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Louis Raymond Eeckhoudt, Elisa Pagani, Eugenio Peluso

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Multidimensional Risk Aversion:

The Cardinal Sin

Louis Eeckhoudt[*], Elisa Pagani[†], Eugenio Peluso[‡]

Abstract. Attitudes towards multidimensional risk depend both on the shape of the indifference map under certainty and on the degree of concavity of the utility function representing preferences under risk. A decomposition of the risk premium is built on the new notion of "compensated risk aversion". The balance between the two components is shown to depend on the association of the risks. This result is then used to disentangle risk attitudes from the strength of the preferences, in the "intrinsic risk aversion" setting (Bell and Raiffa 1979).

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JEL Classification: D01, D11, D81.

[*]IÉSEG School of Management, CNRS-LEM UMR 9221, louis.eeckhoudt@fucam.ac.be

[†]University of Verona, Department of Economics, Via Cantarane, 24, 37129, Verona, Italy, elisa.pagani@univr.it

[‡]University of Verona (Italy), Department of Economics, Via Cantarane, 24, 37129, Verona, Italy, eugenio.peluso@univr.it

1 Introduction

It is well-known in the risk literature (see Kihlstrom and Mirman 1974) that when lotteries are defined on many attributes, the properties of the Von Neumann-Morgenstern (VNM) utility function can be confused with changes in the degree of substitutability between goods, which is an ordinal property of individual preferences. In fact, in the Expected Utility (EU) theory, risk attitudes are modeled through a utility function. Utility should describe an intrinsic appreciation of wealth, but risk attitudes consist of more than just this appreciation. Even if the analysis of multidimensional risk aversion is limited to the case of agents with identical indifference curves under certainty, it remains difficult to separate the consequences of the ordinal properties of preferences under certainty from those depending on risk attitudes. This issue is especially relevant for empirical purposes, raising questions about a possible bridge between the assessment of the degree of substitutability between determinants of individual well-being, the elicitation of risk aversion and the measurement of the strength of preferences.[1]

When choices under risk involve two or more dimensions, the VNM utility functions used to represent individual preferences are defined on bundles of attributes varying across two or more states of the world. The nature of the substitutability/complementarity of the goods and the association between good and bad outcomes in the different dimensions over states of the world then become crucial for understanding the effects of risk on individual utility (Eeckhoudt et al. 2007). A more comprehensive interpretation of the utility function is then required, which is relevant for risky as well as riskless applications.

This paper develops the Pratt (1964) comparative risk aversion analysis in the bidimensional case and shows that the ordinal properties of the utility function can be disentangled from risk attitudes by a suitable decomposition of the risk premium into two components: the first one reflects the degree of substitutability between two attributes in a certainty environment; the second one accounts for the sensitivity of the decision-maker to possible risks associated with each single attribute and for her willingness to compress or spread positive and negative outcomes across different states of the world. Therefore, the premium now incorporates components describing risk attitudes and riskless attitudes, which can be measured by considering the willingness to pay for a reduction of risk and for a substitution of goods. The intuition is that if a risk moves an initial bundle along a direction tangent to its indifference curve, then the

[1]The former topic has been studied in the inequality/poverty literature by Aaberge and Brandolini (2015), or Brambilla et al. (2013), among many others. In risk analysis, Dohmen et al. (2011) recovered measures of individual risk attitudes towards several attributes, ranging from financial matters to leisure and health. See also Abdellaoui et al. (2008) for an elicitation procedure of preferences on the domain of gains jointly with the domain of losses.

effect on an agent's utility mainly depends on the degree of curvature of the indifference curve, which is invariant to concave transformations. The more divergent the risk direction is from the slope of the indifference curve, the higher is the effect of concave transformations of the utility function, which modify the utility level attached to each indifference curve, but leave unchanged its shape.

This analysis is then extended by explicitly accounting for the strength of preferences. Building on the "intrinsic risk aversion" and the "relative risk aversion" concepts due to Bell and Raiffa (1979) and Dyer and Sarin (1982), respectively, a cardinal value function is adopted to measure differences in utility levels in a certainty environment, while a VNM utility function accounts for preferences towards risk. In this setting, the informational content of the utility function is extremely rich. However, the degree of curvature of the indifference curves - a distinctly ordinal property of individual preferences - plays a crucial role both in the representation of the strength of preferences and in the assessment of risk aversion, which are two cardinal concepts.[2] The analysis of household risk aversion and further examples discussed in the last part of the paper suggest a variety of potential applications of the main results.

