The structure of Nash equilibria in Poisson games

Claudia Meroni, Carlos Pimienta

THE STRUCTURE OF NASH EQUILIBRIA IN POISSON GAMES[*]

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ABSTRACT. In finite games, the graph of the Nash equilibrium correspondence is a semialgebraic set (i.e. it is defined by finitely many polynomial inequalities). This fact implies many game theoretical results about the structure of equilibria. We show that many of these results can be readily exported to Poisson games even if the expected utility functions are not polynomials. We do this proving that, in Poisson games, the graph of the Nash equilibrium correspondence is a globaly subanalytic set. Many of the properties of semialgebraic sets follow from a set of axioms that the collection of globaly subanalytic sets also satisfy. Hence, we easily show that every Poisson game has finitely many connected components and that at least one of them contains a stable set of equilibria. By the same reasoning, we also show how generic determinacy results in finite games can be extended to Poisson games.

KEY WORDS. Poisson games, voting, stable sets, o-minimal structures, globaly subanalytic sets.

JEL CLASSIFICATION. C70, C72.

1. INTRODUCTION

The geometric structure of Nash equilibria has been exploited to obtain several game-theoretical results. In particular, the graph of the Nash equilibrium correspondence of a finite game is a *semialgebraic set*, i.e., a set defined by a finite system of polynomial inequalities. Every semialgebraic set is homeomorphic to a finite simplicial complex (van der Waerden, 1939), hence, the set of Nash equilibria of a game consists of finitely many connected components and at least one of them contains a *stable set* (Kohlberg and Mertens, 1986). Blume and Zame (1994) exploit another fundamental result of semialgebraic geometry,

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the *Tarski-Seidenberg Theorem*, to show that the perfect and sequential equilibrium correspondences have a semialgebraic structure and to obtain the generic equivalence between these two equilibrium concepts. Moreover, the *Generic Local Triviality Theorem* (Hardt, 1980; Bochnak et al., 1987) has been used to provide generic finiteness results for equilibria in finite games (Govindan and McLennan, 2001; Govindan and Wilson, 2001).

A defining element of a finite game is the set of players which, furthermore, is assumed to be common knowledge. Games with population uncertainty and, in particular, Poisson games (Myerson, 1998) have been proposed to model economic scenarios where it is more reasonable to assume that agents only have probabilistic information about such a set. Poisson games have been primarily used to study voting games but they are also useful in more general economic environments where the number of economic agents is uncertain (see, e.g., Satterthwaite and Shneyerov, 2007; Makris, 2008, 2009; Ritzberger, 2009; McLennan, 2011; Jehiel and Lamy, 2014). This uncertainty about the number of opponents is factored into players' expected utility functions. One consequence is that expected utilities are no longer polynomials and that the same tools of semialgebraic geometry that are useful to examine finite games are not directly applicable.

The objective of this paper is to examine the geometric structure of Nash equilibria in Poisson games and to analyze its game-theoretical consequences. We show that utility functions in Poisson games are real analytic functions and, correspondingly, the Nash equilibrium conditions define a bounded *semianalytic set* (that is, a bounded set that can be locally given by the solution of a finite system of analytic inequalities). Similarly to the semialgebraic case, semianalytic sets have a special structure that, e.g., allows us to establish that every Poisson game has finitely many connected components and that each such component is itself a semianalytic set. A corollary of this result is that every Poisson game has a *stable set* (Kohlberg and Mertens, 1986; De Sinopoli, Meroni, and Pimienta, 2014) contained in one connected component of Nash equilibria.

We can tighten more the parallelism with finite games. Many properties of semialgebraic sets can be derived from a set of axioms that define *o-minimal structures* (van den Dries, 1998).[1] Examples of o-minimal structures are the collection of semilinear sets, the collection of semialgebraic sets and, as showed by

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[1] In the economic literature, Blume and Zame (1994) apply the properties of o-minimal structures to general equilibrium theory to identify a class of preferences such that, for generic endowments, the corresponding economies have finitely many equilibria. See also Richter and Wong (2000).

van den Dries (1986), the collection of *globaly subanalytic* sets.[2] Every bounded semianalytic set is globally subanalytic, therefore, we can easily show that the set of Nash equilibria of any Poisson game as well as the graph of the Nash equilibrium correspondence for Poisson games are globaly subanalytic sets. We use the general version of the Generic Local Triviality Theorem for o-minimal structures to show the generic finiteness of Nash equilibria in some general Poisson games as well as some relevant Poisson voting models.

In the next section, we review the general description of Poisson games. We discuss the geometric structure of the Nash equilibrium set in Section 3 and prove that stable sets in Poisson games satisfy the same version of connectedness as Kohlberg and Mertens (1986) stable sets for finite games. We give a quick review of o-minimal structures in Section 4 and use some of its basic properties in Section 5 to establish the generic finiteness of Nash equilibria in some Poisson games including plurality, negative plurality, and approval voting games.

2. POISSON GAMES

We adopt the same notation used in De Sinopoli, Meroni, and Pimienta (2014), where the description of Poisson games closely follows the one introduced by Myerson (1998).

A *Poisson game* is a tuple $\Gamma := (n, \mathcal{T}, r, (C_t)_{t \in \mathcal{T}}, \Omega, \theta, v)$. The number of *players* is a Poisson random variable with parameter n. Given n, the probability that there are k players in the game is

$$P(k \mid n) = \frac{e^{-n} n^k}{k!}.$$

The set $\mathcal{T} = \{1, \ldots, T\}$ is the non-empty finite set of possible *types* of players. A player is of type $t \in \mathcal{T}$ with probability r_t. The probabilities that a player is of each type are listed in the vector $r = (r_1, \ldots, r_T) \in \Delta(\mathcal{T})$.[3]

We let C_t be the finite set of *actions* that are available to players of type t. The set of all actions is $C := \bigcup_t C_t$. An *action profile* $x \in Z(C)$ specifies for each action $c \in C$ the number of players $x(c)$ who choose that action. The set of action profiles is $Z(C) := \mathbb{Z}_+^C$.

A player of type $t \in \mathcal{T}$ who chooses action $c \in C_t$ when the action profile is $x \in Z(C)$ induces some *outcome* that belongs to the *outcome set* Ω. This information is specified by the *outcome function* $\theta : T \times C \times Z(C) \to \Delta(\Omega)$. Note that, e.g., the outcome function can be the identity function as in Myerson (1998).

