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Abstract

The aim of this survey is to provide an overview of the main definitions of certainty equivalent and its applications in the one-dimensional and multidimensional framework. We also show the relationships between the concept of certainty equivalent and other definitions related to different fields. In particular, we focus on financial and economic approaches that imply risk or inequality measurement.

Keywords: certainty equivalent, risk, inequality, efficiency, willingness to pay.

JEL Classification: D81, D63, I30.

1 Introduction

The certainty equivalent can be defined as a mean value and it represents a sure amount for which a subject is indifferent to a risky position or, for instance, in the theory of inequality, it can be interpreted as the level of an attribute which, if given to each subject, assures the same level of social welfare as the given distribution. But, it can also be linked either to a premium principle, to a risk measure or to an efficiency measure.

In this survey we present the main definitions of certainty equivalent in the univariate approach of decision making under risk and under uncertainty, and, moreover, we extend the research to the multidimensional framework. We explore the recent advances on the topic and we connect the concept of certainty equivalent to other definitions in different fields. We focus on financial and economic approaches in order to analyze differences and analogies between them. Finally, we consider the definition of certainty equivalent in the multidimensional framework, because we are also concerned with the impact of possible dependence among risks. We will see how this dependence structure influences the definition and the measurement of the certainty equivalent.

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The paper is organized as follows. In Section 2, we present the main definitions of certainty equivalent in the univariate case, in models under risk or under uncertainty and we analyze their interpretations and their connections. In Section 3, we present some fields of application of the concept of certainty equivalent, without the claim of being exhaustive. We consider the notion of efficiency, inequality, risk measure and willingness to pay. The multivariate setting is discussed in Section 4. We concentrate on the multidimensional generalization of the certainty equivalent in inequality and risk theory. Finally, we provide our conclusions.

2 One-Dimensional Case

The certainty equivalent is the fixed amount that one is willing to trade against an uncertain prospect. The concept is born as a mean value, but it has been developed by the theory of decision making under uncertainty, that presupposes the ability to rank random variables and therefore to define a complete order on the space of random variables.

Bonferroni's mean (1924) is defined in the following way

$$\psi(M) = \sum_{i=1}^{n} q_i \psi(x_i) / \sum_{i=1}^{n} q_i, \tag{1}$$

with ψ continuous and strictly increasing function, $x = (x_1, \ldots, x_n)$ vector of values assumed by the distribution and q_i positive weights. De Finetti (1931) has reinterpreted this definition and the axioms of Nagumo (1930, 1931) and Kolmogorov (1930) as natural requirements to determine the certainty equivalent of a distribution, that is

$$M = \psi^{-1}\left(\sum_{i=1}^{n} p_i \psi(x_i)\right), \text{ with } \sum_i p_i = 1. \tag{2}$$

The characterizing properties of the certainty equivalent are: continuity, strict monotonicity, symmetry (invariance to labeling of the variables), reflexivity (if all variables are equal, the value is the mean) and associativity (replacing a subset of the observations with their partial means does not change the overall mean).

Other properties may be required imposing restrictions on ψ. See also Alcantud and Bosi (2003) for a study on the characterizing properties of the certainty equivalent functionals.

Chew (1983) has generalized the quasilinear mean of De Finetti, Nagumo and Kolmogorov allowing for a system of weights given by a continuous function.

In what follows, we distinguish, in Section 2.1, the model of decision making under risk, i.e., the expected utility (EU) model of Von Neumann-Morgenster (1947), and, in Section 2.2, the decision making theory under uncertainty, i.e., the subjective expected utility (SEU) model of Savage (1954). Later, Quiggin (1982) and Schmeidler (1989) proposed a critique based on new axiomatizations. The difference between risk and uncertainty is given by the presence or not of a given probability distribution.

Next, we summarize the basic notation. A random variable $X : \mathcal{X} \to R$ is a function defined on the probability space $(\mathcal{X}, \mathcal{A}, P)$, where $\mathcal{X}$ is the universe, $\mathcal{A}$ is the sigma algebra and P is the probability measure. The domain is a set of acts that associate to each state of nature a possible consequence. The outcome space of each random variable is the set of positive real numbers.

Let $F_X(x)$ be the cumulative distribution function of X, $G_X(x)$ the decumulative distribution function of X, $F_X^{-1}(p) = \inf\{x \in R : F_X(x) \geq p\}$, $\forall p \in [0,1]$, and $G_X^{-1}(p) = \inf\{x \in R : G_X(x) \leq p\}$, $\forall p \in [0,1]$, be their inverse functions, respectively, $\phi(x)$ be the probability distribution function and E denote the expectation.

2.1 Models Under Risk

In the Expected Utility (EU) theory of von-Neumann and Morgernstern (1944) agents facing risk maximize the expected value of the utility of their wealth. Hence, the decision maker is assumed to have a utility function $u : \mathcal{X} \to R$ which is strictly increasing and concave. All preferences are complete, reflexive and transitive.

We recall that the EU framework is based on the idea of Bernoulli (1738) who has stated that individuals act so as to maximize the expected utility of the gain and not the expected gain itself. The utility function is an index that represents the degree of satisfaction of the decision maker. The expected utility is ordinal, because it is invariant to any increasing transformation, whereas the utility function is cardinal and an increasing linear transformation of u will not change the ranking of outcomes. For an individual with utility function u, the preference order between the variables X, Y is defined by

$$X \succeq Y \text{ if and only if } Eu(X) \geq Eu(Y). \tag{3}$$

Because $u(X)$ is strictly increasing, then follows

$$X \succeq Y \iff Eu(X) \geq Eu(Y) \iff C_u(X) \geq C_u(Y), \tag{4}$$

where, for any random variable X, the **certainty equivalent** (CE) in the EU theory is the quantity

$$C_u(X) := u^{-1}Eu(X), \tag{5}$$

i.e., the sure amount for which a decision maker remains indifferent to a lottery X. Note that $C_u(X)$ satisfies a property of consistency, i.e., $C_u(C_u(X)) = C_u(X)$.