To simplify the exposition, the presentation focuses on the bidimensional case. A generalization to the case of many attributes and non-binary lotteries is provided in the Appendix.

2 Comparative risk aversion

2.1 Basic concepts

In the one-dimensional case, an individual is risk averse if she dislikes all zero-mean risks at any initial wealth level x_0:

$$Eu(x_0 + \tilde{\varepsilon}) \leq u(x_0), \ \forall \tilde{\varepsilon} \text{ s.t. } E(\tilde{\varepsilon}) = 0. \tag{2.1}$$

An individual (with utility) v is more risk averse than an individual (with utility) u if v dislikes all lotteries that u dislikes, for any common level of initial wealth. Pratt (1964) characterized comparative risk aversion through absolute and relative measures based on the curvature of the utility function and demonstrated that any increasing and concave transformation $w = f(u(x))$ raises the degree of risk aversion and the amount of the risk premium (see Gollier 2001 for details).

In the bivariate case, let $u : R^2 \to R$ be a componentwise increasing and concave VNM utility function, differentiable as many times as are necessary, defined on two attributes x_1, x_2.

[2]Although the VNM utility function cannot be considered "fully" cardinal, see Luce and Raiffa 1957, p. 31 and Fishburn 1989 for a discussion.

Let u_i denote the marginal utility of u with respect to x_i and u_{ij} the second derivatives with respect to x_i and x_j, for $i, j = 1, 2$.

Richard (1975, p. 13) studied multivariate risk aversion in terms of the preferences of an agent for mixing bad and good outcomes and provided a characterization based on the cross derivative of the utility function. This definition confounds two different aspects: risk aversion and correlation aversion. Kihlstrom and Mirman (1974) defined a risk in terms of a share of the initial vector of the attributes, scaling all the components in the same way. The resulting definition of a risk premium does not fully generalize inequality (2.1) and for this reason what follows refers to the approach of Karni (1979). Karni proposes to complement the Pratt absolute risk aversion coefficient on a single dimensions i, defined as

$$RA_i(x) = -\frac{u_{ii}}{u_i}(x), \tag{2.2}$$

using a matrix measure of risk aversion whose entries $-u_{ij}/u_i$ represent the local aversion to the interaction between any two risks.(Duncan 1977 and Courbage 2001). The "index of absolute correlation aversion"for attribute i with respect to attribute j is

$$CA_i(x) = -\frac{u_{ij}}{u_i}(x), \text{ with } i \neq j. \tag{2.3}$$

The use of the cross derivative and the related notion of aversion to positive (or negative) correlation provides an incomplete view of the decision maker's preferences specification: on one side, risk aversion implies that one prefers with certainty $(5, 5)$ to an equally probable lottery $\{(0, 0), (10, 10)\}$, regardless of the sign of the cross derivatives. On the other side, contrasting the latter with the equally probable lottery $\{(0, 10), (10, 0)\}$, the degree of substituability/ complementarity becomes crucial in the assessment of the risk premium.

2.2 The risk premium

Let $(\tilde{\varepsilon}, \tilde{\delta})$ be a random vector, where $\tilde{\varepsilon}$ is a zero-mean risk on x_1 and $\tilde{\delta}$ is a zero-mean risk on x_2. Let $V = [\sigma_{ij}]$ denote the corresponding symmetric positive semi-definite variance-covariance matrix. If the two risks are simultaneous, then:

$$Eu(x_1 + \tilde{\varepsilon}, x_2 + \tilde{\delta}) \leq u(x_1, x_2), \ \forall \tilde{\varepsilon}, \tilde{\delta} \text{ s.t. } E(\tilde{\varepsilon}) = 0, E(\tilde{\delta}) = 0. \tag{2.4}$$

Assuming small risks, the second order approximation of this expression around (x_1, x_2) leads to:

$$\frac{1}{2}\left[u_{11}(x_1, x_2)\sigma_{11} + u_{22}(x_1, x_2)\sigma_{22} + 2u_{12}(x_1, x_2)\sigma_{12}\right] \leq 0. \tag{2.5}$$

Notice that with negative correlation between the two risks and a positive cross derivative of u, (2.5) is always satisfied and the agent is risk averse for any (x_1, x_2).