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[2] Sometimes also called *finitely subanalytic* sets.

[3] For any set S, we write $\Delta(S)$ for the set of probability distributions on S with finite support.

The utility vector $v = (v_1, \ldots, v_T)$ summarizes players' preferences over outcomes. Each entry v_t is a bounded function $v_t : \Omega \to \mathbb{R}$. The utility function that a player of type t has over elements in $C \times Z(C)$ is computed according to $u_t(c, x; v_t) = \sum_{\omega \in \Omega} \theta(t, c, x)(\omega) v_t(\omega)$.

The set $\Delta(C_t)$ is the set of *mixed actions* for type t players. We identify the mixed action that assigns probability one to action c with the pure action $c \in C$. A *strategy function* $\sigma = (\sigma_1, \ldots, \sigma_T)$ is a function from $\mathscr{T}$ to $\Delta(C)$ that satisfies $\sigma_t \in \Delta(C_t)$ for all $t \in \mathscr{T}$, i.e. a mapping from the set of types to the set of mixed actions available to each corresponding type. We let Σ denote the set of all strategy function. We may sometimes refer to strategy functions simply as *strategies*. Let $\tilde{\tau}(\sigma) \in \Delta(C)$ be the population's "average" behavior induced by the strategy σ, which is given by $\tilde{\tau}(\sigma)(c) := \sum_{t \in \mathscr{T}} r(t) \sigma_t(c)$. Moreover, we define the set $\tilde{\tau}(\Sigma) := \{\tau \in \Delta(C) : \tau = \tilde{\tau}(\sigma) \text{ for some } \sigma \in \Sigma\}$. When the population's average behavior is given by $\tau \in \tilde{\tau}(\Sigma)$, the probability that the action profile $x \in Z(C)$ is realized is equal to

$$\mathbf{P}(x \mid \tau) := \prod_{c \in C} \left(e^{-n\tau(c)} \frac{(n\tau(c))^{x(c)}}{x(c)!} \right).$$

The expected payoff to a player of type t who plays action $c \in C_t$ is then

$$U_t(c, \tau; v_t) := \sum_{x \in Z(C)} \mathbf{P}(x \mid \tau) u_t(c, x; v_t). \tag{2.1}$$

From now on we fix n, $\mathscr{T}$, r, $(C_t)_{t \in \mathscr{T}}$, Ω, and θ. A Poisson game is then given by a utility vector $v \in \mathbb{R}^{\#\Omega T}$. We denote such a game $\Gamma(v)$.

3. The set of Nash equilibria

Let $s := \sum_t \#C_t$. It is convenient to give the following definition of Nash equilibrium.

Definition 1. The vector $\varsigma \in \mathbb{R}^s$ is a Nash equilibrium of $\Gamma(v)$ if and only if it is a solution to the following system of equalities and inequalities:

$$\sum_{c_t \in C_t} \varsigma_t(c_t) - 1 = 0 \qquad \text{for all } t \in \mathscr{T}, \tag{3.1}$$

$$\varsigma_t(c_t) \geq 0 \qquad \text{for all } t \in \mathscr{T}, \; c_t \in C_t. \tag{3.2}$$

$$\varsigma_t(c_t)\big[U_t(c_t, \tilde{\tau}(\varsigma); v_t) - U_t(d_t, \tilde{\tau}(\varsigma); v_t)\big] \geq 0 \qquad \text{for all } t \in \mathscr{T}, \; c_t, d_t \in C_t, \tag{3.3}$$

Conditions (3.1) and (3.2) ensure that ς belongs to the set of strategy functions Σ. Conditions (3.3) ensure that, when every player is playing according to the strategy function ς, if $\varsigma_t(c_t) > 0$ then action c_t is a best response for a player of type t against ς.

In finite normal form games, expected utilities are polynomial functions. The Nash equilibrium conditions define a semialgebraic set which, in turn, is homeomorphic to a finite simplicial complex (van der Waerden, 1939). This fact was used by Kohlberg and Mertens (1986) to show that, for any game, the set of Nash equilibria consists of finitely many connected components. In a Poisson game, the Nash equilibrium conditions (3.3) are clearly not polynomial inequalities. However, Lemma 1 below shows that the expected utility functions are real analytic (i.e., functions that are locally given by a convergent power series) and, therefore, the Nash equilibrium conditions define a *semianalytic set*.

Definition 2. A set $X \subset \mathbb{R}^m$ is *semianalytic* if for each $y \in \mathbb{R}^m$ there is an open neighborhood O such that $O \cap X$ is a finite union of sets of the form

$$\left\{ x \in \mathbb{R}^m : f(x) = 0 \text{ and } g_1(x) > 0, \ldots, g_k(x) > 0 \right\}$$

where $f, g_1, \ldots, g_k$ are real analytic functions on O.

By definition, the class of semianalytic sets includes also sets defined by weak inequalities. This class is closed under finite union, finite intersection, finite product, and complementation. It is also the case that every compact semianalytic set is homeomorphic to a finite simplicial complex (Lojasiewicz, 1964). Therefore, like in the semialgebraic case, compact semianalytic sets also have finitely many connected components.

We now show that expected utility functions in a Poisson game are indeed real analytic.

Lemma 1. *Given a Poisson game* $\Gamma(v)$*, for every type* $t \in \mathcal{T}$ *and every action* $c \in C_t$*, the expected utility function* $U_t(c, \cdot; v_t)$ *is real analytic.*

Proof. If $\#C = K$ we first note that $U_t(c, \cdot; v_t)$ can be considered as a complex function $U_t(c, \cdot; v_t) : \mathbb{C}^K \to \mathbb{C}$. We actually prove that $U_t(c, \cdot; v_t)$ is a complex analytic function, i.e., a complex function that is locally given by a convergent power series. The sum, product and composition of complex analytic functions are complex analytic, and the limit of a sequence of complex analytic functions that converges uniformly on every compact subset of the domain is also complex analytic.[4] Examples of complex analytic functions are polynomials and the exponential function, hence, once we fix $n \in \mathbb{N}_+$ then, for any $y \in \mathbb{N}_+$, the function $z \mapsto e^{-nz}(nz)^y (y!)^{-1}$ is complex analytic because it is the product of compositions of complex analytic functions.

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[4] The latter result follows from the Weierstrass approximation theorem. This is in contrast with the situation in real analysis, where the limit of a sequence of real analytic functions that converges uniformly on a compact subset of the domain is not necessarily real analytic in that subset.