As pointed out by Borch (1963), up to that time, the classical economic theory has been unable to analyze insurance problems, even if Tauber, in 1909, assumed that risk bearing acts as a service and then it must have a price. Despite this, the Bernoulli principle was exploited in 1834 by Barrois, with the purpose to study fire insurance. He has measured the utility of the initial wealth of an individual, reduced with certainty by a quantity of money, and he has compared it with the expected value of the utility in the case in which a house may be destroyed by fire with a given probability. This is, at the best of our knowledge, the first definition of certainty equivalent.

A premium principle was formally defined by Pratt in 1964. He has proposed the **risk premium** π as the monetary amount such that a subject would be indifferent between receiving a risk ε, and receiving the non random amount $E\varepsilon - \pi$, sometimes called cash equivalent. The following relation holds

$$u(x + E\varepsilon - \pi) = Eu(x + \varepsilon). \tag{6}$$

If ε represents a pure risk, then $E\varepsilon = 0$, and the proper definition of π is risk premium, the maximum amount of money that one is ready to pay to escape a pure risk. Otherwise, if $E\varepsilon \neq 0$, then $C_u(X) = E\varepsilon - \pi$ is the certainty equivalent, the sure amount that makes you indifferent between play the lottery or receive a sure gain.

Within the expected utility framework have been developed different notions linked to the concept of certainty equivalent. For instance, we cite the mean proposed by Bühlmann (1970), called **u-mean**, $C_M(X)$, satisfying

$$Eu(X - C_M(X)) = 0. \tag{7}$$

This equation is also known as the **principle of zero utility**. See Fishburn (1986) for a similar definition of the certainty equivalent as an implicit mean, which solves

$$\int \tilde{u}(x, y)d\phi(x) = 0. \tag{8}$$

We have $y = C_M(X)$ if the function $\tilde{u}$ is such that $\tilde{u}(x, y) = \tilde{u}(x) - \tilde{u}(y)$.

Later, Bühlmann et al. (1977) proposed a premium principle to generalise the concept of certainty equivalent and zero utility principle, the so called **Swiss premium principle**

$$u\left((1-z)\pi_z(X)\right) = Eu\left(x - z\pi_z(X)\right), \tag{9}$$

with $z \in [0,1]$. It is possible to recover the certainty equivalent if $z = 0$ and the zero utility principle if $z = 1$.

In Ben-Tal and Teboulle (1986), another certainty equivalent is introduced, called the **Optimized Certainty Equivalent** (OCE), or the recourse CE, and defined by

$$C_O(X) = \sup_{\eta \in R} \left\{\eta + Eu(x-\eta)\right\}, \tag{10}$$

where u is a normalized concave utility function. In this case $Eu(X)$ is interpreted as the sure present value of a future uncertain income X. The definition of the OCE describes this situation: a decision maker expects a future uncertain income of x dollars, and can consume η dollars at present. The resulting present value of x is then $\eta + Eu(x - \eta)$. Thus, the certainty equivalent $C_O(X)$ is the result of an optimal allocation of X between present and future consumption. If the utility is a strictly increasing, concave and normalized function, then the OCE is translation invariant, consistent, monotone, concave and moreover, it preserves the expected utility order (second order stochastic dominance).

2.1.1 Risk aversion

In the literature on decision under risk, the risk aversion is defined by the preference of the expectation EX to the random variable X. In EU theory this concept is characterizable by concave utilities, that imply diminishing marginal utility of wealth. Therefore, two different concepts, the attitude toward risk and wealth, are kept together.

We dedicate a section to risk aversion, because the attitudes toward risk of an agent influence the properties of the utility function and, consequently, of the CE and of the risk premium.

Definition 1. An agent is risk averse if he dislikes all zero-mean risks ε, with $E\varepsilon = 0$, at all wealth level

$$Eu(x+\varepsilon) \leq u(x). \tag{11}$$

Other alternative approaches to the EU model have been given by Machina (1982), Segal (1984), Allais (1987), together with Quiggin (1982), Ben-Tal and Teboulle (1986) and Yaari (1987). These models try to solve some paradoxes of EU model and entail a different approach to risk aversion, that no longer necessarily goes along with a concave utility function.

Before defining the different kind of risk aversion, we recall the following notions.

Definition 2. Two random variables are comonotonic if and only if

$$(X_1(x) - X_1(y))\,(X_2(x) - X_2(y)) \geq 0,\ \forall x, y \in \mathcal{X}. \tag{12}$$

Comonotonicity refers to perfect positive dependence between the random variables X_1 and X_2.

Definition 3. X_1 is a mean preserving spread (MPS) of X_2 if and only if $EX_2 = EX_1$ and $\int_0^{\bar{x}} F_{X_2}(x)dx \leq \int_0^{\bar{x}} F_{X_1}(x)dx,\ \forall x$.

Ranking gambles by mean preserving spreads is a special case of ranking gambles by second order stochastic dominance, that is when the means are equal.

Definition 4. X_1 is a monotone mean preserving spread (MMPS) of X_2 if and only if there exists a random variable z, with $Ez = 0$, such that X_1 has the same probability distribution of $X_2 + z$, and X_2 and z are comonotonic.

Now, different attitudes toward risk can be distinguished:
- **weak risk aversion** (Pratt (1964)): $EX \succeq X$,
- **strong risk aversion** (Hadar and Russel (1969), Rothschield and Stiglitz (1970)): $X_2 \succeq X_1$ if and only if (iff) X_1 is a MPS of X_2,
- **monotone risk aversion** (Quiggin (1992)): $X_2 \succeq X_1$ iff X_1 is a MMPS of X_2.

In the EU model a risk averse decision maker is represented by a concave utility function and it implies that X is less desirable than a sure reward of EX. The concavity of the utility function also expresses the attitude toward wealth, because the marginal utility is decreasing.

According to Montesano (1999), it is possible to give a characterization of risk aversion by means of the CE. Let us denote with $C_A(X)$ an unspecific CE of the variable X for an agent A. Then,
- A is **global risk averse** iff $C_A(X) \leq EX$,
- A is **strong risk averse** iff $C_A(X_1) \leq C_A(X_2)$, with X_1 MPS of X_2,
- A is **more risk averse** than B iff $C_A(X) \leq C_B(X)$.

