaching the extreme case where only non-decreasing tax liabilities are required.

In more recent years questionnaire investigations on the inequality perceptions and normative analysis have suggested alternative models to formalize inequality perceptions ranging from the relative view to the absolute one. This is for instance the case for the path-independent unit-consistent criterion by Zheng (2007a, b).

In this paper we derive the analogous of the JFK result taking into account these different formalizations of the inequality perceptions. We show that when considering the unit-consistent inequality equivalence, differently from the Pfingsten (1988) result, the condition of non-decreasing average tax rate is robust to the alternative views on income inequality with the exception of the absolute view that requires non-decreasing tax liabilities. This result provides novel relevance for the original definition of tax progressivity by JFK.

In next section we illustrate the debate on inequality evaluations and the associated effects on the definition of tax progressivity. We will then present the main novel results that will be proved in the Appendix.

2 Notation and preliminary results

2.1 Inequality evaluations

We consider the set of ordered vectors $\mathbf{x} \in \mathbb{R}^n$ ranked in non-decreasing order[1] where x_i is the generic element of $\mathbf{x}$ covering the i^{th} position, that is $x_i \geq x_{i-1}$. We will call distributions all vectors $\mathbf{x}, \mathbf{y}$ belonging to $X := \{\mathbf{x} : x_i \geq 0, \sum_{i=1}^{n} x_i > 0\}$ where population size is $n \geq 2$, that is all distributions with non-negative income where at least one income is positive. X^n represents the subset of admissible distributions with fixed population n. Where appropriate we will consider a more general parametric set of distributions X_θ where $X_\theta := \{\mathbf{x} : \sum_{i=1}^{n} x_i > -\theta\}$ with $\theta \in \mathbb{R}$, with associated fixed population set X_θ^n. We will make explicit the reference to the more general set of distributions only when such distinction will be relevant for the obtained results.

We consider an (in)equality ordering [i.e., a binary relation that is complete and transitive] as a primitive concept, we denote it by $\succcurlyeq_E$ the (in)equality binary relation over distributions on $X^n \times X^n$, where $\mathbf{y} \succcurlyeq_E \mathbf{x}$ means that distribution $\mathbf{y}$ is considered at least as equal as distribution $\mathbf{x}$. The reflexive part of this relation is $\sim_E$, where $\mathbf{x} \sim_E \mathbf{y}$ means that $\mathbf{x}$ and $\mathbf{y}$ are considered equivalent w.r.t. the concept of equality applied.

Inequality aversion is introduced in the evaluation by referring to the Principle of Transfers (PT). Here we refer to the PT version presented in Fields and Fei (1978) the definition considers the inequality reducing effect of rank-preserving transfers from richer to poorer individuals, called *progressive transfer*.

Axiom 1 (Principle of Transfers (PT)) *Consider distributions $\mathbf{x}, \mathbf{y} \in X^n$, if $\mathbf{y}$ is obtained from $\mathbf{x}$ through a positive income transfer from a richer individual to a poorer one, then $\mathbf{y} \succcurlyeq_E \mathbf{x}$ provided that the ranking of the individuals is not affected.*

Axiom PT refers only to distributions with same total income, in order to extend the comparisons to "cakes of different sizes" is needed some property that posits under which conditions inequality is preserved if we move from an original distribution to distributions of different total income, this is precisely what an Inequality Equivalence Criterion (IEC) does.

The most commonly adopted IEC is the relative criterion that considers (in)equality equivalent all the distributions that can be obtained by scaling all incomes by the same positive factor. An alternative view, originally proposed by Kolm (1976), may consider an absolute IEC that states that equal absolute variations leave inequality unchanged.

A general parametric version of the IEC due to Pfingsten (1986) and Bossert and Pfingsten (1990) reaches for the extreme values of the parameter $0 \leq \eta \leq 1$ the two extreme IECs, the relative and the absolute.

[1] In order to simplify the exposition we consider ordered income vectors from the outset, instead of considering non-ordered vectors and assume invariance of the evaluations w.r.t. permutations of the incomes.

Axiom 2 (Intermediate Inequality Equivalence (IIE)) *Consider* $\mathbf{x} \in X^n$*, if*

$$\mathbf{y} = \mathbf{x} + \varepsilon\left[\eta\mathbf{x} + (1-\eta)\,\mathbf{1}\right]; \ \varepsilon \in R \textit{ such that } \mathbf{y} \in X^n, \tag{1}$$

where $0 \leq \eta \leq 1$, $\mathbf{1} := (1, 1, ..., 1)$ *then* $\mathbf{x} \sim_E \mathbf{y}$.

All distributions $\mathbf{y} \in X^n$ obtained from $\mathbf{x}$ through (1), for a given value of η, are inequality equivalent to $\mathbf{x}$ according to the intermediate criterion associated to η. When $\eta = 1$ then $\mathbf{y} = \mathbf{x}(1+\varepsilon)$ leading to the *relative inequality equivalence,* according to which the perception of inequality is based over comparisons of the income shares. If $\eta = 0$ one gets $\mathbf{y} = \mathbf{x} + \varepsilon\mathbf{1}$ in (1), leading to the *absolute inequality equivalence,* according to which what matters in the perception of inequality is the comparison of income differentials. Relative and absolute positions are commonly experienced by individuals, and at least intuitively, seem to be fundamental elements in any inequality evaluation, unfortunately the relative and absolute criterion are incompatible for inequality evaluations[2]. It is therefore worth to specify criteria which are in between these two extremes, for all the intermediate values of η the perception of inequality is represented by a convex combination of the relative and absolute perceptions according to (1).

Moreover, an alternative formalization[3] of IIE shows that it could be interpreted as a relative criterion applied over traslated distributions, that is distributions where all income are increased by a constant $\theta := \frac{(1-\eta)}{\eta} \geq 0$. The IIE could be formalized therefore as: for $\mathbf{x} \in X^n$, if $\mathbf{y}+\theta\mathbf{1} = \varepsilon\left(\mathbf{x}+\theta\mathbf{1}\right)$, $\varepsilon > 0$ such that $\mathbf{y} \in X^n$ then $\mathbf{x} \sim_E \mathbf{y}$. Equivalently if we denote by $\mu_x := \sum_{i=1}^n x_i/n$ the average income of the distribution $\mathbf{x}$ the relevant condition for IIE could be rewritten as: if

$$\frac{\mathbf{y}+\theta\mathbf{1}}{\mu_y + \theta} = \frac{\mathbf{x}+\theta\mathbf{1}}{\mu_x + \theta} \qquad for\ \mathbf{y}, \mathbf{x} \in X^n, \tag{2}$$

then $\mathbf{x} \sim_E \mathbf{y}$.

