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Alberto Peretti

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Alberto Peretti
Department of Economics, University of Verona
Verona, Italy
e-mail: alberto.peretti@univr.it

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Abstract

A divisibility rule is a shorthand way of determining whether a given number is divisible by a fixed divisor without performing the division, usually by performing a simple calculation on its digits. There is no similarity among these rules, in the sense that for example the rule for testing the divisibility by 3 is very different from the rule for 7. In this working paper we present a general rule for testing divisibility by two digit integer numbers. The general rule is characterized by a pair of conditions, but in some cases just the main condition of the two is necessary and sufficient for divisibility. In the second part we investigate on this aspect and we show in general what are the cases in which just one condition is enough. Although there are divisibility tests for numbers in any base, and they are usually different, in this paper we concentrate just on the base 10.

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AMS Classification. 11A05, 11A51

JEL Classification. C60, C65

1 Introduction

From elementary school we learn that in order to know if a number is even (divisible[1] by 2) we have just to look at the last digit, the digit of units. If this is even then the number is even. In order to know if a number is divisible by 3 we have just to calculate the sum of the digits of that number and if the sum is a multiple[2] of 3 here it comes that our number is a multiple of 3. It must be pointed out that the two conditions are *if and only if* conditions. It means that they are necessary and sufficient conditions for divisibility: in other words it is also true that if the last digit is not even then the number is not even, and if the sum is not a multiple of 3 then the number is not a multiple of 3. These are probably the most famous divisibility rules.

As we have divisibility by a non prime number if and only if there is divisibility by all the primes in its factorization, one could say that divisibility by prime numbers is the real problem. There are rules for testing divisibility by all the prime numbers less than 100 and by some primes greater than 100. These rules all involve some calculation on the digits of the dividend number, and the rules are quite different from each other; divisibility by 2 involves just the last digit, while divisibility by 3 involves the already cited sum of the digits, then all the digits.

To cite another interesting case, a rule for testing divisibility by 11 is the following: compute the alternating sum of digits, namely the sum of even place digits and the sum of odd place digits, and compute the difference of them. If this difference is a multiple of 11 then the number is a multiple of 11 (this also is an if and only if condition).

It is important to remark that there are many equivalent rules for testing divisibility by the same number. Just to provide an example of the large variety of rules, in the case of 11, beside the already cited "alternating sums" rule, we have the following equivalent rules:

- Compute the difference between the two alternating sums. The result must be a

[1] An integer number a is divisible by another non null integer number b if the integer division of a by b has a null remainder.

[2] An integer number a is a multiple of another integer number b if there exists an integer n such that $a = nb$. As a particular case 0 is a multiple of any integer number, as n can be zero. From the definitions of divisibility and multiple it follows that a number a is divisible by a number b if and only if a is a multiple of b.

multiple of 11. 1287 is divisible by 11 as

$$(1+8)-(2+7)=0=0\times 11.$$

- Add the digits in blocks of two from right to left. The result must be a multiple of 11. 1287 is divisible by 11 as

$$12+87=99=9\times 11.$$

- Subtract the last digit from the rest. The result must be a multiple of 11. 1287 is divisible by 11 as

$$128-7=121=11\times 11.$$

- Add the last digit to the hundredth place (that is add 10 times the last digit to the rest). The result must be a multiple of 11. 1287 is divisible by 11 as

$$128+70=198=18\times 11.$$

- If the number of digits is even, add the first and subtract the last digit from the rest. The result must be a multiple of 11. 1287 is divisible by 11 as

$$28+1-7=22=2\times 11.$$

- If the number of digits is odd, subtract the first and last digit from the rest. The result must be a multiple of 11. 12287 is divisible by 11 as

$$228-1-7=220=20\times 11.$$

All these rules can be proved to be equivalent. Each one of them is a necessary and sufficient rule for testing the divisibility of a number by 11.
A common denominator of divisibility rules is that they generally transform a given number into a smaller number, while preserving divisibility by the divisor of interest. Therefore the resulting number should be evaluated in turn for divisibility by the same divisor. Frequently the process can be iterated until the divisibility is obvious. Here it comes out an interesting matter, that is the "speed of convergence" of the various