Note that the following relations hold

$$\begin{gathered}\text{strong risk aversion}\\ \Downarrow\\ \text{monotone risk aversion}\\ \Downarrow\\ \text{global risk averse}\end{gathered}$$

Further, we recall the definitions of absolute and relative risk aversion (Pratt (1964)), because they entail a different definition of risk premium.

Definition 5. The degree of absolute risk aversion is given by

$$A(x) = -\frac{u^{''}(x)}{u^{'}(x)}, \tag{13}$$

where $u^{'}$ and $u^{''}$ denote the first and second order derivative of u.

$A(x)$ is a measure of the degree of concavity of the utility function, and it is positive under risk aversion.

By the absolute risk aversion and the first and second order approximation of equation (6), the risk premium can be recovered in this way

$$\pi \cong \frac{1}{2}\sigma_{\varepsilon}^{2} A(x), \text{ where } \sigma_{\varepsilon}^{2} = E\varepsilon^{2}. \tag{14}$$

Relative risk aversion is referred to multiplicative risks instead. This is simply a unit-free measure of risk aversion, and it can be recovered by the first and second order approximation of the following equation

$$Eu\left(x(1+i)\right) = u\left(x(1-\pi_r)\right), \tag{15}$$

where $\pi_r = \pi/x$ and $1+i$ is a multiplicative risk.

Definition 6. The degree of relative risk aversion is given by

$$r(x) = xA(x) \tag{16}$$

From Definitions 5 and 6, a classification of the utility functions can be done, depending on if they imply decreasing (increasing/constant) absolute (relative) risk aversion.[1]

2.1.2 Rank dependent expected utility models

The rank dependent expected utility (RDEU) model has been proposed by Quiggin (1982) and then developed by Yaari (1987). These are complementary approaches to EU, based on some critiques. In the EU theory, agents' attitudes toward risk and wealth are kept together; in fact, the utility function expresses the decision maker's attitudes toward risk, because the concavity requirement implies risk aversion, but it also implies diminishing marginal utility of wealth. Moreover, in the EU theory, the preference functional is the expected value of a non linear transformation of wealth, while in the RDEU approach it is the expected value of wealth under a non linear transformation of the probability distribution[2].

In the **dual theory** of Yaari (1987) the certainty equivalent is given by

$$C_Y(X) = \int_0^{+\infty} f(G_X(t))dt, \tag{17}$$

where $f : [0,1] \to [0,1]$ is a continuous non decreasing function with $f(0) = 0$ and $f(1) = 1$.

The utility function assingns to each random variable its certainty equivalent, therefore $u(X)$ is equal to the sum of money which, when received with certainty, is considered by the agent equally as good as X, this implies a money metric utilitarian evaluation. Note that $C_Y(X)$, as well as $C_u(X)$, is consistent.

In Yaari (1987), the CE can not always be considered as the expectation of X, because $f(G_X(x))$ is not necessarily right continuous; C_Y is rather a distorted expectation, but if f is concave, then it is a mean value.

Further, weak risk aversion implies $C_Y(X) \leq EX$, and it requires $f(t) \leq t$, $\forall t$, while strong risk aversion requires the convexity of f.

Quiggin (1982) has proposed the **anticipated utility theory**, a generalization of the EU theory that actually is also a generalization of the Yaari's dual theory. By means of the distortion of subjective probability, the CE becomes

$$C_Q(X) = \int_0^{+\infty} f(G_X(t))du(t), \tag{18}$$

where u is a von-Neumann-Morgenstern utility.

Now, we want to concentrate on the two main approaches, the EU and the Dual approach, in order to provide some relationships between the two.

From their definitions of certainty equivalent, in the following proposition, we state a necessary condition for the equality of the two approaches.

Proposition 1. *If* $u(x) = xf'(G_X(x))$, $\forall x \in \mathcal{X}$, *the values of the Certainty Equivalents defined in equations (5) and (17) are equal.*

Proof. From equations (5) and (17) follows

$$u(C_u(X)) = Eu(X) = \int_0^1 u(x)dF_X(x),$$
$$C_Y(X) = \int_0^1 f(G_X(t))dt = \int_0^1 xf'(G_X(x))dF_X(x).$$

Then, if it holds

$$u(x) = xf'(G_X(x)), \ \forall x \in \mathcal{X}, \tag{19}$$

the thesis follows. □

This proposition provides a way to represent the utility function through a modifying function f, or viceversa.

In EU theory, agents choose among random variables so as to maximize the expected value of the utility function. In Yaari's theory, agents choose among random variables so as to maximize the utility function. From equation (19) we can see that the utility so defined is different from that one of the EU theory, because now it depends on the variation of the probability distribution. Moreover, we know that $\int_0^{+\infty} xdF_X(x)$ is the mean of the random variable and $\int_0^1 f'(G_X(x))dF_X(x) = 1$, hence, $f'(G_X(x))$ is a system of nonnegative weights summing to 1. Therefore, the utility function $u(X)$ is a corrected mean of X, in which bad outcomes receive high weights, while good outcomes receive low weights.

This modification of the probability distribution is called **pessimism** if it implies that

$$P_f = \inf_{0<t<1} \left[\frac{1-f(t)}{1-t} / \frac{f(t)}{t} \right] > 1.$$

This index of pessimism has been introduced by Chateauneuf et al. (2005) and it is compared with an index of **greediness** or non-concavity

$$G_u = \sup_{y<x} \frac{u'(x)}{u'(y)},$$

in particular, they state that a decision maker with probability perception f and utility function u is monotone risk averse if and only if $P_f \geq G_u$.

This is a way to characterize the role of f and u by means of their derivatives ratio. We can have a direct comparison of the derivatives of f and u if we suppose the homogeneity of degree h of the utility function u.