The corresponding definition of the risk premium $\pi_1(\tilde{\varepsilon}, \tilde{\delta})$ that must be paid in one dimension, for instance x_1, to get rid of risk in both dimensions is:

$$Eu(x_1 + \tilde{\varepsilon}, x_2 + \tilde{\delta}) = u(x_1 - \pi_1(\tilde{\varepsilon}, \tilde{\delta}), x_2), \ \forall \tilde{\varepsilon}, \tilde{\delta}, \text{ s.t. } E(\tilde{\varepsilon}) = 0, E(\tilde{\delta}) = 0. \tag{2.6}$$

The risk premium is positive if and only if the utility function is concave (Theorem 1, Karni 1979).[3]

By a second order approximation of equation (2.6), it follows that:

$$\pi_1 \approx -1/2 \left[\frac{\sigma_{11} u_{11} + \sigma_{22} u_{22} + 2\sigma_{12} u_{12}}{u_1} \right]. \tag{2.7}$$

Richard (1975) studied the effects of a concave transformation applied to the utility function on the magnitude of the risk premium. Kihlstrom and Mirman (1974) extended the Pratt approach to the bidimensional case characterizing "more risk aversion" in terms of "more concave" utility functions (p. 366). Under a specific definition of a risk premium (directional and multiplicative, see Proposition 1, p. 368), they also generalize the Pratt result that a higher risk premium is paid by a more risk averse agent. However, such transformations only change the utility values attached to the different indifference curves without affecting their shape, which contains the ordinal information about preferences under certainty. In what follows, the ordinal component of individual preferences is shown to contribute to determining the risk premium through a separable component.

2.3 Compensated risk aversion

In what follows, the special case of binary equiprobable lotteries is developed, while the same results can be obtained in a more general setting, as shown in the Appendix.[4] The risk premium π_1 is defined by:

$$1/2[u(x_1 + \varepsilon, x_2 + \delta) + u(x_1 - \varepsilon, x_2 - \delta)] = u(x_1 - \pi_1, x_2), \text{ which is approximated by} \tag{2.8}$$

$$\pi_1 \approx -1/2 \left[\frac{\varepsilon^2 u_{11} + \delta^2 u_{22} + 2\delta\varepsilon u_{12}}{u_1} \right], \tag{2.9}$$

or equivalently,

$$\pi_1 \approx \varepsilon^2/2 \left[RA_1 + \frac{\delta^2 u_2}{\varepsilon^2 u_1} RA_2 + 2\frac{\delta}{\varepsilon} CA_1 \right]. \tag{2.10}$$

[3] In fact, from the property of monotonicity of the utility function $u(x_1 - \pi_1, x_2) \leq u(x_1, x_2), \forall \pi_1 \geq 0$ and using equation (2.6), we get $Eu(x_1 + \tilde{\varepsilon}, x_2 + \tilde{\delta}) \leq u(x_1, x_2)$, which is the definition of concavity of u by means of Jensen's inequality.

[4] More precisely, for expository reasons, the binary equiprobable lottery between $(+\varepsilon, +\delta)$ and $(-\varepsilon, -\delta)$ is presented. An alternative possibility is a 50-50 lottery between $(+\varepsilon, -\delta)$ and $(-\varepsilon, +\delta)$.

The effect of the cross derivative of the utility function on the risk premium depends on the correlation between the outcomes of the risks.

The risk premium locally depends on (i) the sum of the risk aversion coefficient of the reference good, (ii) the risk aversion coefficient of the other good weighted by the marginal rate of substitution and (iii) the correlation aversion index.[5] A positive CA ($u_{12} < 0$) means that a lottery that substitutes to some extent one good with another reduces the total riskiness. The cross derivative of u is not itself a marker of the presence of multivariate risk aversion, but plays a role for the intensity of risk aversion and correlation aversion.

The effect on the risk premium of a concave transformation of the utility function u are now investigated. Considering negatively correlated risks, from equation (2.9), a concave transformation raises the risk aversion coefficient in each dimension and the risk premium. However, this effect might be balanced by a decrease in the cross derivative. As noted by Kannai (1980), for any utility function u, ALEP[6] complements can become substitutes by applying a sufficiently concave transformation f to u.[7]

To study this “compensative effect”, we consider the relative variation of the total differential of u_i with respect to dx_i:

$$-\frac{du_i(x_i,x_j)/dx_i}{u_i(x_i,x_j)} \approx -\frac{u_{ii}}{u_i} - \frac{u_{ij}}{u_i}\frac{dx_j}{dx_i}. \tag{2.11}$$

If this approximation is computed at a fixed level of utility, then $\frac{dx_j}{dx_i} = -\frac{u_i}{u_j}$ and we get:

$$\frac{du_i(x_i,x_j)/dx_i}{u_i(x_i,x_j)} \approx -\frac{u_{ii}}{u_i} + \frac{u_{ij}}{u_j}. \tag{2.12}$$