rules. Of course it is not a convergence in a proper sense (the process is anyway a finite process). To have convergence can be intended as to reach a step at which the divisibility is obvious. So the interesting question could be: starting from a very big number and being interested in its divisibility by some number n, and supposing a certain rule gives me a smaller number but big enough to make divisibility by n non obvious, how long will it take some further iterations of the rule to give me an obvious divisibility? Just consider the first four rules about the divisibility by 11: with the same dividend 1287 the four rules give different multiples of 11, and while the first three may be obvious, the fourth may be not. We will not consider the aspect of speed of convergence in this working paper.

As we will be concentrated in the sequel in the divisibility by two digit numbers, we give here a short summary of some divisibility rules by the prime integers smaller than 20.

✎ Divisibility by 13. Some equivalent rules are:

- ☞ Add 4 times the last digit to the rest;

  663 is a multiple of 13 as $66 + 3 \times 4 = 78$ and again $7 + 8 \times 4 = 39 = 3 \times 13$.

- ☞ Subtract 9 times the last digit from the rest;

  663 is a multiple of 13 as $66 - 3 \times 9 = 39 = 3 \times 13$.

- ☞ Form the alternating sum of blocks of three from right to left.

  11271 is a multiple of 13 as $11 - 271 = -260 = -20 \times 13$.

The first rule, for example, may be easily proved in this way: suppose we have the integer number $10a + b$, where b is the digit of the units (this writing form is unique if we say that b is a digit). The rule says to consider $a + 4b$; suppose this is a multiple of 13, namely $a + 4b = 13k$ where k is some integer number. Then $a = 13k - 4b$ and hence

$$10a + b = 10(13k - 4b) + b = 130k - 40b + b = 130k - 39b = 13(10k - 3b)$$

is a multiple of 13.

The condition is necessary too. Suppose $10a + b = 13n$ for some integer n. Then $b = 13n - 10a$ and hence

$$a + 4b = a + 4(13n - 10a) = 52n - 39a = 13(4n - 3a)$$

is a multiple of 13.

The second rule is clearly equivalent to the first, as $a - 9b = a + 4b - 13b$.

In order to prove the third rule, taking for example an up to six digits number, the number may be written as $10^3 a + b$, where a and b are integer numbers less than 10^3. The rule brings to consider $a - b$; supposing $a - b = 13k$, we have

$$10^3 a + b = 10^3 a + a - 13k = 1001a - 13k = 13 \times 77a - 13k = 13(77a - k),$$

that is a multiple of 13.

✎ Divisibility by 17. A rule is:

☞ Subtract 5 times the last digit from the rest;

867 is a multiple of 17 as $86 - 7 \times 5 = 51$ and again $5 + 1 \times 5 = 0$.

The proof is very similar to the previous case. Suppose we have $10a + b$, where b is the digit of the units. The rule says to consider $a - 5b$; suppose this is a multiple of 17, namely $a - 5b = 17k$ where k is some integer number. Then $a = 17k + 5b$ and hence

$$10a + b = 10(17k + 5b) + b = 170k + 51b = 17(10k + 3b)$$

is a multiple of 17.

The condition is necessary too. Suppose $10a + b = 17n$ for some integer n. Then $b = 17n - 10a$ and hence

$$a - 5b = a - 5(17n - 10a) = 51a - 85n = 17(3a - 5n)$$

is a multiple of 17.

✎ Divisibility by 19. A rule is:

☞ Add twice the last digit to the rest;

10013 is a multiple of 19 as $1001 + 3 \times 2 = 1007$ and $100 + 7 \times 2 = 114$ and $11 + 4 \times 2 = 19$.