Proposition 2. *If the concave utility function $u(X)$ is homogeneous and equation (19) holds, then $f^{'}(G_X(x)) \geq u'(x)$, $\forall x \in \mathcal{X}$.*

Proof. If the utility function is homogeneous of degree h, from Euler's equation we get $hu(x) = xu'(x)$, and, from equation (19) it holds $hf^{'}(G_X(x)) = u'(x)$. Further, we know that if u is homogeneous of degree greater than one, then it cannot be concave, hence, the utility is homogeneous of degree $0 \leq h \leq 1$. The inequality $f^{'}(G_X(x)) \geq u'(x)$ follows. □

The result of Proposition 2 implies, in a discrete distribution, that the choice of the new weights $f^{'}(G_X(x))$ is greater than the slope of the utility function, computed in every x.

Let us note that an homogeneous function of degree h has constant relative risk aversion (CRRA), with $h = 1 - r(x)$. The opposite does not hold in general, in fact, the function $u(x) = \ln(x)$ has relative risk aversion equal to one, but it is not homogeneous.

2.2 Models Under Uncertainty

Decision making under uncertainty can be ascribed to Savage (1954). He proposed a different interpretation of probability and, in particular, he considered the "personal probability", which is not given by a self-reported feeling of a person answering to a question, but it implies a behavioral interrogation.

Ellsberg paradox (Ellsberg (1961)) shows that an expected utility maximization model may fail to describe preferences and, later, Schmeidler (1989) has proposed a model to explain such paradox based on a weaken axiom of independece.

Further, Schmeidler has proved that the preference relation may be represented by means of a non additive probability, and, hence, by means of a Choquet integral.

Definition 7. The Choquet integral (Choquet (1953)) is given by

$$C_v(X) = \int_{\mathcal{X}} u(X)dv = \int_0^{+\infty} v(u(X) > t)dt, \tag{20}$$

where v is a **capacity** function such that $v(\emptyset) = 0$, $v(\mathcal{X}) = 1$, and if $A \subseteq B$ then $v(A) \leq v(B)$.

Let us note also that the concept of uncertainty aversion is implied by a convex capacity, then $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.

The definition of the certainty equivalent is obtained in case of non-additive measures through the Choquet integral representation. Choquet integral is a non-linear mathematical expectation and it is widely used in economics, finance and insurance as an alternative to traditional mathematical expectation (if v is a probability, then C_v is the mathematical expectation).

The RDEU model can also be represented by a Choquet integral

$$C_v(X) = \int_0^{+\infty} v(u(X) > t)dt = \int_0^{+\infty} f(P(u(X) > t))dt, \tag{21}$$

where the distorted probability $f \circ P$ is a capacity.

If the modifying function is the identity $f(p) = p$, then $C_v(X) = u(C_u)$; if $u(x) = x$, then $C_v(X) = C_Y(X)$. Finally, if both f and u are identity functions, then $C_v(X) = EX$.

3 Comparison to Other Fields

The certainty equivalent may be linked to many other definitions, not only to the risk premium, and it may be exploited not only in risky or uncertainty contexts. The main interpretation of the certainty equivalent is as a measure of dispersion around the mean, and, for this reason, it lends itself to several applications. We describe, in what follows, the efficiency measures, the risk measures and the inequality measures. Then, we conclude this section with the willingness to pay, because it also has some links to the definition of certainty equivalent.

3.1 Efficiency Measures

If we move from the setting of EU theory and its generalizations, we can analyze another framework in which the notion of CE or its modifications are considered.

In Debreu (1951), a **coefficient of resource utilization** is introduced. The treatment is based on the vector-set properties in the commodity space. When a situation is nonoptimal it is possible to find some measure of the loss involved, indicating how far it is from being optimal. The loss is a measure of the distance from the actually available complex of resources to the set of optimal ones.

Let us call x_0 the utilizable resource vector, x the total net consumption, then $x_0 - x$ is the non utilized resource vector, moreover, we denote with x^* the collinear vector of

x_0 (i.e., $x^* = cx_0$). The loss is given by

$$x_0 - x^* = x_0(1 - c). \tag{22}$$

The distance function reaches its minimum for an optimal complex resulting from a reduction of all quantities of the nonoptimal complex by a ratio c, the coefficient of resource utilization of the economic system. This coefficient is $c = 1$ if the situation is optimal and lower than this if the situation is nonoptimal and hence, it measures the efficiency of the economy.

The classical reason for studying the comparison of alternatives relative to some set seems to date back to Deputi (1844), in order to measure changes in welfare.

Debreu multiplies, for each commodity, the difference between the available and the optimal quantity by the price derived from the intrinsic price system. Then he sums such expressions for all commodities and divides by a price index in order to eliminate the arbitrary multiplicative factor affecting all the prices.

The mathematical problem is

$$\min_{x \in \mathcal{X}_m} \left\{ p(x_0 - x)/px \right\},$$

where $\mathcal{X}_m$ is the set of minimal resources required to achieve at least a given standard of living, and p is the vector of prices of the resources. This problem is equivalent to $\max_{x \in \mathcal{X}_m} \{px/px_0\}$ and because of collinearity it is equivalent also to

$$c \max_{x \in \mathcal{X}_m} \left\{ px/px^* \right\},$$

that leads to

$$\max_{x \in \mathcal{X}_m} \left\{ px/px_0 \right\} = c,$$

because of the convexity of $\mathcal{X}$, that implies $p(x^* - x) \geq 0$ and $px/px^* \leq 1$.

If in equation (22), we interpret x^* as the utility level reached by the distribution and x_0 is the mean of the distribution, then c represents the scaling of the available resources that let you reach a given utility level, or the expected utility. This is very close to the approach of Atkinson (1970), described in the following subsection.

Ahlheim (1988) investigates the characterizing properties of the Debreu's coefficient and interpretes it as a measure of individual welfare and of efficiency.

It is also possible to consider the case in which efficiency is determined according to a social welfare function. Graaff (1977) calculates an equity index as the ratio between an efficiency and a social welfare index

$$\gamma(x) = \frac{\Gamma(x)}{\varepsilon(x)},$$

where $\Gamma(x)$ is the minimal scale of a vector of total consumption that can be redistributed so as to maintain social welfare and $\varepsilon(x)$ is the minimal scale of a producible consumption bundle that can still increase individual welfare.