Take some $x = (x_1, \ldots, x_K) \in \mathbb{N}_+^K$, and some $(c, x) \in C_t \times Z(C)$. Then $u_t(c, x; v_t) \in \mathbb{R}$ and the function

$$(\tau_1, \ldots, \tau_K) \mapsto \prod_{i=1}^{K} \left(e^{-n\tau_i} \frac{(n\tau_i)^{x_i}}{x_i!} \right) u_t(c, x; v_t)$$

is also complex analytic because it is a composition of complex analytic functions.

Define the set of profiles $Z^b = \{x \in Z(C) : \max_i x_i \leq b\}$. For any bounded function $v_t : \Omega \to \mathbb{R}$, we can use the map $u_t(c, x; v_t) := \sum_{\omega \in \Omega} \theta(t, c, x)(\omega) v_t(\omega)$ to construct the function from $\mathbb{C}^K$ to $\mathbb{C}$:

$$U_t^b(c, \tau; v_t) = \sum_{x \in Z^b} \left(\prod_{i=1}^{K} e^{-n\tau_i} \frac{(n\tau_i)^{x_i}}{x_i!} \right) u_t(c, x; v_t).$$

The function $U_t^b(c, \cdot; v_t)$ is complex analytic because it is the finite sum of complex analytic functions. The sequence of complex analytic functions $\{U_t^b(c, \cdot; v_t)\}_{h=1}^{\infty}$ converges uniformly to $U_t(c, \cdot; v_t)$ on a compact subset $Q \subset \mathbb{C}^K$ if, for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}_{++}$ such that

$$\left| U_t^b(c, \tau; v_t) - U_t(c, \tau; v_t) \right| < \varepsilon \text{ for all } b \geq N \text{ and } \tau \in Q.$$

In our case this holds if for each $\varepsilon > 0$ we can find an $N \in \mathbb{N}_{++}$ such that

$$\left| \sum_{x \in Z(C) \setminus Z^b} \left(\prod_{i=1}^{K} e^{-n\tau_i} \frac{(n\tau_i)^{x_i}}{x_i!} \right) u_t(c, x; v_t) \right| < \varepsilon \text{ for all } b \geq N \text{ and } \tau \in T.$$

Since the utility function is bounded, $u_t^* := \sup_{x \in Z(C)} |u_t(c, x; v_t)| < \infty$. Consider an arbitrary compact subset $Q \subset \mathbb{C}^K$ and let $\alpha := \max_{\tau \in Q} |\tau|$. For every integer h and every $\tau \in Q$, the value to the left of the previous inequality is smaller than or equal to

$$\sum_{x \in Z(C) \setminus Z^b} \left(\prod_{i=1}^{K} \left| e^{-n\tau_i} \frac{(n\tau_i)^{x_i}}{x_i!} \right| \right) u_t^* \leq \sum_{x \in Z(C) \setminus Z^b} \left(\prod_{i=1}^{K} e^{|-n\tau_i|} \frac{|n\tau_i|^{x_i}}{x_i!} \right) u_t^* =$$

$$= \sum_{x \in Z(C) \setminus Z^b} \left(\prod_{i=1}^{K} e^{n\alpha} \frac{(n\alpha)^{x_i}}{x_i!} \right) u_t^* = \sum_{h=0}^{\infty} \left[\sum_{x \in Z^{b+h+1} \setminus Z^{b+h}} \left(\prod_{i=1}^{K} e^{n\alpha} \frac{(n\alpha)^{x_i}}{x_i!} \right) u_t^* \right]. \quad (3.4)$$

If $\xi := \lfloor n\alpha \rfloor$ we have $\frac{(n\alpha)^y}{y!} \leq \frac{(n\alpha)^{\xi}}{\xi!}$ for any integer y. Hence, if $x \in Z^b \setminus Z^{b-1}$ we can write

$$\prod_{i=1}^{K} \left(e^{n\alpha} \frac{(n\alpha)^{x_i}}{x_i!} \right) u_t^* \leq e^{n\alpha} \frac{(n\alpha)^b}{b!} \left(e^{n\alpha} \frac{(n\alpha)^{\xi}}{\xi!} \right)^{K-1} u_t^*.$$

Noting that there are $(b+1)^K - b^K$ profiles in $Z^b \setminus Z^{b-1}$ we have the following bound

$$\sum_{x \in Z^b \setminus Z^{b-1}} \left(\prod_{i=1}^{K} e^{n\alpha} \frac{(n\alpha)^{x_i}}{x_i!} \right) u_t^* \leq ((b+1)^K - b^K) e^{n\alpha} \frac{(n\alpha)^b}{b!} \left(e^{n\alpha} \frac{(n\alpha)^{\xi}}{\xi!} \right)^{K-1} u_t^* \quad (3.5)$$

We now put the sum over elements in $Z^{b+h+1} \setminus Z^{b+h}$ in terms of this bound. Let us start with an arbitrary profile $x' \in Z^{b+1} \setminus Z^b$. We can find a profile $x \in Z^b \setminus Z^{b-1}$ such that $x' = x + (e_1, \ldots, e_K)$ where $e_i \in \{0, 1\}$ for every i and $e_i = 1$ only if $x_i = b$. Therefore, each $x \in Z^b \setminus Z^{b-1}$ cannot generate more than $(2^K - 1)$ different profiles in $Z^{b+1} \setminus Z^b$ in such a way. Let b be large enough so that $n\alpha < b + 1$. Letting H replace the obvious expression in (3.5) that does not depend on b we have

$$\sum_{x \in Z^{b+1} \setminus Z^b} \left(\prod_{i=1}^{K} e^{n\alpha} \frac{(n\alpha)^{x_i}}{x_i!} \right) u_t^* \leq (2^K - 1) \frac{n\alpha}{b+1} H((b+1)^K - b^K) \frac{(n\alpha)^b}{b!}.$$

Iterating the same recursive argument, the last expression in (3.4) is less than

$$H((b+1)^K - b^K) \frac{(n\alpha)^b}{b!} \sum_{h=0}^{\infty} \left[(2^K - 1) \frac{n\alpha}{b+1} \right]^{h+1}. \tag{3.6}$$

As b grows we eventually have $(2^K - 1)n\alpha < (b+1)$, thus, (3.6) converges to zero as b goes to infinity. This implies that the sequence of complex analytic functions $\{U_t^b(c, \cdot; v_t)\}_{h=1}^{\infty}$ converges uniformly to $U_t(c, \cdot; v_t)$ on every compact subset $Q \subset \mathbb{C}^K$, proving that $U_t(c, \cdot; v_t)$ is complex analytic. Since the image of any real vector $(\tau_1, \ldots, \tau_K)$ under $U_t(c, \cdot; v_t)$ is a real number, we can conclude that $U_t(c, \cdot; v_t)$ is real analytic when restricted to the real domain. □

Given our previous discussion, an immediate consequence of this result is the analogue of Proposition 1 in Kohlberg and Mertens (1986).