Following this notation the extreme criteria are obtained for $\theta = 0$ and $\theta \to \infty$. Ebert (2004) characterizes a further parametric linear generalization of the IIE where θ could also be negative. If θ is negative it could be interpreted as a poverty line level and inequality can be considered equivalent between two n dimensional distributions $\mathbf{y}$ and $\mathbf{x}$ if for any position i the ratio $\frac{x_i-d}{y_i-d}$ is constant, with the poverty line $d := -\theta$ in the IIE, that is the incomes gaps of incomes exceeding d are proportionally scaled. Additional parametric formulations of liner IECs have also been proposed by Seidl and Pfingsten (1997), del Río and Ruiz-Castillo (2000), del Río and Alonso-Villar (2008), Azpitarte and Alonso-Villar (2014) Bosmans et al. (2014). However, they are not path-independent for all inequality perceptions that are in between the absolute

[2] See Eichorn and Gehrig (1982) for a discussion on the impossibility of constructing nondegenerate indeces that are simultaneously relative and absolute.

[3] See Besley and Preston (1988).

and the relative views, that is if we move from distribution $\mathbf{x}$ to $\mathbf{y}$ following such a criterion it is not possible to come back from $\mathbf{y}$ to $\mathbf{x}$ using the same criterion with the same parameter formalizing the inequality perception. Indeed, the only linear and path-independent generalization of the IIE criterion is the one proposed in Ebert (2004).

2.1.1 Beyond linear inequality equivalence criteria

One of the most controversial axioms in inequality measurement is the IEC. Individuals shows different perceptions on how inequality should change if we move from one distribution to another with different total income adding surpluses or subtracting taxes.

According to the questionnaire investigations by Amiel and Cowell (1992, 1997, 1999), Ballano and Ruiz Castillo (1993) and Harrison e Seidl (1994a,b) the majority of individual answers are not consistent with the relative perception of inequality. Furthermore, as showed by Amiel and Cowell (1992), questionnaire evidence supports the view that both absolute and proportionate additions to income reduce inequality as suggested by Dalton (1920). This property is satisfied by all the IIE criteria excluding the extremes, but as discussed in Ballano and Ruiz Castillo (1993), the IIE criterion seems not sufficiently flexible to provide a rational for peoples' inequality perceptions. Moreover, Amiel and Cowell (1997) find evidence that *the appropriate equivalence concept depends on the income levels at which inequality comparisons are made.* They show that a typical IEC consistent with empirical evidence *goes from relative to absolute positions as income increases.*

Such attitudes are in line with parametric non-linear IECs that in addition are path-independent as those presented in Krtscha (1994), Yoshida (2005), and Zheng (2007a,b). In particular Zheng IEC, that generalizes those of Krtscha and Yoshida, is path-independent and satisfies the property of "unit-consistency". That is, if the unit of measure of income is scaled for two distributions this operation does not affect the inequality ranking of these distributions.

In order to illustrate the formalization of these criteria we consider a two parameter IEC, the *Flexible Inequality Equivalence* (FIE) criterion, characterized in Zoli (1998, 2012) that incorporates as special cases the IIE and the "unit-consistent" criterion presented by Zheng (2007a,b).

Axiom 3 (Flexible Inequality Equivalence (FIE)) *The (in)equality ordering $\succcurlyeq_E$ satisfies FIE if for all $\mathbf{x}, \mathbf{y} \in X^n$, such that*

$$\frac{y_i - \mu_y}{(\mu_y + \theta)^\lambda} = \frac{x_i - \mu_x}{(\mu_x + \theta)^\lambda} \quad \forall i = 1, 2, ..., n \tag{3}$$

then $\mathbf{x} \sim_E \mathbf{y}$, where $\lambda \in [0, 1], \theta \geq 0$.

The parameters $\lambda \in [0,1]$, and $\theta \geq 0$ of the FIE relates to two families of IEC respectively those that are unit-consistent and the IIE. In fact if we set $\theta = 0$ we obtain the path-independent "unit-consistency" IEC for $\lambda \in [0,1]$.[4] While if we let $\lambda = 0$ we obtain the IIE for $\theta \geq 0$.

For $\lambda = 0$ or if $\theta \to \infty$ we reach the absolute criterion, the relative criterion instead is obtained for $\lambda = 1$ and $\theta = 0$, but for intermediate values of λ the criterion is *non-linear*, and as λ moves from 0 to 1 the inequality perceptions move from the absolute one to the one formalized by the IIE parameter θ.

Following the works of Kolm (1969), Atkinson (1970), Sen (1973), it is possible to derive a characterization of a dominance condition, over distributions in X^n, in terms of a parametric transformation of the Lorenz curves which is equivalent to (in)equality dominance for all orderings satisfying PT and FIE.

Denote with $L_x(i/n)$ the Lorenz curve of distribution $\mathbf{x}$ evaluated at the population quantile i/n, that is $L_x(i/n) := \frac{\sum_{j=1}^{i} x_j}{\sum_{j=1}^{n} x_j}$ for all $i > 0$, with $L_x(0) = 0$. Next proposition is proved in Zoli (1999).

Proposition 1 *Given two distributions $\mathbf{x}, \mathbf{y} \in X^n$, distribution $\mathbf{y} \succcurlyeq_E \mathbf{x}$ for all (in)equality relations satisfying PT and FIE, if and only if*

$$\left(\frac{k+\theta}{\mu_y+\theta}\right)^{\lambda} \mu_y \left[i/n - L_y(i/n)\right] \leq \left(\frac{k+\theta}{\mu_x+\theta}\right)^{\lambda} \mu_x \left[i/n - L_x(i/n)\right] \qquad \forall i = 0, 1, 2, ..., n \tag{4}$$

and for some $k > 0$. Where $1 \geq \lambda \geq 0$, and $\theta \geq 0$.

