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Prudence, risk measures and the Optimized Certainty Equivalent: a note

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Abstract. The notion of prudence was very useful in economics to analyze saving or self protection decisions.

We show in this note that, following Ben-Tal and Teboulle (2007), it is also relevant to develop risk measures.

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JEL Classification: D31, D81, G11.

1 Introduction

In two related papers, Ben-Tal and Teboulle (1986, 2007) have proposed the optimized certainty equivalent (OCE) as an alternative to other notions of the certainty equivalent such as the classical certainty equivalent or the u-mean. It turns out that the OCE is an efficient tool to interpret and develop coherent or convex risk measures.

In section 2 we establish the link between the OCE and the "equivalent prudence premium" as defined by Kimball (1990) and we show that some of its properties recover or extend results presented by Ben-Tal and Teboulle (2007).

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The main result of the note is developed in section 3 where we use the notion of "compensating prudence premium"[1] to explicit its link with convex risk measures.

A short conclusion ends the note.

2 Main results

Using the notation in Ben-Tal and Teboulle (2007), the OCE is obtained by solving

$$S_u(x) = \sup_{\eta \in \mathbb{R}} \{\eta + E\left[u(x-\eta)\right]\}, \tag{2.1}$$

where u is a normalized utility function, such that $u(0) = 0,\ u'(0) = 1$ and $E[u(x)]$ is a sure present value of a future uncertain amount x. The quantity η is the amount allocated for present consumption. Assuming that the supremum in (2.1) is attained and under some further assumptions, this yields the first order condition

$$1 - E\left[u'(x-\eta)\right] = 0 \tag{2.2}$$

(see Remark 2.1 of Ben-Tal and Teboulle (2007)). Under concavity of u the second order condition for a maximum is satisfied.

In his 1990 paper Kimball defines the "equivalent prudence premium" associated to a risk x (ψ_x) by

$$E\left[u'(x)\right] = u'(\mu - \psi_x(\mu)) \tag{2.3}$$

where $\mu = E[x]$ and $\psi_x(\mu)$ indicates that ψ is evaluated at the level of μ.

Notice that - as observed by Kimball - the definition of the prudence premium is very similar to that of the risk premium in Pratt (1964); the only difference is that total utility u in Pratt is replaced by marginal utility (u') in Kimball.

Because of (2.3), (2.2) can also be written:

$$1 - u'(\mu - \eta - \psi_x(\mu - \eta)) = 0 \tag{2.4}$$

where $\psi_x(\mu - \eta)$ stands for the prudence premium attached to x when the decision maker has a wealth equal to $\mu - \eta$. Since the utility function is defined in such a way that $u'(0) = 1$, it follows from (2.4) that

$$\eta = \mu - \psi_x(\mu - \eta), \tag{2.5}$$

[1] Notice that there are also two definitions related to the risk premium (Pratt (1964)): the ask and the bid price of a lottery.

which is a nonlinear equation in the unknown η.

This result has some interesting implications with some of them appearing already in Ben-Tal and Teboulle (2007).

(i) If u is quadratic, $\psi = 0$ and $\eta = \mu$ (see Ben-Tal and Teboulle (2007), p.457).

(ii) If u is exponential, the prudence premium is constant in wealth so that $\psi_x(\mu - \eta) = \psi_x(\mu)$ and

$$\eta = \mu - \psi_x(\mu). \tag{2.6}$$

Kimball (1990) shows that $\psi_x(\mu)$ is approximately equal to $\frac{\sigma_x^2}{2}\left(-\frac{u'''(\mu)}{u''(\mu)}\right)$ when u'' and u''' are respectively the second and the third order derivatives of u.

As a result, if $u(x) = \frac{1-e^{-\lambda x}}{\lambda}$ then $-\frac{u'''(\mu)}{u''(\mu)} = \lambda$ and

$$\eta \cong \mu - \lambda\frac{\sigma_x^2}{2}. \tag{2.7}$$

(iii) In order to extend these results to other utility functions we notice by a first order development that

$$\psi_x(\mu - \eta) = \psi_x(\mu) - \eta\frac{d\psi_x}{d\mu}(t) \tag{2.8}$$

for some $0 \leq t \leq \mu$.

Consequently,

$$\eta = \frac{\mu - \psi_x(\mu)}{1 - \frac{d\psi_x}{d\mu}(t)}. \tag{2.9}$$

A standard assumption in the economics literature is that the prudence premium (as the risk premium) is decreasing in wealth. As a result under decreasing prudence we have

$$\eta < \mu - \psi_x(\mu). \tag{2.10}$$

Notice that since the OCE is cash-additive[2], monotone and concave (see Ben-Tal and Teboulle (2007), Theorem 2.1) it is up to a sign a convex risk measure[3], contrarily to the equivalent prudence premium which is not in general cash-additive (see, for instance, the case of a quadratic utility function).

