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Tricks for maximizing collective utility functions

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Abstract This paper compares different centralized collective models for maximizing the utility of a household from the point of view of the single members that compose it. We describe the collective maximization program as a vector optimization problem and analyze the household welfare and Pareto specification as special scalar cases. The comparisons of these programs reveals interesting relationships among the Lagrangian multiplier and the sharing rule parameters. We also generalize a multiobjective dual problem that leads to the characterization of the individual shares of total income. Hence, we have a measure of the distribution of wealth inside the household.

Keywords: Collective household model, Vector Optimization Problems, Scalarization, Vector Duality.

JEL Classification: C61, C30, D11, D13.

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1 Introduction

We compare different centralized models to solve the utility maximization problem of a collective household from the point of view of the single members that compose it. New approaches have been studied either assuming cooperation or noncooperation among the members of the household [2, 4, 5, 6] . This study analyzes the collective household model, initially, as a general vector optimization problem with many objective functions, and, then, specifying collective household preferences with a unique utility function, obtained as a weighted Bergsonian welfare function, or as a Pareto-constraint specification, known also as ε-constraint method (see [7] and [10]), or as a generalized household welfare function. These approaches are all non-unitary and, hence, they avoid the hypothesis of income pooling.

The main contribution of this study consists in the generalization of the household collective optimization problem, both in its primal and dual form.

We describe the collective maximization program as a vector optimization problem and analyze the household welfare and Pareto specification as special scalar cases. The comparisons of these programs reveals interesting relationships among the Lagrangian multipliers, the Pareto weights and the so-called bargaining weights.

The dual vector program, for a collective consumption problem as in [1], is obtained from the dual linear scheme proposed by Isermann [8], and we extend it to the homogeneous case. There exist many formulations of vector optimization dual problem, but it is difficult to define a nonlinear vector dual useful to intepret the possible applications. Our proposal characterizes a vector minimization dual problem, where the objective functions are given by the share of total income allocated to each household member.

2 Vector optimization model

Consider the following collective consumption model:

$$P: \qquad \max_{\mathbb{R}^\ell_+\setminus\{0\}} U(x) = (U^1(x), \dots, U^\ell(x)) \ \text{ subject to } \ x \in K = \{x \in X : p^T x = y\}, \quad (2.1)$$

where $U = (U^1, \dots, U^\ell)$ is a vector, or multicriteria, function, defined by $U : X \subseteq \mathbb{R}^N \to \mathbb{R}^\ell$, with $U^i(x)$ representing the utility of member i and having the usual properties of continuity, positive monotonicity and at least quasi-concavity; $X \subseteq \mathbb{R}^N_+$ is the set of consumption goods, where $x = (q_1^1, \dots, q_n^1, q_1^2, \dots, q_n^2, \dots, q_1^\ell, \dots, q_n^\ell, Q_1, \dots, Q_m)$ is the allocation vector, with $x \in \mathbb{R}_+^N$, $N = \ell \times n + m$ and where $q_j^i,\ i = 1, \dots, \ell,\ j = 1, \dots, n$ is the demand of individual ith for the jth private good and $Q_k,\ k = 1, \dots, m$ is the kth public good; $K \subseteq \mathbb{R}_+^N$ is the budget set, where $p \in \mathbb{R}_+^N$ is the vector of prices and $y \in R_+$ is the budget of the household, $\max_{\mathbb{R}_+^\ell \setminus \{0\}}$ denotes the maximum made with respect to the cone $\mathbb{R}_+^\ell \setminus \{0\}$, and we consider $\max_{\mathbb{R}_+ \setminus \{0\}} = \max$. Preferences may be either egotistic or not; this fact does not alter the matter of the paper.

Definition 2.1. An allocation vector $\bar{x} = (\bar{q}^1, \bar{q}^2, \dots, \bar{q}^\ell, \bar{Q}) \in K$ is a vector Pareto optimum for problem P, with respect to the positive orthant $\mathbb{R}_+^\ell \setminus \{0\}$, if there is no $x \in K$ such that $U^j(x) > U^j(\bar{x})$, for some j, and $U^i(x) \geq U^i(\bar{x})$, for every $i = 1, \dots, \ell$.

The set of all Pareto efficient allocations is known as the contract curve, that is the set of points that it is possible for the agents to block the trade. Each point on the contract curve is such that any other reallocation could not make one of the person better-off, without making the other person worse-off.