Theorem 1. *The set of Nash equilibria of a Poisson game $\Gamma(v)$ is a compact semianalytic set, therefore, it has finitely many connected components. At least one of those components is such that, for every Poisson game $\Gamma(\hat{v})$ sufficiently close to $\Gamma(v)$, there is a Nash equilibrium of $\Gamma(\hat{v})$ close to it.*

Proof. Lemma 1 implies that the set of Nash equilibria is a semianalytic set. Hence, it is made of finitely many connected components.

For the second part, we prove a stronger result. Recall that the set of fixed points N of an upper semi-continuous convex-valued correspondence F is an *essential* set of fixed points. That is, for every open $W \supset N$ there is a neighborhood O of the graph of F such that every upper semi-continuous convex-valued correspondence G whose graph is a subset of O has a fixed point in W. Kinoshita (1952) proves that if a set of fixed points N is essential and $\{N_1, \ldots, N_k\}$ is a partition of N into disjoint compact sets, then some N_j is essential.[5] Since the best response correspondence is upper semi-continuous and convex valued, and we can choose those compact sets to be connected, every Poisson game has a connected component of Nash equilibria that is essential. □

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[5] See also McLennan (2012, Theorem 8.3.2).

A Kohlberg and Mertens (1986) *stable set* of a finite game is a minimal (in terms of set inclusion) subset of Nash equilibria such that every close-by game that can be generated by pure-strategy perturbations has a Nash equilibrium close to it. Since robustness is only required against strategy perturbations, every member of a stable set is a perfect (hence undominated) equilibrium. Even if Kohlberg and Mertens (1986, p. 1020) list *connectedness* (i.e. every solution should be connected) as one of the main requirements that a set-valued solution concept should satisfy, the form of the definition allows for stable sets that are disconnected. It is also possible that a stable set is made of Nash equilibria that belong to different components.[6] Nevertheless, stable sets satisfy a weaker version of connectedness. Namely, every game has a stable set which is contained in a single connected component of the set of Nash equilibria.

De Sinopoli, Meroni, and Pimienta (2014) propose the analogous definition of stability for Poisson games. For any given Poisson Game $\Gamma(v)$ we can construct a suitable set of perturbed Poisson games $P(v)$ so that robustness against elements in $P(v)$ guarantees that players choose only admissible strategies in $\Gamma(v)$.[7] Then, a stable set of the Poisson game $\Gamma(v)$ is a minimal (in terms of set inclusion) subset of Nash equilibria such that every close-by game in $P(v)$ has a Nash equilibrium close to the stable set.

Again, we can show that the situation in Poisson games parallels that of normal form games. A consequence of Theorem 1 is that, even if not every stable set is connected, every Poisson game has a stable set that is contained in a single connected component of equilibria. (For completeness, the formal definition of stability in Poisson games, as well as an example illustrating the connectedness issue, is contained in the Appendix. We also defer to the Appendix the proof of the following Proposition.)

Proposition 1. *Every Poisson game $\Gamma(v)$ has a stable set contained in a connected component of equilibria. Moreover, every Poisson game has a minimal connected set of Nash equilibria that is robust against every perturbation in $P(v)$.*

From an applied standpoint, a connected component of equilibria (or a stable set contained in it) does not substantially differ from a single-valued equilibrium concept as long as every Nash equilibrium in such a component induces the same probability distribution over outcomes. This is the case in generic finite normal form games (Harsanyi, 1973) and in generic finite extensive form games (Kreps and Wilson, 1982). Govindan and Wilson (2001) show that these results can be

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[6] Alternative definitions of stability have been proposed (Hillas, 1990; Mertens, 1989, 1991) such that every solution is connected.

[7] Games in $P(v)$ cannot be generated by strategy perturbations in the game $\Gamma(v)$. The definition of the appropriate set of perturbations is given in the Appendix.

easily derived from some basic properties of semialgebraic sets and functions. Such properties are shared by some *o-minimal structures*.

4. O-MINIMAL STRUCTURES ON $\mathbb{R}$

The objective of this section is to provide the necessary background to show that the graph of the Nash equilibrium correspondence and the set of Nash equilibria of any Poisson game belong to a family of sets with very convenient finiteness properties. In this section we follow van den Dries (1986, 1998) and Blume and Zame (1992). We simply present the relevant results without their proofs and, where appropriate, we indicate some of their consequences for Poisson games.

Before identifying the family of sets that we are interested in, we need an intermediate step.

Definition 3. A set $X \subset \mathbb{R}^m$ is *subanalytic* if for each $y \in \mathbb{R}^m$ there is an open neighborhood U, a strictly positive integer l, and a bounded semianalytic set $Y \subset \mathbb{R}^{m+l}$ such that $U \cap X$ is the projection of Y onto the first m coordinates.

It is easy to see that every semianalytic set is subanalytic. Like the collection of semialgebraic sets, the collection of subanalytic sets is closed under finite unions, finite intersections, finite products, and complementation (Gabriélov, 1968) but, in contrast with semialgebraic sets, subanalytic sets are not closed under projections. However, the *bounded* subanalytic sets in $\mathbb{R}^m$ are exactly those sets that are the projections of bounded semianalytic sets $X \subset \mathbb{R}^{m+l}$ on the first m coordinates. Following from this and other basic results, van den Dries (1986) shows that the following family of sets is, in addition, closed under projections:

Definition 4. A set $X \subset \mathbb{R}^m$ is *globaly subanalytic* (or *finitely subanalytic*) if given the map

$$f(x_1, \ldots, x_m) = \left(x_1 / \sqrt{1 + x_1^2}, \ldots, x_m / \sqrt{1 + x_m^2} \right)$$

we have that $f(X)$ is a subanalytic subset of $\mathbb{R}^m$.