Take again $10a + b$ as before. The rule says to consider $a + 2b$; suppose this is a multiple of 19, namely $a + 2b = 19k$ where k is some integer number. Then $a = 19k - 2b$ and hence

$$10a + b = 10(19k - 2b) + b = 190k - 19b = 19(10k - b)$$

is a multiple of 19.

The condition is necessary too. Suppose $10a + b = 19n$ for some integer n. Then $b = 19n - 10a$ and hence

$$a + 2b = a + 2(19n - 10a) = 38n - 19a = 19(2n - a)$$

is a multiple of 19.

Here is a general statement on divisibility.

2 A general statement

Proposition 1 *The integer number $10a+b$ is divisible by the integer $10c+d$ if and only if $ad - bc$ is equal to $k(10c + d)$ for some integer k and $a - k$ is a multiple of c.*

Proof.

Necessity. Suppose $10a + b$ is divisible by $10c + d$ and let be $10a + b = m(10c + d)$. Then

$$b = 10mc + md - 10a.$$

Hence

$$\begin{aligned} ad - bc &= ad - (10mc + md - 10a)c \\ &= ad - 10mc^2 - mcd + 10ac \\ &= a(10c + d) - mc(10c + d) \\ &= (a - mc)(10c + d) \end{aligned}$$

and this proves that $ad - bc$ is a multiple of $10c + d$. Moreover, if $k = a - mc$ is the quotient, we have

$$a - k = a - (a - mc) = mc$$

and then $a - k$ is a multiple of c. This proves necessity.

Sufficiency. Suppose that

$$ad - bc = k(10c + d) \tag{1}$$

and $a - k$ is a muliple of c.

From (1) we get

$$bc = ad - 10kc - kd$$

and again

$$b = d \times \frac{a-k}{c} - 10k$$

and by the hypothesis $a - k$ is a multiple of c this is coherent with an integer b.

Hence

$$\begin{aligned} 10a + b &= 10a + d \times \frac{a-k}{c} - 10k \\ &= \frac{1}{c}\big(10ac + ad - dk - 10kc\big) \\ &= \frac{1}{c}\Big(a(10c + d) - k(10c + d)\Big) \\ &= \frac{1}{c}(a - k)(10c + d). \end{aligned}$$

As $a - k$ is a multiple of c, this proves that $10a + b$ is divisible by $10c + d$. □

Remark. From the proof it follows that when $10a+b$ is a multiple of $10c+d$, the quotient is equal to the quotient between $a - k$ and c.

Remark. Divisibility between two numbers is reduced to divisibility by the second of a smaller number than the first. In fact

$$ad - bc \leq ad < 10a \leq 10a + b.$$

This implies that the repeated application of the rule to the numbers it generates each time does come to an end.

Remarks.

The number $ad - bc$ may be seen as the "determinant" of the table

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array}$$

where a and b are respectively the number of tens and units of the dividend and analogously c and d are the numbers of tens and units of the divisor.

The condition that $ad - bc$ is a multiple of $10c + d$ (in the sequel the "condition on the determinant" for short) is necessary for divisibility, but not sufficient on its own. An example is

$$\begin{array}{c|c} 3 & 3 \\ \hline 2 & 2 \end{array}$$

where $ad - bc = 0$ (then a multiple of 22 with $k = 0$) but 33 is not divisible by 22.

But another example is

$$\begin{array}{c|c} 12 & 1 \\ \hline 2 & 2 \end{array}$$

where $ad - bc = 22$ (the $k = 1$) and 121 is not divisible by 22 either.

2.1 Some particular cases

We have seen that in the general statement the divisibility condition is made of two conditions, the one we called the condition on the determinant and the additional one, that $a - k$ is a multiple of c. We have seen also that in general the only condition on the determinant is not enough to guarantee divisibility.

It comes out though that in some cases this condition is enough and we are going to provide examples in order to understand when the additional condition may be redundant. We will show in the last section a general result about this redundancy.