In what follows, we assume that vector optimum points exist. Each agent follows his/her own preferences, but he/she also respects the needs of the other members, because each one is interested in sharing the budget.

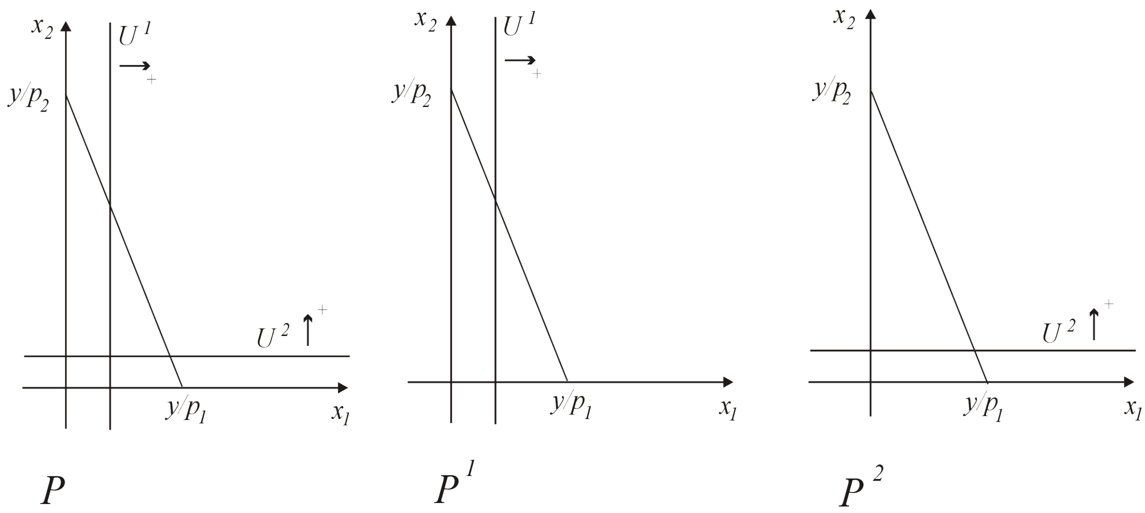
The assumption that all family members have a single set of preferences prevents analysis of conflict or inequality inside the household. As we will stress in Section 6 the inequalities can be of two types: about the allocation of money income and about bargaining power.

Let us now consider an illustrative linear example, with two agents and two goods explaining the difference between problem P and the following scalar problems:

$$P^1 : \ \max\ U^1(x) \ \text{ subject to } \ x \in K, \text{ and } P^2 : \ \max\ U^2(x) \ \text{ subject to } \ x \in K.$$

Example 2.1. Let $U(x) = (ax_1, bx_2),\ a, b > 0$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : p_1x_1 + p_2x_2 = y\}$. Thus, we have $\ell = 2, n = 2, m = 0$. Problems P^1 and P^2 reach the optimum in $(y/p_1, 0)$ and $(0, y/p_2)$, respectively, while, in problem P, all the points, belonging to the segment $p_1x_1 + p_2x_2 = y$, with $x^i \geq 0$, $\forall i = 1, 2$, are Pareto efficient, and hence, are optima of the problem. Problems P^1 and P^2 describe a degenerate situation, where only one subject, within the family, holds the power.

In what follows, we represent the budget set $K \subseteq \mathbb{R}^2$ and the level sets of the functions $U^i(x)$.



Remark 2.1. We observe that if there is concordance among the objective functions, so that they all move in the same direction, then there exists a unique solution where every objective function attains its optimum and there is no need of any particular scalarizing technique to reach the maximum. Otherwise, with conflicting goals, it is admissible that a single solution does not exist and we are dealing with an efficient frontier or an efficient set of choices.

Now we give a necessary optimality condition for a Pareto optimum. For the proof of this result, see, for instance, [13].

Proposition 2.1. Let $U^1, \ldots, U^\ell$ be at least once differentiable functions on X and let $\bar{x} \in X$ be a Pareto optimum for U on K. Then there exists $\ell + 1$ scalars α_i, with $i = 1, \ldots, \ell$, and λ, not all zero, such that $\alpha_i \geq 0, \forall i = 1, \ldots, \ell$ and

$$\sum_{i=1}^{\ell} \alpha_i U'^i(\bar{x}) - \lambda p = 0,$$

where $U'^i(\bar{x})$ is the gradient vector of U^i evaluated in $\bar{x}$.