We will write $\mathbf{y} \succcurlyeq_L^{\lambda,\theta} \mathbf{x}$ to denote that distribution $\mathbf{y}$ is more equal or at least as equal as distribution $\mathbf{x}$ according to the Lorenz dominance criterion associated to the FIE with parameters $1 \geq \lambda \geq 0$, and $\theta \geq 0$. By readjusting the definition in (4) the Lorenz dominance condition can be written as

$$\mathbf{y} \succcurlyeq_L^{\lambda,\theta} \mathbf{x} \Longleftrightarrow \frac{1}{n}\sum_{j=1}^{i}(y_j - \mu_y)\left(\frac{k+\theta}{\mu_y+\theta}\right)^{\lambda} \geq \frac{1}{n}\sum_{j=1}^{i}(x_j - \mu_x)\left(\frac{k+\theta}{\mu_x+\theta}\right)^{\lambda} \quad \forall i = 1, 2, ..., n \tag{5}$$

and for some $k > 0$. The definition of strict equality dominance $\mathbf{y} \succ_L^{\lambda,\theta} \mathbf{x}$ holds if the weak inequality $[\geq]$ in (5) is strict $[>]$ for all $i = 1, 2, ..., n-1$.

Note that the "classical" Lorenz dominance condition that we will denote with $\succcurlyeq_L$ can be derived as a special case of $\succcurlyeq_L^{\lambda,\theta}$ by setting $\lambda = 1$ and $\theta = 0$. In this case we obtain

$$\mathbf{y} \succcurlyeq_L \mathbf{x} \Longleftrightarrow \frac{1}{n}\sum_{j=1}^{i}\frac{y_j}{\mu_y} \geq \frac{1}{n}\sum_{j=1}^{i}\frac{x_j}{\mu_x} \quad \forall i = 1, 2, ..., n. \tag{6}$$

[4] The FIE with $\theta = 0$ and $\lambda \in [0,1]$ coincides with the IEC presented in Yoshida (2005) and includes as special case the one of Krtscha (1994) when $\lambda = 1/2$.

2.2 Evaluating the redistributive effect of income taxation

The results introduced in the previous sections could be helpful in providing normative basis for the evaluation of the *redistributive effects of tax reforms.*

The theorems of Jakobsson (1976), Fellman (1976) and Kakwani (1977) suggest conditions for inequality reducing tax reforms, whatever is the *fixed* underlying gross income distribution. These conditions requires the tax schedule is progressive in the sense of exhibiting non-decreasing average tax rates. Given that it is assumed that the gross income is not affected by the taxation, such reforms are also welfare improving if compared to others associated to less progressive tax schemes ensuring the same revenue.

When pre-tax incomes are exogenous, there is therefore, little doubt on what "progressivity" means or on what are its distributional implications. Although various definitions of "progressivity" have been proposed, it has been shown that they are equivalent indeed. Before making such equivalences clear, let us introduce the terminology. The tax payments (or tax liabilities) of an individual with income x are summarized by the function $T(x)$, where it is assumed that $T(\cdot)$ is *non-decreasing* and that the *marginal tax rate* $T'(\cdot) < 1$, that is taxation preserves the ranking of the individuals in the post-tax income distribution. The post-tax income enjoyed by the individual is then given by the function $y = x - T(x)$ and the net-income distribution across the population is $\mathbf{y} = (y_1, ..., y_n)$.

Unambiguous inequality comparisons are carried out through the Lorenz order, represented by $\succcurlyeq_L$. The crucial equivalence result proved by Jakobsson (1976), Fellman (1976) and Kakwani (1977) (JFK theorem henceforth) could be summarized as follows:[5]

Proposition 2 (JFK theorem) *When pre-tax income x is fixed, then for any continuous and non-decreasing $T(.)$ the following conditions are equivalent:*

(1) schedule progression:

$$T(x)/x$$

is non-decreasing for all x;

(2) distributional progression:

$$\mathbf{y} \succcurlyeq_L \mathbf{x} \quad \textit{for all } \mathbf{x} \in X.$$

Condition (1) simply states the generally accepted notion of tax progression, that the average tax rate is non-decreasing in pre-tax income. Condition (2) offers an alternative and plausible interpretation of progressivity, in that is progressive a tax function that reduces the existing income inequality. This condition also says implicitly that a progressive tax is one that produces a less unequal net income distribution than a (equal-yield) proportional tax, since the latter preserves relative income differentials. The list of equivalences could also be expanded by considering a further

[5] See also Eichorn *et al.* (1984) and Ebert and Moyes (2002).

condition that regards as progressive any tax function that unambiguously guarantees a greater social welfare than a proportional tax producing the same tax revenue. The importance of the JFK theorem is that it ensures that these concepts of progressivity are equivalent and little ambiguity is left on "what a progressive tax" should actually be. It is however important to recall that one of the crucial assumptions underlying such equivalence is that it is based on the notion of *relative inequality* that motivates the use of the "classical" Lorenz dominance condition in statement (2).

2.2.1 Beyond the JFK theorem

The result in Proposition 2 holds under a set of restrictive conditions. The pre-tax income distributions are exogenous and no labour supply responses to taxation are considered. Moreover, the result should hold for any pre-tax income distribution. In what follow we will retain these conditions and we will investigate the role of the inequality equivalence concept adopted in the theorem for deriving the redistributive effects of taxation.

The definition of inequality equivalence applied by JFK is the relative one (which was almost the only one applied in 1976 when the theorem(s) appeared). Pfingsten (1988) provides a definition of progressive tax system which could be extended to the intermediate IEC view of inequality. Let $\succcurlyeq_L^{1,\theta}$ denote the Lorenz order associated to the IIE parameterized by θ, the generalization of the JFK theorem is:

Proposition 3 (Pfingsten (1988)) *When pre-tax income x is fixed, for any individual, then for any continuous and non-decreasing $T(\cdot)$ the following conditions are equivalent:*

(1) θ schedule progression: the function

$$g_{1,\theta}(x) := T(x) \cdot \frac{k+\theta}{x+\theta} \tag{7}$$

is non decreasing with x, for all x, for some $k > 0, \theta \geq 0$.