Particular cases exist in which, when the determinant is a multiple of the divisor, then it is certainly true that $a - k$ is a multiple of c, recalling that k is the quotient of $ad - bc$ divided by $10c + d$. In all these cases the only condition on the determinant is necessary and sufficient for divisibility. In other cases thing are not like that and the condition that $a - k$ is really a multiple of c has to be taken into account to have divisibility. Let's consider now some of these particular cases.

- $c = 1$.

If $c = 1$ trivially $a - k$ is divisible by c. Hence for all the integer numbers between 11 and 19 the necessary and sufficient condition for $10a + b$ is divisible by $10c + d$ is that the determinant is a multiple of $10c + d$.

- $d = 1$.

  If $a - bc = k(10c + 1)$ then $a - k = 10kc + bc = c(10k + b)$ and then $a - k$ is a multiple of c.

  Hence also for the numbers having 1 as the digit of the units it is true that $10a + b$ is divisible by $10c + 1$ if and only if the determinant $a - bc$ is a multiple of $10c + d$.

- With $d = 0$, that is with a divisor multiple of 10, the condition on the determinant is not sufficient by its own. Just consider the division of 30 by 20.

- Some other particular cases are interesting, in which the additional condition that $a - k$ is a multiple of c is certainly verified and others where this condition is not verified instead.

  In order to get conclusions in each case some different equivalent ways exist. We now use a method that considers the possible values of b and k. In the sequel, by giving the general statement, we actually provide a general method to prove all the results.

  ☞ *Divisibility by 23.* In the division by 23 the condition on the determinant is sufficient. In fact this condition is

  $$\begin{array}{c|c} a & b \\ \hline 2 & 3 \end{array} = 3a - 2b = 23k \quad \text{that is} \quad 3a = 23k + 2b.$$ [3]

  Now, if k is an even number, then necessarily a is even, and then $a - k$ is even. hence $a - k$ is divisible by c, because $c = 2$. If k is odd instead, then a is necessarily odd too, and then $a - k$ is even again.

[3] Now we are going to get some consequences from that. In order to avoid the error of getting consequences from a wrong condition, it has to be said that we are supposing the equation is possible for appropriate values of a, b, k, and in fact for example it is trivially true with $10a + b = 23$, that is with $a = 2$, $b = 3$ and $k = 0$. The same remark holds also for all the following cases.

☞ *Divisibility by 25, 27, 29.* With the same arguments we have used with 23 the condition on the determinant is sufficient for divisibility by 25, 27 and 29 also. In the case of 25 we can use an equivalent way, that we shall use in the following for some more complicated case. Let's state a lemma first.

Lemma 1 *If an integer number B is divisible by b then $A+B$ is divisible by b if and only if A is divisible by b.*

Proof. Let be $B = kb$.

If $A+B = mb$, then we can write $A+kb = mb$, so $A = (m-k)b$ and then $A+B$ is divisible by b. On the contrary, if $A = mb$, then $A+B = mb+kb = (m+k)b$ and then $A+B$ is divisible by b. □

Let's consider divisibility by 25. The condition on the determinant is

$$\begin{array}{c|c} a & b \\ \hline 2 & 5 \end{array} = 5a - 2b = 25k \quad \text{that is} \quad 5a = 25k + 2b.$$

Because of the lemma, being $25k$ divisible by 5, then $25k + 2b$ is divisible by 5 if and only if either $b = 0$ or $b = 5$. If $b = 0$ we have $a = 5k$ and then $a - k = 4k$ is divisible by 2. On the contrary, if $b = 5$ we have $a = 5k + 2$ and again $a - k = 4k + 2$ is divisible by 2.

☞ *Divisibility by 22.* Things are different in the division by 22, as

$$\begin{array}{c|c} a & b \\ \hline 2 & 2 \end{array} = 2a - 2b = 22k \quad \text{that is} \quad 2a = 22k + 2b \quad \text{or} \quad a = 11k + b.$$

From that we may get $a-k = 10k+b$, but this integer number is not necessarily divisible by 2 (and the values $a = 3$, $b = 3$, $k = 0$ or $a = 12$, $b = 1$, $k = 1$ give the two counterexamples we have seen before showing that in the 22 case the only condition on the determinant is not sufficient). The same with the other even numbers with 2 tens.