3 Bergson representation

The weighted scalar problem is given by:

$$P_\mu : \quad \max \sum_{i=1}^{\ell} \mu_i U^i(x) \quad \text{subject to} \quad x \in K, \tag{3.1}$$

with $\mu_i \geq 0,\ \forall i,\ \sum_{i=1}^{\ell} \mu_i = 1$. Household utility is a weighted sum of individual utilities. The weight μ_i, for each member i, can be interpreted as the bargaining power that the individual has within the household.

This specification adopts a Bergsonian welfare function, that is $W^B(x) = \sum_{i=1}^{\ell} \mu_i U^i(x)$, but it can also be written as a log-transformation of a Nash welfare function $W^N(x) = \prod_{i=1}^{\ell} [U^i(x)]^{\mu_i}$ and, then, we obtain $\log W^N(x) = \sum_{i=1}^{\ell} \mu_i \log U^i(x)$.

Remark 3.1. The weighting vector μ could be thought as the bargaining rule $\mu = \mu(y, p, s)$, where $s \in \mathbb{R}^t$ are distribution factors, i.e. variables that affect the decision process, but not the preferences or the constraints. When we maximize with respect to $x \in \mathbb{R}^N_+$, in the partial derivatives μ is constant.

Let us introduce the following example.

Example 3.1. Starting from the same data of Example 2.1, problem P_μ is: max $\mu_1(ax_1) + (1 - \mu_1)(bx_2)$ subject to $p_1x_1 + p_2x_2 = y$. If we choose $0 < \mu_1 < \frac{bp_1}{ap_2 + bp_1}$, which is the value that equals the slope of the objective function and the slope of the constraint, we obtain an objective function that reaches the optimum in $(0, y/p_2)$. Otherwise, if $\frac{bp_1}{ap_2+bp_1} < \mu_1 < 1$ the optimum is $(y/p_1, 0)$. If $\mu_1 = \frac{bp_1}{ap_2+bp_1}$, we obtain all the points of the efficient frontier of P.

Now, we present some classical results, but the interested reader may refer to [10, 7] for the proofs of the results that follow.

Proposition 3.1. A maximum point of problem P_μ is Pareto optimal if the weighting coefficients are positive, that is $\mu_i > 0,\ \forall i = 1, \ldots, \ell$.

Proposition 3.2. If a unique maximum point exists for problem P_μ, then it is Pareto optimal.

Let us call, from now on, a problem convex, if all the objective functions are concave and the constraints are convex functions.

Proposition 3.3. Let P be convex. If $\bar{x} \in K$ is Pareto optimal, then there exists a weighting vector μ such that $\mu_i \geq 0,\ i = 1, \ldots, \ell,\ \sum_{i=1}^{\ell} \mu_i = 1$ and $\bar{x}$ is a maximum point of P_μ.

Note that the solution of the weighting method is Pareto optimal if we choose all positive coefficients, i.e., a positive bargaining rule, or if there is a unique solution. Without any further assumption, hence, the power sharing among the members produces a consistent agreement. Besides, if we suppose that problem P is convex, the Pareto optimality condition is sufficient for the existence of a normalized vector μ.

4 Pareto representation

Another scalar representation is the following Pareto-constraint problem.

$$P^j(\bar{u}): \quad \max\ U^j(x) \ \text{ subject to } \ x \in K, U^i(x) \geq \bar{u}^i, \forall i \neq j, \tag{4.1}$$

where the utility functions of all household members, but jth one, are involved in the feasible region, because it has to be assured at least a level $\bar{u}^i$ of utility for individual i. More precisely, $\bar{u}^i$ could be interpreted as the utility that member i would enjoy if there is no bargaining agreement, or if the subjects do not live together, and hence, it is a threat point. Any point along the Pareto frontier can be achieved by the change of the threat point. This scalar model has only one objective function, the utility of agent j, but it takes into consideration all the other members, since their utilities define the feasible region.

Proposition 4.1. A vector $\bar{x} \in K$ is Pareto optimal if and only if it is a maximum point of all problems P^j, $\forall j = 1, \ldots, \ell$, at $\bar{u}^i = U^i(\bar{x})$ for $i = 1, \ldots, \ell, \quad i \neq j$.

Therefore it is possible to find every Pareto optimal solution to P by this method. Further results are the following ones.