(2) θ distributional progression:

$$\mathbf{y} \succcurlyeq_L^{1,\theta} \mathbf{x} \quad \textit{for all } \mathbf{x} \in X.$$

It is immediate to check that this result generalizes those associated to the relative criterion. If $\theta = 0$ the condition becomes the same as in JFK theorem: $g_{1,0}(x) = k\frac{T(x)}{x}$ should be non-decreasing in x. As s $\theta \to \infty$, we approach the absolute IEC and we get the condition that $T(x)$ should be non-decreasing with x, for all x, which is the condition of *minimal progressivity* associated to the absolute view introduced by Moyes (1988).

In all cases if $T(x)$ satisfies the above conditions, then the post-tax distribution dominated the pre-tax according to the intermediate criterion characterized by θ.

As a result, as we move away from the relative inequality view with the parameter $\theta = 0$ the condition on schedule tax progression moves from non-decreasing average tax rate to non-decreasing tax liabilities. The classical condition on schedule tax progressivity becomes a special case and "less redistributive" conditions could be considered as θ increases. This is also the case for the equivalent condition of marginal tax rate not lower than average tax rate, that as shown in Pfingsten (1988) could be more generally rewritten as $T'(x) \geq \frac{T(x)}{x} \cdot \frac{x}{x+\theta}$, where the r.h.s. decreases as θ increases.

3 New results on inequality equivalence and progressive taxation

The result in Proposition 3, may depend on the shape of the parametric IEC considered. Ebert (2010) generalizes the result for all linear and path-independent IEC derived in Ebert (2004), also in these cases the schedule progressive condition depends on the value of the parameter considered and the classical condition of non-decreasing average tax rate holds only when relative inequality is taken into account.

Here we investigate wether the introduction of a more flexible IEC as the FIE that considers as special cases both the IIE and the unit-consistent IEC can contribute to a better understanding of the perception of progressivity and to what extent does it contradicts previous results or allows for new ones.

Since FIE embodies the IIE criterion of course it cannot contradict the related results, but it could lead to a more general setting. The non-linearity exhibited by the criterion when $\lambda \in (0,1)$ is of particular interest.

In order to illustrate first the most interesting result we focus on the subset of FIE criteria where $\theta = 0$. These criteria are those that are *path-independent* and *unit-consistent* according to the terminology in Zheng (2007a, b) [see also del Río and Alonso-Villar, 2008 and Zoli, 2012].

We consider now directly the condition for the path independent unit consistent IEC, that requires that for all $\mathbf{x}, \mathbf{y} \in X^n$, such that

$$\frac{y_i - \mu_y}{{\mu_y}^{\lambda}} = \frac{x_i - \mu_x}{{\mu_x}^{\lambda}} \quad \text{for all } i = 1, 2, ..., n \tag{8}$$

then $\mathbf{x} \sim_{\mathbf{E}} \mathbf{y}$, where $\lambda \in [0,1]$.

We provide here the main intuition behind the result in next Proposition 4 that will be formalized in the proof in the Appendix.

What usually is investigated in the literature on progressivity measurement is a set of conditions that the fiscal system should satisfy in order to lead to an inequality equivalent after-tax distribution, that is a distribution of net incomes that according to the IEC considered can be viewed as "as equal as" the pre-tax income distribution. Then it follows that all tax schedules imposing an higher contribution to the richer individuals as in the limit tax system derived are more progressive according to the IEC considered.

Consider a tax schedule $T_\lambda(x)$ that is non-decreasing and such that the post-tax income is increasingly related to the pre-tax income. The schedule could in principle depend on the parameter λ of the IEC, and considers the information on the individual income. In general the schedule does not take into account nor the distribution of income in the society neither the total/average revenue that should be collected, however we leave open at this stage this possibility to illustrate the problem.

The associated net income for individual i is $y_i = x_i - T_\lambda(x_i)$. Let τ denote the average tax rate imposed by the tax system $T_\lambda(\cdot)$ on the distribution $\mathbf{x}$, that is $\sum_{i=1}^n T_\lambda(x_i) = \tau(n\mu_x)$, therefore the average post-tax income is $\mu_y = (1-\tau)\mu_x$. If we are aiming at deriving a post-tax distribution $\mathbf{y}$ that is inequality equivalent to the pre-tax distribution $\mathbf{x}$, it should be verified the following condition

$$\frac{x_i - T_\lambda(x_i) - (1-\tau)\mu_x}{[(1-\tau)\mu_x]^\lambda} = \frac{x_i - \mu_x}{\mu_x{}^\lambda} \quad \forall i.$$

Simplifying we get

$$x_i - T_\lambda(x_i) - (1-\tau)\mu_x = (1-\tau)^\lambda (x_i - \mu_x) \quad \forall i,$$

which gives

$$\begin{aligned} T_\lambda(x_i) &= \tau x_i + \left[(1-\tau) - (1-\tau)^\lambda\right](x_i - \mu_x) \qquad \forall i \\ T_\lambda(x_i) &= \left(1 - (1-\tau)^\lambda\right) x_i + \left[(1-\tau)^\lambda - (1-\tau)\right]\mu_x \quad \forall i. \end{aligned} \tag{9}$$

That is the "inequality neutral" tax schedule should be

$$T_\lambda(x) := a_\lambda + b_\lambda x, \tag{10}$$

where $a_\lambda = \mu_x\left[(1-\tau)^\lambda - (1-\tau)\right]$, and $b_\lambda = 1 - (1-\tau)^\lambda$, and such that by construction the revenue constraint is satisfied, that is $\frac{1}{n}\sum_{i=1}^n T_\lambda(x_i) = \tau\mu_x$.