In Chambers and Miller (2014), a family of path-based measures is proposed. It is an ordinal approach that suggests some axioms (scale invariance, strong monotonicity, continuity) on efficiency measures, but they do not assume the existence of a functional representation. The Debreu's coefficient of resourse utilization is obtained as a special case. A path-based measure is a monotonic and continuous function from the origin to some fixed point. The point of intersection with the path provides a scale. The further along this path, the more efficient the bundle of resources.

A close field is represented by the performance measures. For a survey on these measures see Caporin et al. (2014).

3.2 Inequality Measures

Another branch of the economic studies, which it has much in common with the theory of risk measurement, is the theory about inequality measurement. It refers to the comparative evaluation of distributions of a single variable. Atkinson (1970) has proposed some criteria to evaluate if an income distribution presents less inequality than a second one. In the meantime, Rothschild and Stiglitz (1970) considered some criteria to classify distributions according to their risk. We are interested in describing the relationship between the notion of certainty equivalent and the notion of **equally distributed equivalent income** introduced by Kolm (1969), Atkinson (1970) and Sen (1973).

The functional used to rank distributions is

$$W(X) = \int_0^{\bar{x}} u(x)\phi(x)dx. \tag{23}$$

For a complete ranking we have to specify the utility function up to a monotonic linear transformation.

Dalton (1920) has suggested that we should use as a measure of inequality the ratio of the actual level of social welfare to that which would be achieved if income were equally distributed, that is $\int_0^{\bar{x}} u(x)\phi(x)dx/u(EX)$, but this is not invariant with respect to linear transformations of the function $u(x)$. It does not happen for the concept of equally distributed equivalent income C_E, i.e., the level of income which, if equally distributed, would give the same level of social welfare as the present distribution. Therefore, we have

$$u(C_E)\int_0^{\bar{x}}\phi(x)dx=\int_0^{\bar{x}}u(x)\phi(x)dx. \tag{24}$$

The quantity C_E is indeed a certainty equivalent. Moreover, the concept of equally distributed equivalent income is interpretable as a Debreu's coefficient, see equation (22).

By equation (24), some measures of inequality can be recovered (Blackorby and Donaldson (1978, 1980))

- the **relative inequality index** $I_r = 1 - C_E/EX$, with $0 \le I \le 1$ and
- the **absolute inequality index** $I_a = EX - C_E$.

As noted by Chew (1987), the measures of income inequality mostly assume the forms I_r or I_a, varying the choice of C_E.

Later, Yaari (1988) has shown the way to derive a measure of income inequality starting from the Lorenz curve. Consider $\bar{X}$ as a non decreasing rearrangement[3] of X. The function

$$L_X(p)=\frac{1}{EX}\int_0^{\bar{p}}\bar{X}(p)dp, \tag{25}$$

with $p \in [0,1]$, is called Lorenz measure and it is increasing and convex.

In order to give a simple utility representation, starting from the form

$$u(X)=\int_0^1\bar{X}(p)d\tilde{\phi}(p), \tag{26}$$

where $\tilde{\phi}(p)$ is a continuous distribution, Yaari defines the equality rating of the variable X as the quantity

$$\eta(X)=\int_0^1 L_X^{-1}(p)d\tilde{\phi}(p), \tag{27}$$

where $L_X^{-1}(p)$ is the left derivative of the Lorenz measure. Doing this, the utility can be represented in this way $u(X) = \eta(X)EX$.

The evaluation measure $\tilde{\phi}(p)$ that makes $\eta(X)$ equals to the Gini's coefficient is $\tilde{\phi}(p) = 2p - p^2$.

3.3 Risk Measures

In recent years has been an increasing interest in the axiomatic approach to risk measures, i.e. functionals assigning a real number to the risk of financial positions. Seminal papers on the axiomatic approach to risk measures have been written by Artzner et al. (1999), Föllmer and Schied (2002, 2004), who have introduced the notions of coherent and convex risk measures, respectively. According to Artzner et al. (1999) a risk measure is **coherent** if it fulfills the following axioms: monotonicity, translation invariance, positive homogeneity and subadditivity.

The representation of such risk measures could be given by means of the acceptance sets, but the dual representation is useful to understand the analogies with the notion of CE

$$\rho(X) = \sup_{p \in P} \{E_p(-X)\}. \tag{28}$$

A coherent risk measure can also be **convex** if it satisfies the axioms of monotonicity, translation invariance and convexity. The dual representation of a convex risk measure is

$$\rho(X) = \sup_{p \in P} \{E_p(-X) - \alpha(p)\}, \tag{29}$$

where α is a penalty function, which can be choosen to be convex and lower semi continuous, with $\alpha(p) \geq -\rho(0)$. The risk measure, or the capital requirement, is thus the expected loss of a position. The decision maker takes the worst penalized expected loss over the probability class P.

In order to analyze the relationships between certainty equivalents and risk measures, we recall the result of Theorem 2.1. of Müller (2007), stating that a functional can be written in the form

$$\rho(X) = \ell^{-1} E\ell(-X) \tag{30}$$

if and only if it satisfies the properties of monotonicity, law invariance, consistentency (with respect to any constant $a \in R$) and quasi linearity[4].

In fact, to any such ℓ function corresponds an increasing concave utility function $u(x) = -\ell(-x)$, therefore the risk measure in (30) in terms of the utility function is defined as $\rho(X) = -u^{-1}Eu(X)$.

Note that increasing concave utility functions do not necessarily lead to convex risk measures. Consider, for instance, the power utility function $u(x) = x^\beta$ with $x \geq 0,\ \beta \leq 1$. The measure $\rho(X) = -\left\|-X\right\|_p$ is subadditve and positively homogeneous, but not translative.

The zero utility principle defines the premium $\pi(X)$ implicitly as the solution of the equation $u(0) = Eu(X - \pi(X))$. The risk measure $\rho(X) = \pi(-X)$ is always a convex risk measure, whereas the CE typically is not, because $\rho(X) = -CE(X)$.