Proposition 4.2. A point $\bar{x} \in K$ is Pareto optimal if it is the unique maximum point of a Pareto-constraint problem for some j with $\bar{u}^i = U^i(\bar{x})$ for $i = 1, \ldots, \ell, \quad i \neq j$.

Proposition 4.3. The unique maximum point of the jth Pareto-constraint problem, i.e. $P^j(\bar{u})$, is Pareto optimal for any given upper bound vector $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^{j-1}, \bar{u}^{j+1}, \ldots, \bar{u}^\ell)$.

From Propositions 4.1, 4.2 and 4.3, to ensure that a solution given by problem $P^j(\bar{u})$ is Pareto optimal, we have either to solve ℓ different problems or to obtain a unique solution. These results can be interpreted as follows: a Pareto optimum results from ℓ problems where all the members of the household propose the same consumption vector, or at least one member, maximizing his/her own utility, bound by the level of the utilities of the other members, reaches a unique optimum.

Example 4.1. Consider once again Example 2.1. Let us start from an optimum point of P, such as $(y/p_1, 0)$. We derive the problems $P^1(\bar{u}): \quad \max\ ax_1 \ \text{ subject to } \ p_1x_1 + p_2x_2 = y, bx_2 \geq 0$ and $P^2(\bar{u}): \quad \max\ bx_2 \ \text{ subject to } \ p_1x_1 + p_2x_2 = y, ax_1 \geq ay/p_1$, which have the unique solution $(y/p_1, 0)$. If we start from a non optimal solution, for instance $(0, 0)$, we have $P^1(\bar{u}):$

max ax_1 subject to $p_1x_1+p_2x_2 = y, bx_2 \geq 0$ and $P^2(\bar{u})$: max bx_2 subject to $p_1x_1+p_2x_2 = y, ax_1 \geq 0$. The solution of $P^1(\bar{u})$ and $P^2(\bar{u})$ is not the same, but it is unique for each problem. In fact, they are optimal points.

Proposition 4.4. Let $\bar{x} \in K$ be a maximum point of P_μ and $\mu_i \geq 0, \ \forall i = 1, \ldots, \ell$. The following assertions hold.

i) if $\mu_j > 0$, then $\bar{x}$ is a solution to $P^j(\bar{u})$ for $\bar{u}^i = U^i(\bar{x})$, with $i = 1, \ldots, \ell, \ i \neq j$;

ii) if $\bar{x}$ is the unique solution to P_μ, then $\bar{x}$ is a solution to $P^j(\bar{u})$, $\forall j = 1, \ldots, \ell$, for $\bar{u}^i = U^i(\bar{x})$, with $i = 1, \ldots, \ell, \ i \neq j$.

Proposition 4.4 i) states that, if the sharing rule gives a positive power to the jth subject, then he/she reaches its maximum utility, as if he/she was facing the problem individually, considering the bound of the utilities of the other members; while ii) tells us that the unique solution of a weighting problem also is the optimum for all the Pareto-contraint problems, where the individuals are maximizing their own utility, given the threat point of the others.

Further, the following proposition holds [3].

Proposition 4.5. If P is convex and $\bar{x} \in K$ is a maximum point of $P^j(\bar{u})$ and $\bar{u}^i = U^i(\bar{x})$, with $i = 1, \ldots, \ell, \ i \neq j$, then there exists a weighting vector $\mu \in \mathbb{R}^\ell$ with $\mu_i \geq 0, \ \forall i = 1, \ldots, \ell$, and $\sum_{i=1}^{\ell} \mu_i = 1$, such that $\bar{x}$ is also a maximum point of P_μ.

5 Functional representation

The functional representation can be used to consider the cases in which we refer to a social welfare function, in order to get a unitary approach, but always in a collective framework.
The weighting method may be regarded as a special case of this approach, i.e. when W is a linear function.

Definition 5.1. A function $W = W(U)$, with $W : \mathbb{R}^\ell \to \mathbb{R}$, representing the preferences of the decision maker among the objective vectors, is called social welfare function.

We recall also some useful notions about monotonicity for vector-valued functions.

Definition 5.2. A function $W : \mathbb{R}^\ell \to \mathbb{R}$ is increasing (decreasing) if for $z^1, z^2 \in \mathbb{R}^\ell$, such that $z^1_j \geq z^2_j$, $\forall j = 1, \ldots, \ell$, imply $W(z^1) \geq W(z^2)$ $(W(z^1) \leq W(z^2))$.