Note that if λ reaches its extreme values we obtain $a_0 = \tau\mu_x$, $b_0 = 0\ \forall i$ for $\lambda = 0$, that is all individuals are taxed using a uniform lump sum tax, and if in $\lambda = 1$ then $a_1 = 0$, and $b_1 = \tau$, that is all individuals are taxed proportionally. These are the limit values of the extreme definitions of minimal progressivity (Moyes, 1988) and of classical progressivity (as in the JFK theorem). Moreover, the relative weights of the two components (a_λ and b_λ) in the tax schedule are independent from the changes in values of τ keeping μ_x fixed, or the other way round changing μ_x keeping τ fixed. This result works even for the IIE view in Pfingsten (1988) according to which it could be shown that the inequality neutral tax system is

$$T_\theta(x) = a_\theta + b_\theta x \tag{11}$$

where $a_\theta = \tau\frac{\theta\mu_x}{(\mu_x+\theta)}$, and $b_\theta = \tau\frac{\mu_x}{(\mu_x+\theta)}$, therefore as τ changes given θ and μ_x then a_θ and b_θ change proportionally, if μ_x changes for fixed θ and τ the changes are again proportional for a_θ and b_θ. The relative weight between the absolute and relative component of the tax scheme can be formalized by looking at the ratio $r := \frac{a}{b}$ that in the case of the IIE becomes

$$r_\theta := \frac{a_\theta}{b_\theta} = \theta. \tag{12}$$

This properties comes from the linearity of the IIE.

Since FIE is not linear then the relative weights of the two components of the limit tax system in this case could be sensitive to both changes in μ_x and changes in τ. More precisely as μ_x increases the absolute component a_λ increases its value, while as τ increases the relative component becomes more important. Consider $b_\lambda > 0$ [that is set $\lambda \in (0,1]$] and let

$$r_\lambda := \frac{a_\lambda}{b_\lambda} = \mu_x \frac{(1-\tau)^\lambda - (1-\tau)}{1-(1-\tau)^\lambda} \qquad \text{for } \lambda \in (0,1]. \tag{13}$$

Note that $\lim_{\tau \to 0} r_\lambda = \mu_x \frac{1-\lambda}{\lambda}$, and similarly $\lim_{\tau \to 1} r_\lambda = 0$ which implies that as $\tau \to 1$ the relative weight of a_λ vanishes and we approach a proportional taxation scheme. Most interestingly the proportional tax scheme is approached, irrespective of τ, when μ_x approaches 0. That is, for very low average income distributions.

The insight form this comparisons is that the non-linear equivalence criteria (when $\lambda \in (0,1)$) specify tax schemes that are sensitive both to the total income of the distribution considered and to the total revenue to collect. Roughly speaking the minimally redistributive taxation schemes according to FIE *increase the degree of progressivity as the revenue increases*, while the limit tax scheme associated to tax schedules insuring the same revenue from distribution with different total income should be *"more progressive" for distribution with low incomes and approaches minimal progressivity as incomes increase.*

These implications of course are valid if we fix a distribution or we fix a possible revenue, but the main theorems in progressivity measurement like the JFK theorem provide results that could be applied to all feasible distributions and all possible taxation schemes irrespective of the revenue they are supposed to collect. As proved in the Appendix the following result holds for FIE when $\theta = 0$.

Proposition 4 *When pre-tax income x is fixed for any individual, then for any continuous and non-decreasing $T(\cdot)$ the following conditions are equivalent:*

(1) λ schedule progression: the function

$$g_{\lambda,0}(x) \; : \; = T(x)/x \qquad \textit{if} \quad \lambda \in (0,1] \tag{14}$$

$$g_{0,0}(x) \; : \; = T(x) \qquad \textit{if} \quad \lambda = 0 \tag{15}$$

is non-decreasing with x, for all x.

(2) λ distributional progression:

$$\mathbf{y} \succcurlyeq {}_L^{\lambda,0} \mathbf{x} \quad \textit{for all } \mathbf{x} \in X \quad \textit{if} \quad \lambda = 1, \lambda = 0$$

$$\mathbf{y} \succ {}_L^{\lambda,0} \mathbf{x} \quad \textit{for all } \mathbf{x} \in X \quad \textit{if} \quad \lambda \in (0,1)$$

According to Proposition 4, when inequality perceptions move from the relative view to the absolute view following the path-independent unit-consistent criterion then the notion of schedule progression as non-decreasing average tax rate is robust for all inequality perceptions except for the absolute one [i.e., for all $\lambda \in (0, 1]$.

The result clarifies that Pfinsgten (1988) definition of progressive taxation is dependent on the specific IEC used to encompass the inequality view that ranges from relative to absolute.

Next proposition generalizes the results considering the two parameters definition of FIE. In line with the findings in Proposition 4 for $\lambda \in (0, 1]$ the results converges to those related to the IIE in Proposition 3. Minimal progression is still obtained for $\lambda = 0$.

Proposition 5 *When pre-tax income x is fixed for any individual, then for any continuous and non-decreasing $T(\cdot)$ the following conditions are equivalent:*

(1) λ, θ schedule progression: the function

$$g_{\lambda,\theta}(x) \ : = T(x) \cdot \frac{k+\theta}{x+\theta} \qquad if \quad \lambda \in (0,1] \tag{16}$$

$$g_{0,\theta}(x) \ : = T(x) \qquad if \quad \lambda = 0 \tag{17}$$

is non-decreasing with x, for all x, for some $k > 0, \theta \geq 0$.

(2) λ, θ distributional progression:

$$\mathbf{y} \succcurlyeq {}_L^{\lambda,\theta} \mathbf{x} \quad for\ all\ \mathbf{x} \in X_\theta \quad if \quad \lambda = 1, \lambda = 0$$
$$\mathbf{y} \succ {}_L^{\lambda,\theta} \mathbf{x} \quad for\ all\ \mathbf{x} \in X_\theta \quad if \quad \lambda \in (0,1)$$

where $\theta \geq 0$.

See the Appendix for the proof.

4 Concluding remarks

We have derived a generalization of the JFK theorem that accommodates a variety of inequality equivalence views. The two parameter inequality equivalence criterion considered is incorporating as special cases the intermediate inequality equivalence and the class of path-independent unit-consistent equivalence criteria. As shown in Pfingsten (1988) as one moves from an relativist view to the absolutist one within the intermediate inequality equivalence criteria, the condition on the inequality reducing tax schedule move from requiring non-decreasing average tax rate to require non-negative marginal taxation. We show here that when it is taken the perspective of considering the class of path-independent unit-consistent equivalence criteria, only when the absolutist view is taken into consideration it is required that the marginal tax rate should not be negative, according to all the other views ranging from the absolutist to the relative it is always required the average tax rate should not be decreasing. This result provides further relevance to the classical view introduced in the JFK theorem that considers progressive tax schedules with non-decreasing average tax rate.