In the one-dimensional case, (2.11) reduces to the well-known Pratt absolute coefficient. With two dimensions, the relative change in the marginal utility resulting from the Pratt coefficient is fully compensated for by adding the indirect effect on the other variable. The Compensated Risk Aversion (CRA) coefficients are then defined as:

$$CRA_1 = -\frac{u_{11}}{u_1} + \frac{u_{12}}{u_2} = RA_1 + MRS_{21} \times CA_1 \tag{2.13}$$

$$CRA_2 = -\frac{u_{22}}{u_2} + \frac{u_{12}}{u_1} = RA_2 + MRS_{12} \times CA_2 \tag{2.14}$$

A central property of the CRA coefficients is now stated.

[5]Following Karni (1979), CA_1 measures the excess of the risk premium due to the simultaneous randomness of $\tilde{\varepsilon}$ and $\tilde{\delta}$, with respect to the sum of the unidimensional risk premia when only $\tilde{\varepsilon}$ and only $\tilde{\delta}$ are random.

[6]Two goods are ALEP substitutes (complements) at x if and only if $u_{12}(x) < 0$ ($u_{12}(x) > 0$). See Samuelson 1974.

[7]In fact, given $w = f(u(x))$, its second cross derivative is $w_{ij} = f_{uu}u_iu_j + f_uu_{ij}$ and its sign can be changed by selecting an appropriate concave function f.

Lemma 2.1. *The CRA_i measures are invariant to any strictly monotonic differentiable transformation of the utility function.*

Proof. By considering the transformation $w = f \circ u$, from the definition of absolute risk and correlation aversion we get:

$$-\frac{w_{ij}}{w_j} = -\frac{f_{uu}}{f_u}u_i - \frac{u_{ij}}{u_j} \text{ and } -\frac{w_{ii}}{w_i} = -\frac{f_{uu}}{f_u}u_i - \frac{u_{ii}}{u_i}, \text{ which imply } \frac{w_{ij}}{w_j} - \frac{w_{ii}}{w_i} = \frac{u_{ij}}{u_j} - \frac{u_{ii}}{u_i}. \quad (2.15)$$

□

The two CRA coefficients then describe an ordinal feature of individual preferences. They are invariant to concave transformations of the utility function because the increase of the absolute risk aversion coefficient on each single variable is perfectly compensated for by an increase in the correlation aversion $-u_{ij}/u_i$. The definition of the risk premium stated in equation (2.9), together with equations (2.13) and (2.14), proves the following proposition.

Proposition 2.1. *Given the risks $(\tilde{\varepsilon}, \tilde{\delta})$, the risk premium $\pi_1(\tilde{\varepsilon}, \tilde{\delta})$ can be decomposed as $\pi_1 \approx P + \rho CA_2$, with*

$$P = \frac{1}{2}\varepsilon^2 \left(CRA_1 + \frac{\delta^2 u_2}{\varepsilon^2 u_1} CRA_2 \right) \text{ and } \rho = \left[\frac{1}{2}\varepsilon^2 \left(1 + \frac{\delta u_2}{\varepsilon u_1} \right)^2 \right]. \quad (2.16)$$

The P and ρ components are invariant to strictly monotonic transformations of the utility function, while CA_2 is sensitive to non-linear transformations of the utility function.

In the Appendix, the result of Proposition 2.1 is generalized to the n-dimensional case with non-binary random risks.

Proposition 2.1 allows one to disentangle the ordinal component, that describes individual preferences under certainty, from individual preferences under risk. The magnitude of these components depends on the direction of the risk. For instance, when $\varepsilon/\delta = -u_2/u_1$ (the risk follows exactly the direction of the marginal rate of substitution) then $\pi_1 = P$ and the risk premium remains invariant to concave transformations of the utility function. As the risk direction moves away from the MRS, a more concave utility function generates a stronger impact on the risk premium. This feature is illustrated by the following example.

Example 2.1. *Consider a risk averse agent with utility function $u(x_1, x_2) = (x_1 x_2)^{1/2}$ and the concave transformation $w(u; \gamma)$*

$$w(u;\gamma) = \begin{cases} \frac{u^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1; \\ ln(u) & \text{if } \gamma = 1. \end{cases} \quad (2.17)$$

The cases with $\gamma = 0, 1, 2$ correspond to a correlation prone, neutral or averse agent, respectively:

Table 1: Risk premium decomposition.