Definition 5.3. A function $W : \mathbb{R}^\ell \to \mathbb{R}$ is strictly increasing (strictly decreasing) if for $z^1, z^2 \in \mathbb{R}^\ell$, such that $z_j^1 > z_j^2$, $\forall j = 1, \ldots, \ell$, imply $W(z^1) > W(z^2)$ $(W(z^1) < W(z^2))$.

Definition 5.4. A function $W : \mathbb{R}^\ell \to \mathbb{R}$ is strongly increasing (strongly decreasing) if for $z^1, z^2 \in \mathbb{R}^\ell$, such that $z_j^1 \geq z_j^2$, $\forall j = 1, \ldots, \ell$ and $z_i^1 > z_i^2$, for some i, imply $W(z^1) > W(z^2)$ $(W(z^1) < W(z^2))$.

The scalar problem that we obtain is the following:

$$P_W : \quad \max\ W(U(x)) \ \text{ subject to } \quad x \in K. \tag{5.1}$$

The scalar objective function, given by $W(U(x))$, provides a complete ordering in the objective space.

Proposition 5.1. The solution of P_W is Pareto optimal if the function W is strongly increasing.

Proof. Let $\bar{z} \in X$ be a maximum of a strongly increasing function W, but let us assume that $\bar{z}$ is not Pareto optimal. Then there exists a vector $z \in X$ such that $z_i \geq \bar{z}_i$, $\forall i = 1, \ldots, \ell$ and $z_j > \bar{z}_j$ for at least one j. Since W is strongly increasing $W(z) > W(\bar{z})$ and W does not attain its maximum at $\bar{z}$. But this is a contraddiction. □

This approach is very powerful, particularly when the model needs a kind of cardinalization among the utility functions. From Proposition 5.1, we know that the optimal solution is Pareto optimal if the chosen functional is strongly increasing. Therefore, it is not necessary to specify the cardinalization. What is important is the strong positive monotonicity property of W. Nevertheless, we know that it is not always necessary to mathematically define a social welfare function, because we do not need to admit the existence of a social observer with the responsibility of choosing a specific cardinalization. Thus, the invariance of Pareto optimality, with respect to the choice of one type of cardinalization, is a remarkable result.

Now, given a social welfare function W, we can define the marginal rate of substitution between U^i and U^j as follows.

Definition 5.5. A marginal rate of substitution $m_{ij} = m_{ij}(\bar{x})$, between utilities U^i and U^j, represents the preferences of the decision maker at a decision vector $\bar{x} \in K$. If the partial derivatives exist, then $m_{ij}(\bar{x}) = \frac{\partial W(\bar{x})}{\partial U_i} / \frac{\partial W(\bar{x})}{\partial U_j}$, with $\frac{\partial W(\bar{x})}{\partial U_j} \neq 0$.

The rate m_{ij} is the decrease in the value of the objective function U^i that compensates the decision maker for the one-unit increment in the value of the objective function U_j, while the values of all the other objectives remain unaltered.

Alternatively, we can think about the notion of a partial trade-off rate as stated in the following definition.

Definition 5.6. Let the utility functions be continuously differentiable at an allocation vector $\bar{x} \in K$. Then, we can define a partial trade-off rate at $\bar{x}$ in this way: $\gamma_{ij} = \gamma_{ij}(\bar{x}) = \partial U_j(\bar{x})/\partial U_i$.

If we consider an increasing value function W, we obtain that $\gamma_{ij} = m_{ij}$. The weighting coefficients of problem P_μ are indeed the marginal rate of substitution, that is $m_{ij} = \mu_i/\mu_j$. They are constant for every solution if W is linear.

6 Comparisons of vector and scalar representations

We know that the mathematical formulations, used to study the household decision process, generate a demand function $x = x(p, y)$, which has to be the same through the different scalarization choices. Therefore, we compare the various approaches and we give the conditions that guarantee the invariance of the demand. Let us start from considering the Lagrangian functions of problems $P_\mu, P^j(\bar{u}), P_W$, which are respectively:

$$L_\mu(x, \lambda_\mu) = \sum_{i=1}^{\ell} \mu_i U^i(x) - \lambda_\mu(p^T x - y),$$

$$L^j(x, \lambda^j, \lambda_i^j) = U^j(x) - \lambda^j(p^T x - y) + \sum_{i \neq j} \lambda_i^j(U^i(x) - \bar{u}^i),$$