It has to be pointed out that the result is driven by the fact that the inequality reducing properties of a progressive tax schedule should hold for all possible pre-tax distributions and irrespective of the revenue requirements.

If the tax scheme can take into account information on the pre-tax income distribution and on the revenue requirement then also tax schedules exhibiting increasing average tax rate could reduce the inequality even for values of $\lambda \in (0,1)$ when $\theta = 0$. Suppose for instance that we want to collect a revenue that is 19% of the overall pre-tax income, and that $\lambda = 0.5$ then $(1-\tau) = 0.81$ and $(1-\tau)^{\lambda} = 0.9$. Then according to (10) the inequality neutral taxation requires that $T_{0.5}(x) = (1-0.9)\,x + [0.9-0.81]\,\mu_x = 0.10 \cdot x + 0.09 \cdot \mu_x$ with a positive marginal tax rate of 10% but exhibiting a decreasing average tax rate. In this case a tax schedule $T(x) = 0.11 \cdot x + 0.08 \cdot \mu_x$ will be revenue neutral but inequality reducing given that it has reduced the average tax rate for the poorer individual and increased it for the richer ones. Even under the inequality reducing tax scheme the average tax rate will then be decreasing.

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Appendix

Proof of Proposition 4

For $\lambda = 0$ and $\lambda = 1$ the result presented in the proposition have been already proved by Moyes (1988) [$\lambda = 0$] and in the JFK theorem [$\lambda = 1$]. Both results are also special cases of the IIE generalization of JFK theorem in Pfingsten (1988).

We consider therefore only the cases where $\lambda \in (0, 1)$.

(1) $\Longrightarrow$ (2)

Consider a proportional tax schedule $T(x) = \tau x$, that is obtained when $g_\lambda(x)$ is constant for all x. Let $y_i = x_i - T(x_i)$, and compute $\frac{y_i - \mu_y}{\mu_y{}^\lambda}$ as required by the inequality equivalence criterion. We obtain $\frac{y_i - \mu_y}{\mu_y{}^\lambda} = \frac{x_i - \tau x_i - (1-\tau)\mu_x}{[(1-\tau)\mu_x]^\lambda} = (1-\tau)^{1-\lambda} \frac{x_i - \mu_x}{\mu_x{}^\lambda}$ for all i, that is $\frac{y_i - \mu_y}{\mu_y^\lambda} < \frac{x_i - \mu_x}{\mu_x{}^\lambda}$ for all i. It follows that inequality is reduced and $\mathbf{y} \succ_L^{\lambda,0} \mathbf{x}$. Moreover if $T(x)/x$ is non-decreasing then, for a given average tax rate the income differentials w.r.t. the average of the richer individuals will be proportionally reduced more than those of the poorer individuals and these changes for a fixed average of the post-tax distribution will be equivalent to a sequence of progressive transfers from richer to poorer individuals thereby further reducing inequality [see also Pfingsten (1988), p. 244] and therefore leading to $\mathbf{y} \succ_L^{\lambda,0} \mathbf{x}$.

(2) $\Longrightarrow$ (1)

Note that the conditions (1) and (2) should hold for all pre-tax distributions $\mathbf{x} \in X$ where also the population size n could vary. Consider the index r_λ derived in (13), recall that $r_\lambda = \mu_x \frac{(1-\tau)^\lambda - (1-\tau)}{1-(1-\tau)^\lambda}$ for $\lambda \in (0, 1]$, take a generic distribution $\mathbf{x} := (\varepsilon, \varepsilon, ..., \varepsilon, \varepsilon + x)$ with $n-1$ incomes $\varepsilon > 0$ and one income $\varepsilon + x$ where $x > 0$. Then $\mu_x = \varepsilon + x/n$. As $\varepsilon \to 0$ and $n \to \infty$ we have that $\mu_x \to 0$. Note that in the distribution the higher income converges to $x > 0$ as $\mu_x \to 0$. As a result for sufficiently large values of n it is possible to construct distributions where incomes can take any positive value [as is the case for the higher income $x > 0$] and $\mu_x \to 0$ leading to the condition $r_\lambda \to 0$. In this case according to the construction of (13) the "inequality neutral" tax schedule approaches the proportional taxation where $T_\lambda(x)/x$ is constant. This result of neutrality is obtained for $\lambda = 1$, while it is approximated for all values of $\lambda \in (0,1)$. That is, as shown in the $[(1) \Longrightarrow (2)]$ part of the proof, for all $\lambda \in (0,1)$ the proportional tax schedule $T(x) = \tau x$ approaches the inequality neutrality condition as $\mu_x \to 0$ but still lead to a strict reduction in inequality. We have shown then that $T(x)/x$ constant is a limit condition that guarantees inequality reduction. We now show that $T(x)/x$ non-decreasing is also necessary to obtain inequality reduction according to $\succcurlyeq_L^{\lambda,0}$.

Suppose that it is not the case that $T(x)/x$ is non-decreasing. Then there is at least an income level such that the associated average tax rate is lower than the one at the lower income levels. We show that in this case the dominance in terms of $\succcurlyeq_L^{\lambda,0}$ of the post-tax distribution with respect to the pre-tax distribution is violated.

Consider the following pre-tax distribution $\mathbf{x} := (\varepsilon, \varepsilon, ..., \varepsilon, \varepsilon+z_1, \varepsilon+z_2)$ with $n-2$ incomes $\varepsilon > 0$ and two incomes $\varepsilon + z_j$ where $z_j > 0$ for $j = 1, 2$ and $z_2 > z_1$.

In order to prove the result we:

(i) consider the relation in (9) leading to the inequality neutral tax schedule $T_\lambda(x_i) = \left(1 - (1-\tau)^\lambda\right) x_i + \left[(1-\tau)^\lambda - (1-\tau)\right] \mu_x$ for all i.

(ii) Modify the schedule in order to decrease the average tax rate of the income of richer individual while keeping the overall average revenue unaffected.

(iii) Go back to the limit conditions derived above where we have obtained $T(x)/x$ constant and argue that the lower average tax rate for the income of the richer individual generates a violation in the inequality reduction result.