The function f is an analytic isomorphism onto the bounded open set $(-1, 1)^m$. Hence, every bounded subanalytic set is globaly subanalytic. The collection of globaly subanalytic sets forms an o(rder)-minimal structure:

Definition 5. An *o-minimal structure on* $\mathbb{R}$ is a family $\mathscr{S} = \{\mathscr{S}_m\}_{m \in \mathbb{N}_+}$ such that
(1) For each m, $\mathscr{S}_m$ is a nonempty collection of subsets of $\mathbb{R}^m$ that is closed under formation of finite unions, finite intersections, and complements.
(2) If $X \in \mathscr{S}_m$ then $\mathbb{R} \times X \in \mathscr{S}_{m+1}$ and $X \times \mathbb{R} \in \mathscr{S}_{m+1}$.

(3) For each m we have $\{(x_1,\ldots,x_m) : x_1 = x_m\} \in \mathscr{S}_m$.
(4) If $X \in \mathscr{S}_{m+1}$ and $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the projection onto the first m coordinates then $\pi(X) \in \mathscr{S}_m$.
(5) The ordering of the real line $\{(x_1, x_2) : x_1 < x_2\}$ belongs to $\mathscr{S}_2$.
(6) The sets in $\mathscr{S}_1$ are exactly the finite unions of points and intervals.

Examples of o-minimal structures are the collection of semialgebraic sets, the collection of semilinear sets, and the collection of finitely semianalytic sets. Following standard terminology, once we fix an o-minimal structure $\mathscr{S} = \{\mathscr{S}_m\}_{m \in \mathbb{N}_+}$, we say that X is a *definable set* if $X \in \mathscr{S}_m$ for some m. We say that a function or correspondence is *definable* if its graph is a definable set.

To later prove that the graph of the Nash equilibrium correspondence in Poisson games is globaly subanalytic we need to use the fact that o-minimal structures are *closed under definability*. This means that if $\Phi(x_1,\ldots,x_m)$ is a first order formula, that is, a formula that uses the free variables $(x_1,\ldots,x_m)$, the universal and existential quantifiers, any finite number of quantified variables that range over the definable sets, the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), and the definable sets themselves, then

$$\Big\{(x_1,\ldots,x_m) \in \mathbb{R}^m : \Phi(x_1,\ldots,x_m) \text{ is true}\Big\}$$

is a definable set.[8]

We now present some of the finiteness properties satisfied by o-minimal structures.

Theorem 2. *Each $X \in \mathscr{S}_m$ has only finitely many connected components, and each component also belongs to $\mathscr{S}_m$. If further $f : X \to \mathbb{R}^l$ is definable then there is a positive integer N_f such that, for each $x \in \mathbb{R}^l$, the set $f^{-1}(x)$ has at most N_f components.*

We already established that the set of Nash equilibria of any Poisson game has finitely many connected components. Lemma 4 below shows that, if the outcome set Ω is finite, the graph of the Nash equilibrium correspondence in Poisson games graph(NE) is a globaly subanalytic set. We can apply the second part of the theorem to the projection $\pi : \text{graph(NE)} \to \mathbb{R}^{\#\Omega T}$ to the space of games to conclude that there is a global bound N^* on the number of connected components of equilibria that any Poisson game can have.

Similarly to the family of semialgebraic sets, the family of globaly subanalytic sets also contains the graphs of addition and multiplication:

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[8] Intuitively, this expression can be replaced by another one involving definable sets and the set theoretic operations allowed by Axioms 1 to 4. We have that $\wedge$ corresponds to the intersection of sets, $\vee$ to the union of sets, and $\neg$ to the complement of a set. Additionally, we can replace the universal and existential quantifiers by suitable projections.

(7) $\{(x_1, x_2, x_3) : x_3 = x_1 + x_2\} \in \mathscr{S}_3$, and

(8) $\{(x_1, x_2, x_3) : x_3 = x_1 x_2\} \in \mathscr{S}_3$.

O-minimal structures satisfying (7) and (8) have more useful properties.

Theorem 3 (Triangulability). *Let $\mathscr{S}$ satisfy (7) and (8). If $X \in \mathscr{S}_m$ then there is a finite simplicial complex $\mathscr{K}$ in $\mathbb{R}^m$ such that X is homeomorphic to a union of (open) simplices of $\mathscr{K}$.*

If X is a definable set it follows that we can unambiguously define $\dim(X)$, the dimension of X, to be the largest dimension of any such simplex.

Theorem 4 (Generic Triviality). *Let $\mathscr{S}$ satisfy (7) and (8) and let $f : X \to Y$ be a continuous definable function. There is a lower-dimensional definable set $Z \subset Y$ such that for each of the finitely many connected components C of $Y \setminus Z$ there is a definable set F and a definable homeomorphism $h : C \times F \to f^{-1}(C)$ with $f(h(c, f)) = c$ for every $(c, f) \in C \times F$.*

The next two lemmas are used in Section 5 to prove generic determinacy results of Nash equilibria in Poisson games. They follow immediately from the Generic Triviality Theorem. See Govindan and McLennan (2001) for the proof of the following result in the context of semialgebraic geometry.

Lemma 2. *Let $\mathscr{S}$ also satisfy (7) and (8) and let $f : X \to Y$ be a continuous definable function. Then*

$$\dim X \le \dim Y + \max_{y \in Y} \dim f^{-1}(y).$$

We say that a definable set is generic if its complement is a closed and lower-dimensional definable set. Furthermore, we say that a point satisfying some property is generic if it resides in a generic definable set where every point satisfies such a property. See Govindan and Wilson (2001) for the semialgebraic version of the following lemma.

Lemma 3. *Let $\mathscr{S}$ also satisfy (7) and (8) and let $f : X \to Y$ be a continuous definable function. If $\dim(X) \le \dim(Y)$ then, for generic $y \in Y$, $f^{-1}(y)$ is a finite or empty set.*

5. GENERIC DETERMINACY OF EQUILIBRIA IN POISSON GAMES

To only consider finite-dimensional spaces we assume that the set of outcomes Ω is finite. This assumption is satisfied in most applications of Poisson games.

Assumption 1. The set of outcomes Ω is finite.

We have already seen that given a Poisson game $\Gamma(v)$ the Nash equilibrium conditions define a semianalytic set. Since the set of Nash equilibria is bounded, they also define a globaly subanalytic set. We now show that the Nash equilibrium correspondence is also globaly subanalytic.