☞ *Divisibility by a multilple of 11.* Trivially, for all the multiples of 11 the condition on the determinant alone is not sufficient. Just consider that, with $c \geq 2$,

$$\begin{array}{c|c} c-1 & c-1 \\ \hline c & c \end{array} = 0 \quad \text{but} \quad 10(c-1)+c-1 = 11(c-1) \quad \text{is not divisible by } 11c.$$

☞ *Divisibility by 24, 26, 28.* With the same arguments we have used in the 22 case it is easy to see that also for divisibility by 24, 26, 28 the condition on the determinant alone is not sufficient.

☞ *Divisibility by 32.* In the division by 32 the condition on the determinant is sufficient. In fact the condition is

$$\begin{array}{c|c} a & b \\ \hline 3 & 2 \end{array} = 2a - 3b = 32k \quad \text{that is} \quad 2a = 32k + 3b.$$

From the lemma we get that b must be even, and let's say $b = 2b'$. Then

$$a = 16k + 3b' \quad ; \quad a - k = 15k + 3b' \quad ; \quad a - k = 3(5k + b')$$

and then $a - k$ is divisible by 3.

☞ *Divisibility by 34.* Also in the division by 34 the condition on the determinant is sufficient. In fact the condition is

$$\begin{array}{c|c} a & b \\ \hline 3 & 4 \end{array} = 4a - 3b = 34k \quad \text{and then} \quad 4a = 34k + 3b.$$

We can get that b must be even, and suppose $b = 2b'$. Then the condition becomes

$$4a = 34k + 6b' \quad ; \quad 2a = 17k + 3b'.$$

We may have both the possibilities here (k and b' may be either even or odd). If k and b' are both even, and let be $k = 2k'$ and $b' = 2b''$, we have

$$2a = 34k' + 6b'' \quad ; \quad a = 17k' + 3b''$$

and then

$$a - k = 17k' + 3b'' - 2k' = 15k' + 3b'' = 3(5k' + b'') \text{ is a multiple of 3.}$$

If k and b' are both odd, and let be $k = 2k' + 1$ and $b' = 2b'' + 1$, we have

$$2a = 17(2k'+1)+3(2b''+1) \quad ; \quad 2a = 34k'+17+6b''+3 \quad ; \quad a = 17k'+3b''+10$$

and then

$$a-k = 17k'+3b''+10-2k'-1 = 15k'+3b''+9 = 3(5k'+b''+3) \text{ is a multiple of 3.}$$

☞ *Divisibility by 35.* Also in the division by 35 the condition on the determinant is sufficient.

Now the condition is

$$\begin{array}{c|c} a & b \\ \hline 3 & 5 \end{array} = 5a - 3b = 35k \quad \text{that is} \quad 5a = 35k + 3b.$$

From the lemma, the right hand side is divisible by 5 if and only if $3b$ is divisible by 5, which means if and only if either $b = 0$ or $b = 5$. If $b = 0$ the equation becomes $5a = 35k$, that is $a = 7k$ and then $a - k = 6k$ is divisible by 3. If otherwise $b = 5$ the condition becomes

$$5a = 35k + 15 \quad ; \quad a = 7k + 3 \quad ; \quad a - k = 6k + 3 = 3(2k + 1),$$

that is divisible by 3.

☞ *Divisibility by 36.* In the division by 36 the condition on the determinant alone is not sufficient. A possible counterexample is

$$\begin{array}{c|c} 13 & 2 \\ \hline 3 & 6 \end{array} = 72 \quad \text{that is divisible by 36, with } k = 2,$$

but

$$a - k = 13 - 2 = 11, \text{ is not divisible by 3.}$$

And in fact 132 is not a multiple of 36.