Part (i). Note that $\mu_x = \frac{n-2}{n}\varepsilon + \frac{\varepsilon+z_1+\varepsilon+z_2}{n} = \varepsilon + \frac{z_1+z_2}{n}$, let $x_i = \varepsilon$ for $i = 1, 2, ..., n-2$, $x_{n-1} = \varepsilon + z_1$, $x_n = \varepsilon + z_2$, substituting into $T_\lambda(x_i)$ we obtain for instance for x_{n-1}

$$\begin{aligned} T_\lambda(x_{n-1}) &= T_\lambda(\varepsilon+z_1) = \left(1-(1-\tau)^\lambda\right)(\varepsilon+z_1) + \left[(1-\tau)^\lambda - (1-\tau)\right]\left(\varepsilon + \frac{z_1+z_2}{n}\right) \\ &= \tau\varepsilon + \left(1-(1-\tau)^\lambda\right) z_1 + \left[(1-\tau)^\lambda - (1-\tau)\right]\left(\frac{z_1+z_2}{n}\right). \end{aligned}$$

To summarize, for all the pre-tax income levels we have:

$$\begin{aligned} T_\lambda(\varepsilon) &= \tau\varepsilon + \left[(1-\tau)^\lambda - (1-\tau)\right]\left(\frac{z_1+z_2}{n}\right) \\ T_\lambda(\varepsilon+z_1) &= \tau\varepsilon + \left(1-(1-\tau)^\lambda\right) z_1 + \left[(1-\tau)^\lambda - (1-\tau)\right]\left(\frac{z_1+z_2}{n}\right) \\ T_\lambda(\varepsilon+z_2) &= \tau\varepsilon + \left(1-(1-\tau)^\lambda\right) z_2 + \left[(1-\tau)^\lambda - (1-\tau)\right]\left(\frac{z_1+z_2}{n}\right). \end{aligned}$$

Part (ii). Suppose we increase the average tax burden for the richer individual, while keeping the overall average revenue fixed at τ. We then reduce proportionally the taxation of the individual with income $\varepsilon + z_2$, by $\delta(\varepsilon + z_2)$ where $\delta > 0$, while increasing proportionally the taxation of all the other individuals by the factor $\beta > 0$ that is we add $\beta(\varepsilon + z_1)$ to the taxation of the individual with income $(\varepsilon + z_1)$ and add

$\beta\varepsilon$ to the taxation of all the other $n-2$ poorer individuals. This operation should keep the average tax burden unaffected and therefore should satisfy $\beta\varepsilon(n-2)+\beta\left(\varepsilon+z_1\right)=\delta\left(\varepsilon+z_2\right)$.

We then obtain

$$
\begin{aligned}
T_\lambda(\varepsilon) &= \tau\varepsilon+\beta\varepsilon+\left[(1-\tau)^\lambda-(1-\tau)\right]\left(\frac{z_1+z_2}{n}\right) \\
T_\lambda(\varepsilon+z_1) &= \tau\varepsilon+\left(1-(1-\tau)^\lambda\right)z_1+\beta\left(\varepsilon+z_1\right)+\left[(1-\tau)^\lambda-(1-\tau)\right]\left(\frac{z_1+z_2}{n}\right) \\
T_\lambda(\varepsilon+z_2) &= \tau\varepsilon+\left(1-(1-\tau)^\lambda\right)z_2-\delta\left(\varepsilon+z_2\right)+\left[(1-\tau)^\lambda-(1-\tau)\right]\left(\frac{z_1+z_2}{n}\right).
\end{aligned}
$$

Part (iii). We now consider the richer individual and go back to the notion of inequality equivalence in order to compare his/her post-tax income to the pre-tax income. Recall that the taxation modification in part (ii) has not affected the average tax burden $\tau\mu_x$, we can now calculate the "normalized" inequality gap for the individual with pre-tax income $\varepsilon+z_2$, that is $\frac{\varepsilon+z_2-T_\lambda(\varepsilon+z_2)-(1-\tau)\mu_x}{[(1-\tau)\mu_x]^\lambda}$, where $\mu_x=\varepsilon+\frac{z_1+z_2}{n}$. It follows

$$
\frac{\varepsilon+z_2-T_\lambda(\varepsilon+z_2)-(1-\tau)\left(\varepsilon+\frac{z_1+z_2}{n}\right)}{\left[(1-\tau)\left(\varepsilon+\frac{z_1+z_2}{n}\right)\right]^\lambda}.
$$

After substituting the definition of $T_\lambda(\varepsilon+z_2)$ we obtain the following formulation $\frac{\varepsilon+z_2-\left[\tau\varepsilon+\left(1-(1-\tau)^\lambda\right)z_2-\delta(\varepsilon+z_2)+\left[(1-\tau)^\lambda-(1-\tau)\right]\left(\frac{z_1+z_2}{n}\right)\right]-(1-\tau)\left(\varepsilon+\frac{z_1+z_2}{n}\right)}{(1-\tau)^\lambda\left(\varepsilon+\frac{z_1+z_2}{n}\right)^\lambda}$ that could be simplified as

$$
\begin{aligned}
&\frac{z_2-\left(1-(1-\tau)^\lambda\right)z_2+\delta\left(\varepsilon+z_2\right)-\left[(1-\tau)^\lambda-(1-\tau)\right]\left(\frac{z_1+z_2}{n}\right)-(1-\tau)\left(\frac{z_1+z_2}{n}\right)}{(1-\tau)^\lambda\left(\varepsilon+\frac{z_1+z_2}{n}\right)^\lambda} \\
=\;&\frac{(1-\tau)^\lambda z_2+\delta\left(\varepsilon+z_2\right)-(1-\tau)^\lambda\left(\frac{z_1+z_2}{n}\right)}{(1-\tau)^\lambda\left(\varepsilon+\frac{z_1+z_2}{n}\right)^\lambda}=\frac{z_2+\frac{\delta(\varepsilon+z_2)}{(1-\tau)^\lambda}-\left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon+\frac{z_1+z_2}{n}\right)^\lambda}.
\end{aligned}
$$

We compare now this gap with the one associated to the pre-tax income distribution for the same individual. It can be obtained by setting $\tau=\delta=0$, that is

$$
\frac{z_2-\left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon+\frac{z_1+z_2}{n}\right)^\lambda}.
$$

Note that according to the definition of Lorenz dominance $\succcurlyeq_L^{\lambda,0}$ in order to guarantee that the post-tax income distribution $\mathbf{y}$ is less unequal than the pre-tax income distribution $\mathbf{x}$ it is necessary that for the richer individual we have that $\frac{x_n-T(x_n)-(1-\tau)\mu_x}{[(1-\tau)\mu_x+\theta]^\lambda}\leq\frac{x_n-\mu_x}{[\mu_x+\theta]^\lambda}$ otherwise the inequality dominance $\mathbf{y}\succcurlyeq_L^{\lambda,0}\mathbf{x}$ is violated.