Lemma 4. *The Nash equilibrium correspondence* $\mathrm{NE} : \mathbb{R}^{\#\Omega T} \to \Sigma$ *is globaly subanalytic.*

Proof. By definition, we have to show that the graph of NE is a globaly subanalytic set. Denote by E the set of $(\varsigma, v) \in \mathbb{R}^s \times (0, 1)^{\#\Omega T}$ that satisfy Conditions (3.1), (3.2), and (3.3). That is, E is the graph of the Nash equilibrium correspondence restricted to the domain where every utility value lies in the bounded interval $(0, 1)$. The set E is a bounded semianalytic set, therefore, globaly subanalytic. The Nash equilibrium conditions are not altered under an affine transformation of the utility functions $v_t : \Omega \to \mathbb{R}$. Therefore, the graph of the Nash equilibrium correspondence satisfies

$$\mathrm{graph(NE)} = \Big\{ (\varsigma, v) \in \mathbb{R}^s \times \mathbb{R}^{\#\Omega T} : \exists (\alpha, \beta, \tilde{v}) \in \mathbb{R}^T_{++} \times \mathbb{R}^T \times \mathbb{R}^{\#\Omega T},$$

$$\forall t (\tilde{v}_t = \alpha_t \cdot v_t + \beta_t) \wedge ((\varsigma, \tilde{v}) \in E) \Big\}.$$

Since the family of globaly subanalytic sets is closed under definability, the correspondence $\mathrm{NE} : \mathbb{R}^{\#\Omega T} \to \Sigma$ is globaly subanalytic. $\qquad\square$

We now derive a result analogous to the generic finiteness of equilibria in normal form games (Harsanyi, 1973) and of equilibrium outcomes in finite extensive form games (Kreps and Wilson, 1982). Given $\tau \in \tilde{\tau}(\Sigma)$ and an action $c \in C_t$ we can compute a probability distribution $p_t(\cdot \mid c, \tau)$ on Ω where, for every $\omega \in \Omega$, we have $p_t(\omega \mid c, \tau) := \sum_{x \in Z(C)} \mathbf{P}(x \mid \tau) \theta(t, c, x)(\omega)$. Hence, we can write a type t player's expected utility if she chooses action c while the rest of the population plays according to τ as $U_t(c, \tau; v_t) = \sum_{\omega \in \Omega} p_t(\omega \mid c, \tau) v_t(\omega)$.

Definition 6. We say that $\tau \in \tilde{\tau}(\Sigma)$ is a *maximal dimension point* if for every $t \in \mathcal{T}$ the rank of the matrix whose columns are the vectors $(p_t(\cdot \mid c, \tau))_{c \in C_t}$ is $\#C_t$.

The argument below does not substantially differ from the one used by Govindan and McLennan (2001) and illustrates how many semialgebraic proofs in finite games can be readily extended to Poisson games. We say that τ is a *Nash equilibrium behavior* if there is a Nash equilibrium $\sigma \in \Sigma$ such that $\tau = \tilde{\tau}(\sigma)$.

Proposition 2. *If every Nash equilibrium behaviour $\tau \in \tilde{\tau}(\Sigma)$ is a maximal dimension point then, for generic utilities, the set of Nash equilibria is finite.*

Proof. It is enough to focus on Nash equilibria where every action of every type of player is used with positive probability. Write CNE for the correspondence that assigns to each utility vector the set of completely mixed Nash equilibria. For every $t \in \mathcal{T}$ take some action $d \in C_t$. Let $\sigma \in \text{CNE}(v)$ and $\tau = \tilde{\tau}(\sigma)$. The set

$$\left\{ v_t : \sum_\omega \left[p_t(\omega \mid c, \tau) - p_t(\omega \mid d, \tau) \right] v_t(\omega) = 0 \text{ for every } c \in C_t \right\}$$

has dimension $\#\Omega - (\#C_t - 1)$ because τ is a maximal dimension point.

Consider the projection $\pi : \text{graph}(\text{CNE}) \to \Sigma$. The previous argument implies that $\dim \pi^{-1}(\sigma) \leq \#\Omega T - \dim(\Sigma)$. Therefore, by Lemma 2, $\dim(\text{graph}(\text{CNE})) \leq \#\Omega T - \dim(\Sigma) + \dim(\Sigma) = \#\Omega T$. We obtain the desired result applying Lemma 3 to the projection from graph(CNE) to the space of utilities $\mathbb{R}^{\#\Omega T}$. $\qquad\square$

The next assumption implies that every $\tau \in \tilde{\tau}(\Sigma)$ is a maximal dimension point. It says that, for each type t and each action $c \in C_t$, from the viewpoint of a player of type t, there is an outcome $\omega_{t,c}$ that can be induced with positive probability only if she plays c_t.

Assumption 2. For every type t and every action $c \in C_t$ there is an outcome $\omega_{t,c} \in \Omega$ such that we can find an $x \in Z(C)$ satisfying $\theta(t, c', x) = \omega_{t,c}$ if and only if $c' = c$.

In a voting model, for instance, this assumption is satisfied if voters care not only about the outcome of the election but also about the ballot they cast. That is, if for every player the situation where candidate A wins and she voted for A is different from the situation where candidate A wins and she voted for B. The reason we discuss Assumption 2 is only to notice that, under such assumption, the simple semialgebraic proof given by Govindan and Wilson (2001) to show the generic finiteness of Nash equilibria in normal form games applies almost verbatim to Poisson games. In fact, using such a proof, we can also conclude that the graph of CNE is a real analytic manifold of dimension $\#\Omega T$.

It makes sense to impose more structure on the outcome function θ so that outcomes are not defined from the viewpoint of a player of a particular type but have a universal description. This is the case in political economy models where the outcome of the game simply corresponds to the winner of the election.