The certainty equivalent or the risk measure derived from a utility indifference principle is also known as quasi-linear mean, see Muliere and Parmigiani (1993) for a review of the early history of mean values.

Therefore, any certainty equivalent CE inducing a preference order on random variables generates a corresponding risk measure ρ, inducing the same preference order by the simple relation, $\rho(X) = -CE(X)$, and vice versa. As an example, let us consider the shortfall risk introduced by Föllmer and Schied (2002, 2004), and defined by

$$\rho_{FS}(X) = inf\left\{\eta|Eu(X-\eta)\geq 0\right\}. \tag{31}$$

This measure is related to the u-mean $C_M(\cdot)$, in fact, $\rho_{FS}(X) = -C_M(X)$.

Further, Ben-Tal and Teboulle (2007) have shown that $\rho_u(X) := -C_O(X)$, is a convex risk measure if the utility function u is concave, nondecreasing, normalized and with closed nonempty domain.

Denneberg (1990) and Wang (1996) have noted that the distortion risk measures are based on Yaari's (1987) dual theory of choice. Let f be normalized, bounded, non decreasing and non negative, then

$$\rho_d(X) = \int_0^1 f\left(G_X(t)\right) dt \tag{32}$$

is a distortion premium principle.

Another interesting work about the connections between risk measurement and economic decision theory has been proposed by Denuit et al. (2006). The authors analyze equivalent utility risk measures following different approaches, the expected utility theory, the rank dependent expected utility, the Choquet expected utility and the maximin expected utility of Gilboa and Schmeidler (1989).

3.4 Willingness to Pay

According to the approach of Eeckhoudt et al. (1997), we can analyze the links with the willingness to pay for a risk reduction and the certainty equivalent. Given a negative event x_0 with probability p_0, from the EU theory we have

$$p_0u(x_0)+(1-p_0)u(x_0+\varepsilon) = u(p_0(x_0)+(1-p_0)(x_0+\varepsilon)-\pi_0) = u(x_0+(1-p_0)\varepsilon-\pi_0), \tag{33}$$

where $\pi_0 = \pi_0(x_0, p_0)$ is the risk premium that an agent is willing to pay to obtain with certainty the expectation of x.

If an agent is optimistic, he gives low weights to negative events, then we consider a decrease in the probability $p_1 < p_0$. Comparing the different certainty equivalents we can recover the willingness to pay $w = w(x_0, p_0, p_1)$ due to agents risk reduction

$$\begin{aligned} p_1 u(x_0 - w) + (1 - p_1) u(x_0 + \varepsilon - w) = \\ = u(p_1(x_0 - w) + (1 - p_1)(x_0 + \varepsilon - w) - \pi_1) = u(x_0 + (1 - p_1)\varepsilon - \pi_1 - w), \end{aligned} \tag{34}$$

where $\pi_1 = \pi_1(x_0 - w, p_1)$. If an agent is pessimistic, the situation described by equations (33) and (34) is reversed and p_0 is the subjective probability of the agent, that is greater than p_1 for negative events.

Now, equating the right hand side of equation (33) and (34), Eeckhoudt et al. (1997) obtain

$$w = (p_0 - p_1)\varepsilon + (\pi_0 - \pi_1). \tag{35}$$

While, in contrast, the difference between the two certainty equivalents is given by

$$\begin{aligned} u(CE_0) = u(x_0 + (1 - p_0)\varepsilon - \pi_0) = p_0 u(x_0) + (1 - p_0) u(x_0 + \varepsilon) \\ u(CE_1) = u(x_0 + (1 - p_1)\varepsilon - \bar{\pi}_1) = p_1 u(x_0) + (1 - p_1) u(x_0 + \varepsilon), \end{aligned} \tag{36}$$

where $\bar{\pi}_1 = \bar{\pi}_1(x_0, p_1)$ and hence

$$CE_1 - CE_0 = (p_0 - p_1)\varepsilon + (\pi_0 - \bar{\pi}_1). \tag{37}$$

Using (35) and (37), the variation of the CE can alternatively be written

$$CE_1 - CE_0 = w + (\bar{\pi}_1 - \pi_1). \tag{38}$$

This difference depends on both the pessimism of the decision maker and his attitude toward risk. In fact, an increase in pessimism causes a decrease of CE, because the pessimism increases with the probability of negative events, whereas if the the risk premium decreases (for instance, with DARA[5] utility functions), then CE increases.

Further, if we consider a mean preserving spread of a given distribution, then $CE_1 - CE_0 = \pi_0 - \bar{\pi}_1$ and $w = \pi_0 - \pi_1$. An example could be given by a redistribution of the income, such that there are less people gaining few money. Then, $CE_1 - CE_0 > 0$ in a more egalitarian society.

4 Multivariate Certainty Equivalent

For multivariate distributions, risk or inequality measurement may be associated with dispersion or positive dependence measurement. Therefore, the properties of the utility funtions now must involve the cross derivatives. Since the paper of Eisner and Strotz (1961), the literature on choice under risk with multidimensional utilities has shown that the sign of cross derivatives of the utility functions play an important role, not entirely clear or deepened. If we interpret the utility function as a welfare function, these arguments have a great applicability in the theory of multidimensional inequality, see Atkinson and Bourguignon (1982).

In what follows, we present an historical excursus on the different concepts that define, directly or indirectly, the certainty equivalent in the multidimensional case. Our aim is to trace some guidelines that illustrate how the certainty equivalent concept was born and developed in the different fields.

We consider a random vector $\mathbf{X} = (X^1, \ldots, X^n)$, that is a collection of n univariate random variables $X^i : \mathcal{X} \to R$, defined on the same probability space $(\mathcal{X}, \mathcal{A}, P)$.

Keeney (1973) has defined a conditional utility function $u(x_i, \bar{x}_j)$ for x_i, given $x_j = \bar{x}_j$, meaning that all the risk is associated with only one attribute and the other one is fixed. Therefore, the decision maker's attitudes toward risk depend only on one attribute.