In our case we have $\frac{x_n - T(x_n) - (1-\tau)\mu_x}{[(1-\tau)\mu_x + \theta]^\lambda} = \frac{z_2 + \frac{\delta(\varepsilon+z_2)}{(1-\tau)^\lambda} - \left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon + \frac{z_1+z_2}{n}\right)^\lambda}$ and $\frac{x_n - \mu_x}{[\mu_x+\theta]^\lambda} = \frac{z_2 - \left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon + \frac{z_1+z_2}{n}\right)^\lambda}$. It then follows that

$$\frac{z_2 + \frac{\delta(\varepsilon+z_2)}{(1-\tau)^\lambda} - \left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon + \frac{z_1+z_2}{n}\right)^\lambda} > \frac{z_2 - \left(\frac{z_1+z_2}{n}\right)}{\left(\varepsilon + \frac{z_1+z_2}{n}\right)^\lambda} \tag{18}$$

that violates $\mathbf{y} \succcurlyeq_L^{\lambda,0} \mathbf{x}$.

Note that condition (18) is satisfied irrespective of the values of n and ε, in fact it requires that $\frac{\delta(\varepsilon+z_2)}{(1-\tau)^\lambda} > 0$. If $n \to \infty$ and $\varepsilon \to 0$ we obtain the distribution where $\mu_x \to 0$ as in the first part of the proof of (2) $\Longrightarrow$ (1). We then obtain the limiting proportional tax schedule with constant average tax rate. Considering this case we have shown that if for some pre-tax income levels the average tax rate is decreasing then the inequality reduction condition is violated.

Proof of Proposition 5

The result is a generalization of the proof of Proposition 4. It should follow the same logic and consider all distributions in X_θ compared to the origin $-\theta\mathbf{1}$.

We leave to the reader the modification of the previous proof that could be obtained by translating all incomes from the space with origin 0 to the one with origin $-\theta$, thereby considering incomes $x_i + \theta$ and $y_i + \theta$.

Here we replicate the steps for the derivation of the limit condition of the "inequality neutral tax" schedule in parallel with what done in the first part of the (2) $\Longrightarrow$ (1) section of the proof of Proposition 4. In doing so we provide a generalization of the formula of r derived in (13) for the "unit consistent" IEC.

In what follow we:

(i) find the formula for the taxation limit schemes associated to FIE.

(ii) Consider the extreme limits of $\frac{a_{\lambda,\theta}}{b_{\lambda,\theta}}$ associated to such schemes, both for $\tau \in [0,1]$ and for all $\mu_x > -\theta$.

(iii) Select the limit that shows a lower ratio $\frac{a_{\lambda,\theta}}{b_{\lambda,\theta}}$, and apply the values to the characterization associated to the IIE.

Part (i). Substituting from FIE we get that any inequality neutral tax scheme $T_{\lambda,\theta}(x)$ should satisfy the condition

$$\frac{x_i - T_{\lambda,\theta}(x_i) - (1-\tau)\mu_x}{[(1-\tau)\mu_x + \theta]^\lambda} = \frac{x_i - \mu_x}{[\mu_x + \theta]^\lambda}$$

for all $i = 1, 2, ..., n$.

The condition cold be rearranged such that

$$T_{\lambda,\theta}(x_i) = a_{\lambda,\theta} + b_{\lambda,\theta} x_i$$

where $a_{\lambda,\theta} = \mu_x \left(\frac{[(1-\tau)\mu_x+\theta]^\lambda}{[\mu_x+\theta]^\lambda} - (1-\tau) \right)$, and $b_{\lambda,\theta} = \left(1 - \frac{[(1-\tau)\mu_x+\theta]^\lambda}{[\mu_x+\theta]^\lambda} \right)$. It follows, after simplifications, that

$$r_{\lambda,\theta} = \frac{a_{\lambda,\theta}}{b_{\lambda,\theta}} = \mu_x \frac{[(1-\tau)\mu_x+\theta]^\lambda - (1-\tau)[\mu_x+\theta]^\lambda}{[\mu_x+\theta]^\lambda - [(1-\tau)\mu_x+\theta]^\lambda}. \tag{19}$$

Part (ii). We have to compute the value of the limit taxation schemes both for $\tau \to 1$, and $\tau \to 0$, and also for $\mu_x \to -\theta$ [recall that we consider distributions in X_θ] and $\mu_x \to \infty$. Given the monotonic behavior of $r_{\lambda,\theta}$ within this interval it will be sufficient to select the lowest values of these limits in order to find the parameter for the tax reform that could always be inequality reducing no matter the distribution and the amount of the revenue. Letting $\mu_x \to -\theta$, we obtain

$$\lim_{\mu_x \to -\theta} r_{\lambda,\theta} = -\frac{[-(1-\tau)\theta+\theta]^\lambda}{-[-(1-\tau)\theta+\theta]^\lambda} = \theta.$$

This is the result that one obtains with the IIE criterion, it coincides with $r_{1,\theta}$ in (12). Instead, considering $\mu_x \to \infty$ one obtains $\lim_{\mu_x \to \infty} r_{\lambda,\theta} = \infty$ that is associated to the absolute inequality equivalence criterion. These limits do not depend from τ.

Part (iii). It follows that when the average income converges to its minimum value the inequality neutral taxation converge to the one associated to the IIE with parameter θ. This finding generalizes the one obtained in Proposition 4 and motivates the definition of schedule progression in Proposition 5.