(a) $\varepsilon = \frac{1}{\sqrt{1.09}}$, $\delta = \frac{0.3}{\sqrt{1.09}}$

	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$
π_1^N	0.1123	0.1697	0.2270
π_1^P	0.0573	0.1697	0.2821
P	0.1697	0.1697	0.1697
$\rho^N CA_2$	-0.0573	0	0.0573
$\rho^P CA_2$	-0.1123	0	0.1123

(b) $\varepsilon = \frac{1}{\sqrt{1.0025}}$, $\delta = \frac{0.05}{\sqrt{1.0025}}$

	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$
π_1^N	0.01	0.01	0.01
π_1^P	0	0.01	0.02
P	0.01	0.01	0.01
$\rho^N CA_2$	0	0	0
$\rho^P CA_2$	-0.01	0	0.01

i. $w(\cdot; 0) \equiv u(x_1, x_2) = (x_1 x_2)^{1/2}$;

ii. $w(\cdot; 1) = ln(x_1 x_2)^{1/2}$;

iii. $w(\cdot; 2) = -(x_1 x_2)^{-1/2}$.

Given the initial allocation $\bar{x}_1 = 100$ *and* $\bar{x}_2 = 5$*, the left panel represents an equiprobable lottery with risks along the normalized directions* $\varepsilon = \frac{1}{\sqrt{1.09}}$ *and* $\delta = \frac{0.3}{\sqrt{1.09}}$*. Using (2.16) and (2.9), the risk premia* π_1^P *and* π_1^N *and the values of the components* P*,* $\rho^P CA_2$ *and* $\rho^N CA_2$ *are computed in the case of positively and negatively correlated risks, respectively.*

As expected, the risk premium increases with γ *regardless of the type of risk. To illustrate the effect of correlation aversion, consider first the correlation neutral case* $\gamma = 1$ *when the utility is additively separable. The risk premium then coincides with the ordinal component. If the agent is correlation prone (*$\gamma = 0$*), she has a lower risk premium when the risks are positively correlated:* $\pi_1^N > \pi_1^P$*. Conversely,* $\pi_1^N < \pi_1^P$ *when the agent is correlation averse (*$\gamma = 2$*).*

To show the consequences of a change in the direction of the risk, in the right panel, the direction $\varepsilon = \frac{1}{\sqrt{1.0025}}$ *and* $\delta = \frac{0.05}{\sqrt{1.0025}}$ *is chosen, which coincides with the direction of the MRS. For negatively correlated risks, the risk premium* π_1^N *collapses to the ordinal component and* ρ^N *is equal to* 0*. For positively correlated risks, the risk premium* $\pi_1^P = 0$ *when* $\gamma = 0$ *and the ordinal component* P *is exactly compensated for by* $\rho^P CA_2$*.*

Further, notice that empirical evidence shows that it is possible to be risk seeking with a diminishing marginal utility of wealth (Chateauneuf and Cohen 1994 and Abdellaoui et al. 2008), but in the unidimensional EU theory this is not allowed. While, in the multidimensional case, both these aspects can be considered, because the impact of the P component of π_1 can be reduced by ρCA_2 if $CA_2 < 0$, reflecting a more risk seeking behavior due to correlation proneness.

The ordinal component P depends on the elasticity of substitution. The elasticity of substitution is always equal to 1 for Cobb-Douglas utility functions. A suitable property for the class of the CES utility functions is stated next.[8]

Remark 2.1. *Given a CES utility function:*

$$u(x_1, x_2; s, r) = \left[r_1 x_1^{\frac{s-1}{s}} + r_2 x_2^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}}, \tag{2.18}$$

the "relative" compensated risk aversion $x_i CRA_i$ is equal to the reciprocal of the elasticity of substitution.

From (2.13) and (2.14), it follows that $CRA_i = 1/sx_i$, $\forall i = 1, 2$ where s is the elasticity of substitution. From (2.16), if $s \to +\infty$ (perfect substitutability), then $P \to 0$, and the ordinal component of the risk premium disappears.

The next section investigates the risk attitudes under the assumption that the strength of preferences is measurable in a certainty environment. Exploiting the previous results, risk aversion will be disentangled from the strength of preferences.

3 An application to the theory of intrinsic risk aversion

Inspired by the works of Bell and Raiffa (1979) and Dyer and Sarin (1982), preferences under risk of an agent can be modeled by means of the utility function $U(x) = u(v(x))$, where u is a VNM utility and $v : R_+^2 \to R_+$ is a cardinal value function representing preferences under certainty.[9] More precisely, the change in the strength of preferences over a good after small changes in the other good is measured through the coefficient[10]

$$CA_i^v = -\frac{v_{ij}}{v_i}. \tag{3.1}$$

This measure can be interpreted as the bidimensional extension of the "value satiation" coefficient of Dyer and Sarin (1982, p. 877).