$$L_W(x, \lambda_W) = W(U(x)) - \lambda_W(p^T x - y),$$

and their corresponding systems of necessary conditions:

$$S_\mu = \begin{cases} \sum_{i=1}^{\ell} \mu_i U_1'^i(x) - \lambda_\mu p_1 = 0 \\ \vdots \\ \sum_{i=N}^{\ell} \mu_i U_N'^i(x) - \lambda_\mu p_N = 0 \\ p^T x - y = 0 \end{cases} \qquad S^j = \begin{cases} U_1'^j(x) - \lambda^j p_1 + \sum_{i \neq j} \lambda_i^j U_1'^i(x) = 0 \\ \vdots \\ U_N'^j(x) - \lambda^j p_N + \sum_{i \neq j} \lambda_i^j U_N'^i(x) = 0 \\ p^T x - y = 0 \\ \lambda_i^j (U^i(x) - \bar{u}^i) = 0, \ \forall i \neq j \\ \lambda_i^j \geq 0, \ \forall i \neq j \\ U^i(x) - \bar{u}^i \geq 0, \ \forall i \neq j \end{cases}$$

$$S_W = \begin{cases} \sum_{i=1}^{\ell} \frac{\partial W(U(x))}{\partial U^i} U_1'^i(x) - \lambda_W p_1 = 0 \\ \vdots \\ \sum_{i=1}^{\ell} \frac{\partial W(U(x))}{\partial U^i} U_N'^i(x) - \lambda_W p_N = 0 \\ p^T x - y = 0 \end{cases} .$$

Now, we want to compare P_μ and $P^j(\bar{u})$. Let us consider Proposition 4.4. If we have the optimal demand $\bar{x}$ of P_μ and $\mu_i > 0, \ i = 1, \dots, \ell$, then that demand is optimal for $P^j(\bar{u})$, when $\bar{u}^i = U^i(\bar{x})$, with $i \neq j$. In this case, we will have $\lambda_i^j = \mu_i/\mu_j$ with $\lambda_i^j > 0$ and $\lambda_\mu = \lambda^j$ and the last three conditions of S^j are always satisfied.

From Proposition 4.5, if the problem is convex, then the viceversa is also true.
We interpret the multipliers as follows: λ_μ is the marginal change in social welfare arising from a marginal increase in income, when the social welfare function is linear; λ^j is a marginal change in the utility of agent j with respect to a change in wealth, when the bargaining power is left to agent j (i.e. $\mu_j = 1$) and λ_i^j describes how change the utility of member j, caused by a change in the utility of member i, thus it is a marginal rate of substitution as γ_{ij} of Definition 5.6. Further, the weight μ_i shows how the social welfare varies with respect to a variation of the utility of the ith subject.

Let us now analyze P_μ and P_W. We note that if there exists a solution $\bar{x}$ of system S_W, then there exists also a vector $\mu \in \mathbb{R}_{++}^{\ell}$, with $\mu_i = \partial W(U(\bar{x}))/\partial U^i$, and $\lambda_W = \lambda_\mu$, that satisfy the conditions of S_μ (and of S^j), for the same demand functions. On the contrary, if we have the demand functions of P_μ, we already have a function W that well performs for P_W (i.e. a linear approximation), but there may exist other kind of functions that serve this purpose.

Moreover, if we call $\tilde{\mu}_i = \partial W(U(\bar{x}))/\partial U^i$ the weight when the function W is not linear, then $\tilde{\mu}_i$ is no longer a constant, but it is a function of (p, y), because it varies with respect to the demand function $x(p,y)$. Therefore, one can pass through P_μ, $P^j(\bar{u})$ and P_W making the proper substitutions.

Finally, we consider problem P. From Proposition 2.1, we can compare the demand of the previous problems with the demand of the initial vector optimization problem.