Thus, let us consider a family of Poisson voting games. Let $\Omega = \{\omega_1, \ldots, \omega_K\}$ be the finite set of candidates. Every type t has the same set of ballots (actions) C available. A ballot $c \in C$ is a K-dimensional vector that specifies for each candidate the number of votes c_i given to each candidate ω_i. An electoral system specifies the set of permissible ballots C. For instance, under plurality rule, such a set is the collection of $(c_1, \ldots, c_K)$ such that $c_i \in \{0, 1\}$ for every i and $\sum_{i=1}^K c_i \in \{0, 1\}$. Under negative plurality, C is the collection of $(c_1, \ldots, c_K)$ such

that $c_i \in \{0,1\}$ for every i and $\sum_{i=1}^K c_i \in \{0, K-1\}$. Under approval voting, C is simply the set of all $(c_1, \ldots, c_K)$ such that $c_i \in \{0,1\}$ for every i. Thus, aggregating the ballots cast by the players, we can consider the set of ballot (action) profiles to be $\mathbb{N}_+^K$. The outcome function $\theta : \mathbb{N}_+^K \to \Delta(\Omega)$ selects for each ballot profile the candidate that has received the most votes. Ties are broken using the uniform probability distribution over the winning candidates. A type t player has preferences over candidates represented by the utility function $v_t \in \mathbb{R}^\Omega$. Thus, a type t player who casts ballot c when the ballot profile is x obtains utility $u_t(c,x;v_t) = \sum_{\omega \in \Omega} \theta(c+x)(\omega)v_t(\omega)$.

De Sinopoli (2001) shows that in generic normal form plurality voting games, the set of Nash equilibria such that more than one candidate wins with positive probability is finite. In turn, De Sinopoli, Iannantuoni, and Pimienta (2015) show that (1) in generic normal form negative plurality voting games the set of Nash equilibrium outcomes is finite, and (2) in generic normal form approval voting games with three candidates the set of Nash equilibrium outcomes is also finite. Similarly to Proposition 2, we show that analogous results can be easily established in Poisson games. Furthermore, the fact that in a Poisson game every pivotal event receives positive probability allows us to establish such results in a stronger form and with a simpler proof.

Theorem 5. *For generic utility vectors in $\mathbb{R}^{\#\Omega T}$*

(1) *the corresponding plurality voting game has finitely many Nash equilibria, and*

(2) *the corresponding negative plurality voting game has finitely many Nash equilibria.*

(3) *Moreover, if $\#\Omega = 3$, the corresponding approval voting game also has finitely many Nash equilibria.*

Proof. We want to establish a result that holds for generic utilities, so we can restrict the attention to utility vectors in $\mathbb{R}^{\#\Omega T}$ such that every type has a strict ordering over the set of candidates $\Omega = \{\omega_1, \ldots, \omega_K\}$.

We start with part (1). With abuse of notation, denote also by ω the action of voting for candidate $\omega \in \Omega$. Take a Nash equilibrium behavior τ of a Poisson plurality voting game and, without loss of generality, assume that $\tau(\omega) > 0$ for every $\omega \in \Omega$, otherwise, consider the reduced game where candidates for which $\tau(\omega) = 0$ are eliminated. Relabeling if necessary, let $\omega_1, \ldots, \omega_L$ be the actions that are best response against τ for players of type t. Note that abstention is strictly dominated, so we can also consider the plurality game where $\omega_1, \ldots, \omega_L$ are the only actions available to type t players. We have $L < K$ because type t players are not indifferent between any two candidates and $\tau(\omega) > 0$ for every $\omega \in \Omega$.

Consider the square matrix

$$
M_t = \begin{pmatrix} p_t(\omega_1 \mid \omega_1, \tau) & \dots & p_t(\omega_L \mid \omega_1, \tau) \\ \vdots & \ddots & \vdots \\ p_t(\omega_1 \mid \omega_L, \tau) & \dots & p_t(\omega_L \mid \omega_L, \tau) \end{pmatrix}.
$$

Letting $\varnothing$ represent abstention, we construct a new matrix subtracting from each row the row vector $(p_t(\omega_1 \mid \varnothing, \tau), \dots, p_t(\omega_L \mid \varnothing, \tau))$. For each two candidates ω and ω' let $\pi_t(\omega \mid \omega', \tau) = p_t(\omega \mid \omega', \tau) - p_t(\omega \mid \varnothing, \tau)$. We obtain:

$$
M_t' = \begin{pmatrix} \pi_t(\omega_1 \mid \omega_1, \tau) & \dots & \pi_t(\omega_L \mid \omega_1, \tau) \\ \vdots & \ddots & \vdots \\ \pi_t(\omega_1 \mid \omega_L, \tau) & \dots & \pi_t(\omega_L \mid \omega_L, \tau) \end{pmatrix}.
$$

The expression $\pi_t(\omega \mid \omega', \tau)$ represents the increase, as perceived by a player of type t, in the probability that candidate ω wins the election if she votes for candidate ω' compared to the situation where she abstains. Hence, $\pi_t(\omega \mid \omega', \tau) > 0$ if $\omega = \omega'$ and $\pi_t(\omega \mid \omega', \tau) < 0$ if $\omega \neq \omega'$. Furthermore, for every row r, we have $|\pi_t(\omega_r \mid \omega_r, \tau)| > \sum_{\ell=1, \ell \neq r}^{L} |\pi_t(\omega_\ell \mid \omega_r, \tau)|$ because (1) the increase in the probability that candidate ω_r wins has to be equal to the decrease in the probability that the winner of the election is in $\Omega \setminus \{\omega_r\}$, and (2) $L < K$. Thus, M_t' is a strictly diagonally dominant matrix which implies that it has full rank. The same is true for every principal submatrix of M_t'. It follows that M_t' is an M-matrix (Ostrowski, 1955, p. 95) since it is a square matrix such that all diagonal elements are strictly positive, all nondiagonal elements weakly (actually, strictly) negative, and all principal minors of all orders strictly positive. Ostrowski (1955, p. 97) shows that if we add the same non-negative row vector to each row of an M-matrix then the determinant of the resulting matrix is strictly positive. Thus, the matrix M_t has full rank, τ is a maximal dimension point, and Proposition 2 implies the desired result.

Part (2) can be handled in a similar way, therefore, we skip this part.

For part (3), let τ be a Nash equilibrium behavior. Let $\Omega = \{a, b, c\}$ and let type t players have preferences $a \succ b \succ c$. It is easy to see that action c (approving only candidate c) and abstaining are both strictly dominated, so we can consider the reduced game where a type t player has only action a (approving candidate a) and action ab (approving both candidate a and b) available. Consider the matrix

$$
M_t = \begin{pmatrix} p_t(a \mid a, \tau) & p_t(b \mid a, \tau) & p_t(c \mid a, \tau) \\ p_t(a \mid ab, \tau) & p_t(b \mid ab, \tau) & p_t(c \mid ab, \tau) \end{pmatrix}.
$$

The second row minus the first row represents the change in probability resulting from the additional approval vote for candidate b. This additional approval

vote for b strictly decreases the probability of winning of candidates a and c and strictly increases the probability of winning of candidate b. Hence, the second row cannot be a multiple of the first row and, therefore, the rank of M_t is two. We conclude that every Nash equilibrium behavior τ is a maximal dimension point. Proposition 2 implies the desired result. □

APPENDIX A. STABLE SETS IN POISSON GAMES

In this section we revise the definition of stable set in Poisson games (De Sinopoli, Meroni, and Pimienta, 2014) and prove Proposition 1.