The impact of the introduction of risk on the preferences of an agent "can then be thought of as a basic psychological (personality) trait of the individual" (Bell and Raiffa 1979, p. 393) called the "intrinsic" risk aversion attitude ("relative" risk aversion in Dyer and Sarin 1982), defined as

$$RA^v = -\frac{u_{vv}}{u_v}. \tag{3.2}$$

[8]Many empirical applications in macroeconomics and finance assume CES preferences and interpret u as a discounted expected utility (Mankiw et al. 1985 and Choi et al. 2008).

[9]Olson (1990), in Theorem 5, separates preferences over commodities from preferences under risk.

[10]In what follows, with an abuse of notation, the CA_i and CRA_i terms are also used for the value function.

Proposition 2.1 can be applied in this setting to separate the effect of the strength of preferences on the risk premium from that of risk aversion.

Proposition 3.1. *Given the risks* $(\tilde{\varepsilon}, \tilde{\delta})$ *and the utility function* $U(x) = u(v(x))$*, the risk premium* π_1 *can be decomposed as* $\pi_1 \approx P + \rho(CA_2^v + RA^v v_1)$*, where*

$$P = \frac{1}{2}\varepsilon^2 \left(CRA_1 + \frac{\delta^2 v_2}{\varepsilon^2 v_1} CRA_2 \right) \text{ and } \rho = \left[\frac{1}{2}\varepsilon^2 \left(1 + \frac{\delta v_2}{\varepsilon v_1} \right)^2 \right]. \tag{3.3}$$

Proof. Proceeding as in equations (2.8) and (2.9), the risk premium becomes

$$\pi_1 \approx -1/2 \left[\frac{\varepsilon^2 U_{11} + \delta^2 U_{22} + 2\delta\varepsilon U_{12}}{U_1} \right]. \tag{3.4}$$

If $U(x) = u(v(x))$, then

$$\pi_1 \approx -1/2 \left[\frac{\varepsilon^2 (u_{vv} v_1^2 + u_v v_{11}) + \delta^2 (u_{vv} v_2^2 + u_v v_{22}) + 2\delta\varepsilon (u_{vv} v_1 v_2 + u_v v_{12})}{u_v v_1} \right], \tag{3.5}$$

that, rearranged, leads to

$$\pi_1 \approx -\frac{1}{2} \left[\frac{\varepsilon^2 v_{11} + \delta^2 v_{22} + 2\varepsilon\delta v_{12}}{v_1} \right] - \frac{1}{2} \frac{u_{vv}}{u_v} v_1 \varepsilon^2 \left(1 + \frac{\delta}{\varepsilon} \frac{v_2}{v_1} \right)^2. \tag{3.6}$$

The result is then obtained by applying Proposition 2.1 to the component $-\frac{1}{2} \left[\frac{\varepsilon^2 v_{11} + \delta^2 v_{22} + 2\varepsilon\delta v_{12}}{v_1} \right]$.

□

Notice that P, ρ and CA_2^v depend only on the derivatives of the value function v, accounting for different aspects of the preferences of the agent under certainty. P depends on ordinal characteristics of the value function, ρ depends on the MRS (and on the characteristics of the risk).

The cross derivative of the utility funtion U determines the correlation aversion coefficient $CA_2(x) = -U_{12}/U_2$, which can be rewritten as

$$CA_2(x) = -\frac{U_{12}}{U_2} = -\frac{v_{12}}{v_2} - \frac{u_{vv}}{u_v} v_1 = CA_2^v(x) + RA^v(x) v_1(x). \tag{3.7}$$

The shape of the VNM utility u only affects the correlation aversion coefficient $CA_2(x)$. Equation (3.7) shows that $CA_2(x)$ is the sum of CA_2^v (which measures the effect of changes in the attributes under certainty) and RA^v, the risk aversion coefficient, weighted by the marginal value function v_1.

Under the intrinsic risk aversion setting, it is then possible to provide a finer decomposition of the risk premium, by distinguishing in a neat way the different constituents of individual preferences. This result is illustrated by the following extension of Example 2.1.

Table 2: Premium decomposition with $\varepsilon = \frac{1}{\sqrt{1.09}}$, $\delta = \frac{0.3}{\sqrt{1.09}}$.