☞ *Divisibility by 37 and 38.* In the division by both 37 and 38 the condition on the determinant is sufficient again.

The two conditions are respectively

$$\begin{array}{c|c} a & b \\ \hline 3 & 7 \end{array} = 7a - 3b = 37k \quad \text{that is} \quad 7a = 37k + 3b$$

and

$$\begin{array}{c|c} a & b \\ \hline 3 & 8 \end{array} = 8a - 3b = 38k \quad \text{that is} \quad 8a = 38k + 3b.$$

Here it is not easy to apply the same techniques we have used so far, as the two conditions may be verified with a large variety of possible cases (k and b both even, both odd but also one even and odd the other). For these two cases we shall see the proof in the general result, that comes shortly.

☞ *Divisibility by 39.* In order to conclude the analysis of these particular cases, for the division by 39 again the condition on the determinant is not sufficient, and the additional condition is necessary. A counterexample is given by

$$\begin{array}{c|c} 5 & 2 \\ \hline 3 & 9 \end{array} = 39 \quad \text{that is divisible by 39, with } k = 1,$$

but

$$a - k = 5 - 1 = 4, \text{ is not divisible by 3.}$$

And in fact 52 is not a multiple of 39.

2.2 About the sufficiency of the main condition

In the particular cases we have considered in the previous subsection it appears that the condition on the determinant is sufficient by itself to show divisibility in all and just those cases in which the two digits of the divisor are coprime integers.[4] We show the proof in a while.

Let's see first an example of the proof in a case that was difficult before, the one of divisibility by 37. The condition on the determinant is

$$\begin{array}{c|c} a & b \\ \hline 3 & 7 \end{array} = 7a - 3b = 37k \quad \text{that is} \quad 7a = 37k + 3b.$$

If the identity holds the right hand side is divisible by 7 and then

$$a = \frac{37k + 3b}{7},$$

that gives

$$a - k = \frac{37k + 3b}{7} - k = \frac{37k + 3b - 7k}{7} = \frac{30k + 3b}{7} = 3 \times \frac{10k + b}{7}.$$

By observing that necessarily $10k + b$ is divisible by 7 (3 and 7 are coprime), we have that the fraction is an integer number and then $a - k$ is divisible by 3.

We introduce a lemma before we give the proof of the general result.

[4]Two integers are said to be coprime (also relatively prime or mutually prime) if the only positive integer that evenly divides both of them is 1. That is, the only common positive factor of the two numbers is 1.

Lemma 2 *If an integer number $A \times B$ is divisible by p and A and p are coprime, then B is divisible by p.*

Proof. By the hypothesis we have that $A \times B = m \times p$ for some integer m. The fact that A and p are coprime may be expressed by saying that in the decomposition of A in prime factors there are no divisors of p. Hence in the decomposition of $A \times B$ in prime factors all the prime factors of p have to be in B and then B is divisible by p, as we wanted. $\square$

Proposition 2 *If $ad - bc$ is a multiple of $10c + d$ and if the digits c, d of the divisor are coprime, then $10a + b$ is divisible by the integer number $10c + d$.*

Proof. It is a generalization of the arguments we have just used in the 37 example. By hypothesis we may write

$$ad - bc = k(10c + d), \text{ and then } ad = k(10c + d) + bc$$

If the identity holds the right hand side is divisible by d and then

$$a = \frac{k(10c + d) + bc}{d},$$

that gives

$$a - k = \frac{k(10c + d) + bc}{d} - k = \frac{10kc + kd + bc - kd}{d} = \frac{10kc + bc}{d} = c \times \frac{10k + b}{d}.$$

Now from the lemma, being c and d coprime, we necessarily have that $10k + b$ is divisible by d. Hence the fraction is an integer number and then $a - k$ is divisible by c. $\square$

Remark. As a consequence of Proposition 2 the divisibility by a prime number is equivalent to the condition on the determinant alone.

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