We begin with some preliminary concepts. Let $\mathscr{B}$ be the σ-algebra of Borel sets in $\tilde{\tau}(\Sigma) \subset \Delta(C)$ and let $\mathscr{M}$ be the set of all Borel probability measures over the measurable space $(\tilde{\tau}(\Sigma), \mathscr{B})$. Moreover, let $\mathscr{M}^\circ$ be the subset of measures $\mu \in \mathscr{M}$ that satisfy $\mu(O) > 0$ for every nonempty open set $O \subset \tilde{\tau}(\Sigma)$. For each $\tau \in \tilde{\tau}(\Sigma)$ we let $\delta(\tau)$ be the Dirac measure on $\tilde{\tau}(\Sigma)$ that assigns probability one to $\{\tau\}$. With abuse of notation, if $\sigma \in \Sigma$ we write $\delta(\sigma)$ instead of $\delta(\tilde{\tau}(\sigma))$.

Let us simply write $U_t(c, \tau)$ instead of $U_t(c, \tau; v_t)$. The domain of the utility functions can be extended to $\mathscr{M}$:

$$\overline{U}_t(c, \mu) := \int_{\tau(\Sigma)} U_t(c, \tau) d\mu.$$

Finally, given a subset $A \subset \tilde{\tau}(\Sigma)$ and a point $\tau \in \tilde{\tau}(\Sigma)$, the distance between τ and A is $d(\tau, A) := \inf\{d(\tau, a) : a \in A\}$.

We are ready to define stable sets in Poisson games. We define a *perturbation* as a pair $(\varepsilon, \mu^\circ) \in (0, 1) \times \mathscr{M}^\circ$. A perturbation acts "moving" the average behavior of the population towards the completely mixed measure μ° with vanishing probability ε. Hence, a Nash equilibrium of such a perturbed game is a strategy function σ that satisfies $\sigma \in \overline{\mathrm{BR}}((1 - \varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$, where the correspondence $\overline{\mathrm{BR}} : \mathscr{M} \to \Sigma$ is defined in the obvious way.

Given a perturbation $(\varepsilon, \mu^\circ) \in (0, 1) \times \mathscr{M}^\circ$, it can be easily shown that $\sigma \in \overline{\mathrm{BR}}((1 - \varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$ if and only if σ is a Nash equilibrium of a utility perturbed Poisson game with utility functions:[9]

$$u_t(c, x \mid \varepsilon, \mu^\circ) := (1 - \varepsilon)u_t(c, x) + \varepsilon \int_{\tau(\Sigma)} U_t(c, \tau) d\mu^\circ.$$

Definition 7. A set of equilibria of a Poisson game Γ is *stable* if it is minimal with respect to the following property:

Property (S). $S \subset \Sigma$ *is a closed set of Nash equilibria of Γ satisfying: for any $\varepsilon > 0$ there is a $\bar{\eta} > 0$ such that for any perturbation (η, μ°) with $0 < \eta < \bar{\eta}$ we can find a σ that is ε-close to S and satisfies $\sigma \in \overline{\mathrm{BR}}((1 - \eta)\delta(\sigma) + \eta\mu^\circ)$.*

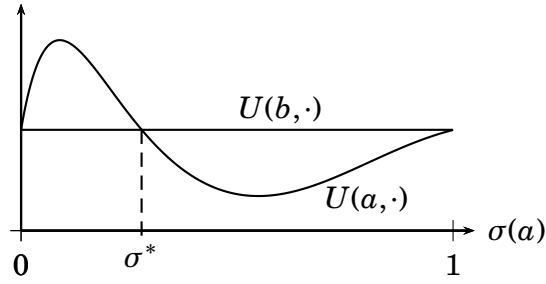[9] This defines the set of games $P(v)$ used in page 8.

FIGURE 1. Utility functions in a Poisson game with a disconnected stable set.

De Sinopoli, Meroni, and Pimienta (2014) prove that stable sets in Poisson games satisfy existence, admissibility, and are robust against iterated deletion of dominated strategies and inferior best responses. Nonetheless, De Sinopoli, Meroni, and Pimienta (2014) show that a stable set is not necessarily connected by means of the example illustrated in Figure 1. It shows the expected utility functions of the unique type of player in the game. The function $U(a, \tau)$ is the expected utility accrued by a player if she chooses action a and the average member of the population plays according to τ. The constant function $U(b, \tau)$ represents the corresponding utility if she chooses action b. The game has two stable sets, $\{\sigma^*\}$ and the disconnected set $\{(a),(b)\}$. The Nash equilibrium strategy a is robust against any perturbation that "lifts" the function $U(a, \tau)$ more than $U(b, \tau)$ while the Nash equilibrium strategy b is robust against any perturbation that "lifts" the function $U(b, \tau)$ more than $U(a, \tau)$. Note that these two strategies belong to different components of the set of Nash equilibria of the game.

Of course, this game has the connected stable set $\{\sigma^*\}$. We can now easily show that any Poisson game has at least one such component.

Proposition 1. *Every Poisson game $\Gamma(v)$ has a stable set contained in a connected component of equilibria. Moreover, every Poisson game has a minimal connected set of Nash equilibria that satisfies Property (**S**).*

Proof. Stable sets in Poisson games are an example of *Q-robust* sets of fixed points (McLennan, 2012, Definition 8.3.5). In broad terms, a set of fixed points X of a correspondence F is *Q-robust* if every correspondence close to F that can be obtained through a perturbation in some class Q has a fixed point close to X. The proof of Theorem 1 shows that every Poisson game has a connected set of Nash equilibria that is essential and, therefore, Q-robust. Moreover, McLennan (2012, Theorem 8.3.8) shows that, if F is an upper semi-countinuous and convex valued correspondence, every Q-robust set contains a minimal Q-robust set and

that every connected Q-robust set contains a minimal connected Q-robust set. The desired result follows. □

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