In Theorem 1 he states that

$$u(x_1, x_2) = u(\bar{x}_1, x_2) + u(x_1, \bar{x}_2) + ku(\bar{x}_1, x_2)u(x_1, \bar{x}_2),$$

with k constant.

The expected value is

$$\begin{aligned} Eu(x_1, x_2) = Eu(\bar{x}_1, x_2) + Eu(x_1, \bar{x}_2) + kEu(\bar{x}_1, x_2)Eu(x_1, \bar{x}_2) = \\ = u(\bar{x}_1, C_c(x_2)) + u(C_c(x_1), \bar{x}_2) + ku(\bar{x}_1, C_c(x_2))u(C_c(x_1), \bar{x}_2), \end{aligned}$$

where $C_c(x_i)$ is the conditional certainty equivalent of x_i.

The inequality measurement, extended to the multidimensional case, dates back to Kolm (1977), Atkinson and Bourguignon (1982) and Maasoumi (1986). Their works lay the theoretical foundations. Later, Tsui (1995, 1999) has extended the work of Kolm (1977).

Kolm suggestes a multidimensional inequality index based on an extension of the equally distributed equivalent income of Atkinson (1970). It is the fraction $\theta(\mathbf{X})$ such that

$$W(\mathbf{X}) = W(\theta(\mathbf{X})\mu), \tag{39}$$

and Tsui (1995) axiomatizes the properties of relative and absolute inequality indices built starting from this fraction.

Abul Naga and Geoffard (2006) exploit the definition of equally distributed equivalent income of Kolm (1977) and the approach of Keeney (1973) of the conditional certainty equivalent to study the possibility to decompose a bivariate inequality measure $\theta(\mathbf{X})$, as explained next.

They define the welfare level $w_0 = u(\theta(\mathbf{X})\mu) = \frac{1}{n}\sum_i u(x_i)$, where $\mu = (EX_1, \ldots, EX_2)$ is the mean vector. Then, they compute a redistribution of each attribute in the population, holding constant the other attribute. If only x_2 is redistributed, from solving $w_0 = \frac{1}{n}\sum_i u(x_{i1}, \rho_2\mu_2)$ and later $w_0 = u(\gamma_1\mu_1, \rho_2\mu_2)$, one gets two measures of equality for the given distribution, ρ_i and γ_i.

If the utility function is additively separable, then $\rho_i = \gamma_i$ and $\theta(\mathbf{X})$ is decomposed in terms of γ_1 and γ_2, otherwise one needs to introduce a third index that captures the correlation.

In the financial context, the concept of multidimensional risk premium and risk aversion has been introduced by Kihlstrom and Mirman (1974) and Richard (1975) and, then, deepened, for instance, by Duncan (1977) and Karni (1979).

Courbage (2001), following Kihlstrom and Mirman (1974), defines the bivariate risk premium, referred to the first argument, starting from

$$Eu(x_1 + \varepsilon_1, x_2 + \varepsilon_2) = u(x_1 - \pi_C, x_2), \tag{40}$$

and he gets

$$\pi_C(x) = -\frac{1}{2}\sum_{i=1,2}\sum_{j=1,2}\sigma_{ij}\frac{u_{ij}}{u_1}(x), \tag{41}$$

where u_{ij} is the mixed partial derivative, and σ_{ij} are the covariance elements of the risks associated to each variable.

However, in the bivariate case, we can consider the vector of risks $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and the risk premia vector $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$. Now, let us compute the first and second order approximations

$$\begin{gathered} u(x + E\varepsilon - \pi) \cong u(x) - \tilde{\pi}_1 u_1(x) - \tilde{\pi}_2 u_2(x), \\ Eu(x + \varepsilon) \cong u(x) + \tfrac{1}{2}\sum_{i=1,2}\sum_{j=1,2}\sigma_{ij}u_{ij}(x). \end{gathered} \tag{42}$$

Comparing the right hand sides of (42), the relationship between the components of the bivariate risk premium is given by

$$\tilde{\pi}_1 u_1(x) + \tilde{\pi}_2 u_2(x) = -\frac{1}{2}\left(u_{11}(x)\sigma_{11} + u_{22}(x)\sigma_{22} + 2u_{12}(x)\sigma_{12}\right). \tag{43}$$

The risk premium proposed by Courbage (2001) is equal to the sum $\tilde{\pi}_1 + \tilde{\pi}_2 \frac{u_2}{u_1}(x)$. The sum of the components of the risk premium vector[6], weighted by the marginal utilities, vary depending on a variation of the risk variances, and of the correlation among risks.

See Nau (2003) for a definition of certainty equivalent in a nonexpected utility framework, where the Pratt risk premium and the certainty equivalent are selling prices.

Finally, we recall that, in a bivariate context, Drèze and Modigliani (1972) and Caapera and Eeckhoudt (1975) have proposed a model to identify the sure level of total time adjusted income such that the individual is indifferent between this level of income and the delayed prospect. Drèze and Modigliani require to compensate only in the second period, whereas Caapera and Eeckhoudt partition the compensation in the two periods, and, in this case, they find that an agent is willing to pay an higher premium.

Another definition that lends itself to generalization in the multidimensional case is that of Yaari. Dhaene et al. (2000) give a characterization of the CE considering the Yaari's CE of the sum of the random variables which compose the vector. In Lemma 1, the authors propose different notions of CE, with respect to the choice of the distortion function f.

Let us now recall the following definitions for mutually comonotonic risks and stop-loss order.

Definition 8. The risks $X_i,\ i = 1, \ldots, n$, are mutually comonotonic if one of the following conditions hold:

i) $F_{\mathbf{X}}(\mathbf{x}) = \min\left\{F_{X_i}(x_i)\right\};$

ii) There exist a random variable Z and a function $u = (u_1, \ldots, u_n)$, with non decreasing components, such that $\mathbf{X} =_d u(Z)$;

iii) $\mathbf{X} =_d \left[F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right]$, $\forall U$ uniformly distributed on $[0, 1]$.

Two distributions are comonotonic if all the attributes are ranked identically. This is a very strong requirement, because, for instance, in a distribution of income and health, we are saying that the richest person is also the most healthy.