	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$
π_1^P	0.0573	0.1697	0.2821
P	0.1697	0.1697	0.1697
ρ^P	22.47	22.47	22.47
CA_2^v	-0.005	-0.005	-0.005
$RA^v v_1$	0	0.005	0.01

Example 3.1. *Given* $U(x;\gamma) = u(v(x);\gamma) = \begin{cases} \frac{v^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1; \\ ln(v) & \text{if } \gamma = 1, \end{cases}$

and the positively correlated risks along the direction $\varepsilon = \frac{1}{\sqrt{1.09}}$, $\delta = \frac{0.3}{\sqrt{1.09}}$. *The risk premium can be decomposed as shown in Table 2.*

If $\gamma = 0$, *the premium does not depend on the intrinsic risk aversion coefficient* $RA^v v_1$. *With* $\gamma = 1$, *the effect of the intrinsic risk aversion is exactly compensated for by the strength of preferences, as measured by* CA_2^v. *The agent is intrinsically risk averse but correlation neutral. Finally, when* $\gamma = 2$, *the agent is both risk averse and intrinsically risk averse because the* $RA^v v_1$ *term dominates the* CA_2^v *term.*

In the following subsection, the intrinsic risk aversion setting is applied to the Samuelson (1956) household unitary model, which inspired the "social welfare function" literature and the development of "household economics".

3.1 Household intrinsic risk aversion

Let us consider two individuals living in a household whose preferences are represented by an additively-separable utility à la Samuelson (1956):

$$v(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2), \tag{3.8}$$

where ψ_i is the cardinal utility assessing the value of income x_i for individual i.[11] The household's preferences over risks on both individual incomes are then described by the VNM utility function:

$$u(v(x_1, x_2)) = u(\psi_1(x_1) + \psi_2(x_2)). \tag{3.9}$$

[11]Here it is assumed that ex-ante risk-sharing schemes cannot be adopted.

From the result of Proposition 3.1, the risk premium paid by individual 1, if she bears the risks $(\tilde{\varepsilon}, \tilde{\delta})$ of the whole family, is:

$$\pi_1 \approx \underbrace{\frac{\varepsilon^2}{2}\left[-\frac{\psi_1''(x_1)}{\psi_1'(x_1)} - \frac{\delta\psi_2'(x_2)}{\varepsilon\psi_1'(x_1)}\frac{\psi_2''(x_2)}{\psi_2'(x_2)}\right]}_{P} - \underbrace{\frac{u_{vv}}{u_v}}_{RA^v}\underbrace{\psi_1'(x_1)}_{v_1}\underbrace{\frac{\varepsilon^2}{2}\left(1+\frac{\delta\psi_2'(x_2)}{\varepsilon\psi_1'(x_1)}\right)^2}_{\rho}. \tag{3.10}$$

P is a weighted sum of the individuals' income value satiation coefficients $\frac{\psi_i''}{\psi_i'}$. The term RA^v, representing the intrinsic risk aversion of the household as a whole, is multiplied by the marginal utility of the first individual and by ρ, which accounts for the risk composition and the MRS between individual incomes.

This example also shows better the role of the cross derivative. In fact, in the case of additively-separable utilities, CRA_i coincides with RA_i and $CA_2(x) = RA^v(x)v_1(x)$ and there are no compensatory effects on the income value satiation coefficients and not even on the risk aversion coefficient.

4 Conclusions

A new decomposition of the risk premium is established. Three different features of individual preferences are shown to drive risk attitudes in the multidimensional case: the degree of substitutability among goods, the intensity of risk aversion on each single dimension and the degree of correlation aversion of the decision-maker.[12]

Following the Bell and Raiffa (1979) and Dyer and Sarin (1982) analyses of the strength of preferences under risk, the cardinal component of individual preferences is shown to depend both on the strength of preferences and on intrinsic risk aversion. Several empirical studies suggest that risk measures built using a cardinal value function are more flexible compared to the traditional ones (Currim and Sarin 1984 and Pennings and Smidts 2000).

Further, the intrinsic risk aversion is used to evaluate risk in a unitary household welfare setting. In this framework, the risk premium accounts for both the strength of individual preferences and intrinsic risk aversion of the household as a whole. These findings can be extended to the theory dealing with collective decisions on consumption or saving.[13] Future research can focus on empirical investigations on risk attitudes that require integrating data collected from different sources: individual preferences on several goods in a risk-free framework,

[12] Epstein and Tanny (1980) provided a way to compare two random variables by means of correlation aversion. See also Tsetlin and Winkler 2009.