[2]With cash-additivity of f one means that $f(x+c) = f(x) + c$ (resp. $f(x+c) = f(x) - c$) for any $c \in \mathbb{R}$ and any x for increasing (resp. decreasing) f.

[3]See Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) for the definition and a detailed treatment of convex risk measures.

3 Prudence and Risk Measures

In this section, we explicit the link between OCE, prudence premium and risk measures.

We remind that any convex, monotone and cash-additive risk measure $\rho : \mathcal{X} \to \mathbb{R}$ satisfying $\rho(0) = 0$ (where $\mathcal{X}$ denotes a set or space of profit and loss of financial risky positions) can be interpreted as the minimal capital needed to secure a financial position, i.e.

$$\rho_A(x) = \inf\{m \in \mathbb{R} : x + m \in A\}, \tag{3.1}$$

where $A = \{x \in \mathcal{X} : \ \rho(x) \leq 0\}$ is the acceptance set that can be based on the expected utility. In this case, the position is acceptable if $E[u(x)]$ is bounded from below by a given threshold u_0, hence the acceptance set becomes $A_u = \{x \in \mathcal{X} : E[u(x)] \geq u_0\}$ and the associated measure ρ_{A_u} is a utility-based shortfall risk measure. Moreover, if $u(x) = -e^{-\theta x}$ for some $\theta > 0$, then we have the entropic risk measures. See Föllmer and Schied (2004) and the references therein for further details.

According to Ben-Tal and Teboulle (2007):

- the u-mean of Bühlmann (1970)

  $$M_u(x) = \sup\{\eta \in \mathbb{R} : E[u(x - \eta)] \geq 0\}$$

  is the opposite of the bounded shortfall risk (see also Föllmer and Schied (2004))

  $$\rho_\ell(x) = \inf\{\eta \in \mathbb{R} : E[\ell(-x - \eta)] \leq \ell_0\}$$

  with $\ell(x) - \ell_0 = -u(-x)$ and ℓ nondecreasing and convex for a nondecreasing and concave u.

- $M_u(x) = \inf_{\lambda>0} S_{\lambda u}(x) \leq S_u(x)$ and $M_u(x)$ is concave.

Now, let us recall the definition of the compensating prudence premium of Kimball (1990), that is denoted by ψ_x^* and is solution of

$$u'(\mu) = E[u'(x + \psi_x^*(\mu))]. \tag{3.2}$$

Since u' is nonincreasing because of concavity of u, we obtain

$$\psi_x^*(\mu) = \inf\{\eta \in \mathbb{R} : E[u'(x + \eta)] \leq u'(\mu)\} = \inf\{\eta \in \mathbb{R} : E[-u'(x + \eta)] \geq -u'(\mu)\}. \tag{3.3}$$

Even if the previous formulation of the compensating prudence premium is very close to (3.1), ψ_x^* fails to be cash-additive (for instance, for a quadratic utility). This can be understood by noticing that, instead of having a level u_0 independent of x, it appears here the term $-u'(\mu)$ that depends on x via its mean μ.

Let us modify now the definition of prudence premium similarly to the u-mean of Bühlmann (1970), that is denoted by ψ_x^{**} and solution of

$$E[u'(x+\psi_x^{**})] = u'(0). \tag{3.4}$$

Additionally to monotonicity and concavity of u, assume now that u' is convex (or, equivalently, $u''' \geq 0$ if u is three times differentiable). The modified prudence premium defined in (3.4) can be seen also as a convex risk measure. Indeed,

$$\psi_x^{**} = \inf\{\eta \in \mathbb{R} : E[-u'(x+\eta)] \geq -u'(0)\},$$

where $-u'$ is nondecreasing and concave. Hence the modified prudence premium can be also written as in (3.1) by means of an acceptable set $A = A_{-u'} = \{x \in \mathcal{X} : E[-u'(x)] \geq -u'(0)\}$ depending on u'. By Föllmer and Schied (2004), it follows that the modified prudence premium is a convex, monotone, normalized and cash-additive risk measure.

It is worth to emphasize that, while both the prudence premium and the compensating prudence premium in general do not satisfy cash-additivity, by construction the modified prudence premium satisfies such a property. Furthermore, while for uncertain amount x with zero mean the modified prudence premium ψ_x^{**} coincides with $\psi_x^*(0)$, for x with mean μ the relation between the modified and the compensating prudence premium is the following

$$\psi_x^{**} = \psi_{x-\mu}^*(0) - \mu.$$

By cash-additivity of ψ_x^{**}, indeed, we get $\psi_x^{**} = \psi_{x-\mu+\mu}^{**} = \psi_{x-\mu}^{**} - \mu = \psi_{x-\mu}^*(0) - \mu$.

4 Conclusions

So far the notion of prudence was used mostly in economics because it turned out to be crucial to analyze saving decisions (Kimball (1990)) or prevention choices (Eeckhoudt and Gollier (2005)).

We have shown in this note that the concept is also useful in other contexts such as the development of convex risk measures.

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