Definition 9. X_i is said to precede X_j in the stop-loss order $(X_i \preceq_{sl} X_j)$ if and only if $E\left(X_i - d\right)_+ \leq E\left(X_j - d\right)_+$, $\forall d \geq 0$, where $E\left(X - d\right)_+ = \max\left\{0, X - d\right\}$ is the stop-loss premium.

Dhaene et al. (2000) prove that the certainty equivalent of the sum of mutually comonotonic risks is equal to the sum of the certainty equivalents of the different risks. If the risks $X_1, X_2, ..., X_n$ are mutually comonotonic, then $C_Y(\sum_{i=1}^n X_i) = \sum_{i=1}^n C_Y(X_i)$, and the stop-loss premiums are maximal.

Moreover, a condition equivalent to Definition 9 implies that $C_Y(X_i) \leq C_Y(X_j)$, for all concave functions f.

Within the framework of expected utility theory, stop-loss order is also equivalent to say that one risk is preferred over the other by all risk averse decision makers. Further, if f is concave, then C_Y is subadditive for any $X_1, X_2, ..., X_n$ and, hence,

$$C_Y(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n C_Y(X_i). \tag{44}$$

Another extension of the Yaari's certainty equivalent has been proposed by Rüschendorf (2006 a-b). He has defined the multivariate distortion type risk measures, but without the intent to apply these measures to the study of the rank dependent expected utility models. Later, Cardin and Pagani (2010) have provided different characterizations for these measures and their properties. Starting from the decumulative distribution function of the vector $\mathbf{X}$, $G_{\mathbf{X}}(\mathbf{x})$, the definition is given by

$$\Psi_f(\mathbf{X}) = \int_{R_+^n} f\left(G_{\mathbf{X}}(\mathbf{x})\right) d\omega(\mathbf{x}), \tag{45}$$

where $\omega(\mathbf{x})$ is a weighting measure.

Analogously, starting from the multivariate cumulative distribution $F_{\mathbf{X}}(\mathbf{x})$, they have

$$\bar{\Psi}_f(\mathbf{X}) = \int_{R_+^n} f\left(1 - F_{\mathbf{X}}(\mathbf{x})\right) d\omega(\mathbf{x}). \tag{46}$$

Note that, in the multivariate case, $\Psi_f(\mathbf{X})$ is not necessarily equal to $\bar{\Psi}_f(\mathbf{X})$.

Another alternative representation could be given by means of the quantiles

$$\tilde{\Psi}_f(\mathbf{X}) = \int_0^1 G_{\mathbf{X}}^{-1}(p) df\left(p\right), \tag{47}$$

where $G_{\mathbf{X}}^{-1}(p) = \omega\left(Q_p\right)$ and $Q_p = \left\{x \in R_+^n : G_{\mathbf{X}}(\mathbf{x}) > 1 - p\right\}$. The boundary of Q_p is a multivariate quantile.

Note that the function $f(p)$ gives a measure of risk aversion and, if the agent is risk neutral, then it is the distribution of a constant a. Then it follows

$$\tilde{\Psi}_f(\mathbf{X}) = f(a) \int_0^1 G_{\mathbf{X}}^{-1}(p) dp. \tag{48}$$

In this case, the weight $f(a)$ immediately recall the Lorenz measure or Yaari's equality rating of equations (25) and (27). Further, $\tilde{\Psi}_f(\mathbf{X})$ has an analogy with the argument of the welfare function W in equation (39). In particular, note the parallelism between $f(a)$ and $\theta(\mathbf{X})$.

See also Galichon and Henry (2012) for another representation of the Yaari's Dual Theory in the multidimensional framework, by means of quantiles.

5 Conclusions

In this paper, we have shown the connections of the definitions of univariate certainty equivalent in many different fields, such as efficiency, inequality and risk measures and the linkage between the certainty equivalent and the willingness to pay.

Our purpose was not to review all the interesting interpretations and applications of the certainty equivalents, but rather to recall the definitions of certainty equivalent that have laid the foundation of many measurement and analysis tools in economics, such as risk premia and inequality indices.

Further, in the multidimensional case, there is still some way to go. Concerning the definition of risk premium, we can consider a vector form, or a premium in terms only of one variable (e.g. Kihlstrom and Mirman (1974), Courbage (2001)), but both these concepts are not very used in practice. Further research on the multidimensional inequality measurement could exploit a different way to choose the certainty equivalent, not only as a one-dimensional mean vector scaling quantity. It could be interesting to understand the relationship among different scaling measures, for instance, if we choose as certainty equivalent a vector that scales different attributes in different ways.

Notes

[1]Decreasing absolute risk aversion implies $A^{'}(x) \leq 0$ and it is equivalent to $P(x) \geq A(x)$, where $P(x) = -u^{'''}(X)/u^{''}(X)$ is the degree of absolute prudence (Kimball (1990)).

[2]See Allais (1953) for the Allais paradox, that give rise to these new models

[3]If F_X is invertible, then $\bar{X}$ is simply F_X^{-1}.

[4]It means that for all cumulative distribution functions $F_{X^1}, F_{X^2}, F_{X^3}$, with the same expected value, if $\rho(F_{X^1}) = \rho(F_{X^2})$, then $\rho(\lambda F_{X^1} + (1-\lambda)F_{X^3}) = \rho(\lambda F_{X^2} + (1-\lambda)F_{X^3})$, $\forall \lambda \in (0,1)$.

[5]DARA means decreasing absolute risk averse

[6]Note that, if the relationship between the CE and π is $C_u = EX - \pi$, componentwise, as in the unidimensional case, then the proportionality between C_1 and C_2 is given by $C_1 = -\frac{u_2(\mu)}{u_1(\mu)}C_2 + \frac{u_2(\mu)}{u_1(\mu)}EX_2 + EX_1 + \frac{1}{2}\left(u_{11}(\mu)\sigma_{11} + u_{22}(\mu)\sigma_{22} + 2u_{12}(\mu)\sigma_{12}\right)$.

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