[13] See Peluso and Trannoy 2004 for an introduction to the sharing rule in a unitary setting, Chavas et al. 2018 for recent advances on the collective approach literature and Apps et al. 2014 for a study on precautionary saving in two-person households.

risk preferences elicited on single goods, and preferences for correlation. This result strengthens the bridge between inequality and risk measurement, whose formal analogy is well-known in the economics literature (see Gajdos and Weymark 2012 for an introduction). Further links can be explored also in production theory, dealing with the study of the strategies for the reduction of risk due to internationalization (Elango et al. 2013). Notice further that Qazi et al. (2018) proposed a method for the mitigation of the risk based on the elicitation of individual preferences over changes in risk in many attributes.

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Appendix

The paper develops the case of equiprobable binary lotteries. In this Appendix, the results are generalized to the case of a risk represented by any n-dimensional random vector. Let $x \in R_+^n$ and $\tilde{\varepsilon}$ be a random vector, where $\tilde{\varepsilon}_i$ is a zero-mean risk on x_i, with $i = 1, \ldots, n$, and let $V = [\sigma_{ij}]$ denote the corresponding symmetric positive semi-definite variance-covariance matrix. With simultaneous risks, we know that:

$$Eu(x + \tilde{\varepsilon}) \leq u(x), \ \forall \tilde{\varepsilon} \text{ s.t. } E(\tilde{\varepsilon}) = 0. \tag{4.1}$$

Assuming small risks, the second order approximation of this expression around x leads to:

$$\sum_i \sum_j u_{ij}(x)\sigma_{ij} \leq 0. \tag{4.2}$$

The definition of the risk premium $\pi_1(\tilde{\varepsilon})$ that must be paid in the first dimension x_1, to get rid of risk in all dimensions is:

$$Eu(x + \tilde{\varepsilon}) = u(x_1 - \pi_1(\tilde{\varepsilon}), x_2, \ldots, x_n), \ \forall \tilde{\varepsilon} \text{ s.t. } E(\tilde{\varepsilon}) = 0. \tag{4.3}$$

Computing the second order approximation of equation (4.3), it follows:

$$\pi_1 \approx -\frac{1}{2}\frac{\sum_i \sum_j u_{ij}\sigma_{ij}}{u_1}. \tag{4.4}$$

This equation can be equivalently written in terms of the indices of risk aversion and correlation aversion:

$$\pi_1 \approx \frac{1}{2}\sum_i \left[RA_i \sigma_{ii}\frac{u_i}{u_1} + \sum_j CA_i \sigma_{ij}\frac{u_i}{u_1} \right]. \tag{4.5}$$

The Compensated Risk Aversion (CRA) coefficients are defined as:

$$CRA_i = -\frac{u_{ii}}{u_i} + \frac{1}{n-1}\sum_{j\neq i}\frac{u_{ij}}{u_j}. \tag{4.6}$$

As in the bidimensional case, the change in the marginal utility measured by the Pratt coefficient is fully compensated for by the indirect effect on the other variables. This indirect effect is measured by a mean of the correlation aversion coefficients.

The central property of the CRA coefficients still holds.

Lemma 4.1. *The CRA_i measures are invariant to any strictly monotonic transformation of the utility function.*

Proof. By considering the transformation $w = f \circ u$, we get:

$$-\frac{w_{ij}}{w_j} = -\frac{f_{uu}}{f_u}u_i - \frac{u_{ij}}{u_j}, \text{ which implies } \quad -\frac{w_{ii}}{w_i} + \frac{1}{n-1}\sum_{j\neq i}\frac{w_{ij}}{w_j} = -\frac{u_{ii}}{u_i} + \frac{1}{n-1}\sum_{j\neq i}\frac{u_{ij}}{u_j}. \tag{4.7}$$

□

The decomposition of the premium can be derived from equations (4.4) and (4.6).

Proposition 4.1. *Given the risk $\tilde{\varepsilon}$, the risk premium $\pi_1(\tilde{\varepsilon})$ can be decomposed as $\pi_1 \approx P + \sum_i \sum_{j\neq i} CA_j \rho_{ij}$, with*

$$P = \frac{1}{2}\sum_i \sigma_{ii}\frac{u_i}{u_1}CRA_i \text{ and } \rho_{ij} = \frac{1}{2}\left(\frac{\sigma_{ii}}{n-1}\frac{u_i}{u_1} + \sigma_{ij}\frac{u_j}{u_1}\right). \tag{4.8}$$

P and ρ_{ij} are invariant to strictly monotonic transformations of the utility function, while CA_j is sensitive to non-linear transformations of the utility function.