Since $\bar{x}$ is a Pareto optimum of U, we have the following Lagrangian function $L(x,\lambda) = \alpha^j U^j(x) + \sum_{i\neq j} \alpha^i (U^i(x) - U^i(\bar{x})) - \lambda(p^T x - y)$ and the resulting necessary conditions:

$$S = \begin{cases} \alpha^j U_1'^j(x) + \sum_{i\neq j} \alpha^i U_1'^i(x) - \lambda p_1 = 0 \\ \vdots \\ \alpha^j U_N'^j(x) + \sum_{i\neq j} \alpha^i U_N'^i(x) - \lambda p_N = 0 \\ p^T x - y = 0 \\ \alpha^i (U^i(x) - U^i(\bar{x})) = 0, \ \forall i \neq j \\ \alpha^i \geq 0, \ \forall i \neq j, \ \lambda \geq 0 \\ U^i(x) - U^i(\bar{x}) \geq 0, \ \forall i \neq j \end{cases}.$$

Inspecting the necessary conditions S^j and S reveals that they are equal, for $\bar{u}^i = U^i(\bar{x})$, $\lambda_i^j = \alpha^i, \forall i \neq j$ and $\alpha^j = 1$. Further, all problems P, P_μ, $P^j(\bar{u})$ and P_W are comparable crossing the observations done.

Remark 6.1. We note that, if Proposition 2.1 holds, then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, with $\alpha_i \geq 0$, $\forall i = 1, \ldots, \ell$, and we can always derive a vector $\mu = (\mu_1, \ldots, \mu_\ell)$, with $\mu_i \geq 0$, $\forall i = 1, \ldots, \ell$, in this way: $\mu_i = \frac{\alpha_i}{\sum_{i=1}^{\ell} \alpha_i}$.

7 Vector collective duality

We propose a more general approach to the dual problem in a scalar dimension, which exploits the Lagrangian function of a vector optimization problem. Let us consider a vector Lagrangian function: $L_v(x,\lambda) : \mathbb{R}^{N+\ell} \to \mathbb{R}^\ell$, defined by:

$$L_v(x,\lambda) = \begin{pmatrix} U^1(x) - \lambda_1(p^T x - y) \\ \vdots \\ U^\ell(x) - \lambda_\ell(p^T x - y) \end{pmatrix}$$

with $\lambda \in \mathbb{R}_+^\ell$ and $p^T x - y \in \mathbb{R}_-$, such that $L_v(x,\lambda) \in \mathbb{R}^\ell$. We recall that by the Walras' law the budget is fully spent, then $p^T x - y = 0$.

We now define the dual convex multicriteria optimization problem: $\mathrm{Min}_{\mathbb{R}^\ell_+\setminus\{0\}}\Phi(\lambda)$, subject to $\lambda \in \mathbb{R}^\ell_+ \setminus \{0\}$, where $\Phi(\lambda) = \max_{\mathbb{R}^\ell_+\setminus\{0\}} L_v(x,\lambda)$ subject to $x \in X$ and $\mathrm{Min}_{\mathbb{R}^\ell_+\setminus\{0\}}$ is referred to the minimum of a set-valued function, with respect to the cone $\mathbb{R}^\ell_+ \setminus \{0\}$. Therefore, the multicriteria dual problem is:

$$\mathrm{Min}_{\mathbb{R}^\ell_+\setminus\{0\}}\max_{\mathbb{R}^\ell_+\setminus\{0\}}[U(x) - \lambda(p^T x - y)]. \tag{7.1}$$

If we suppose that the utility functions $U^i(x)$ are homogeneous of degree one, and apply Euler's equation to (7.1), we obtain:

$$\mathrm{Min}_{\mathbb{R}^\ell_+\setminus\{0\}}\max_{\mathbb{R}^\ell_+\setminus\{0\}}[U'(x)x - \lambda(p^T x - y)], \tag{7.2}$$

where $U'(x) \in \mathbb{R}^{\ell\times N}$ is the Jacobian matrix of U.

Now, the linear approximation of $U(x)$ at $\bar{x} \in M$, where $M \subseteq K$ is the set of maximal solution of P, is $U(x) = U(\bar{x}) + U'(\bar{x})(x - \bar{x}) + o||x - \bar{x}||$. But, $U'(\bar{x})\bar{x} = U(\bar{x})$, because of the homogeneity of U, and hence, $U(x) = U'(\bar{x})x + o||x - \bar{x}||$. We exploit this approximation instead of the original utility function, since it leads us to a useful formulation of the vector dual program. Therefore, let us consider the primal program

$$\overline{P}: \quad \max_{\mathbb{R}^\ell_+\setminus\{0\}}U'(\bar{x})x = (U^{'1}(\bar{x})x, \ldots, U^{'\ell}(\bar{x})x) \ \text{ subject to } \ x \in K = \{x \in X : p^T x = y\}, \tag{7.3}$$

and its vector dual problem, for $\bar{x} \in M$,

$$\overline{D}: \ \mathrm{Min}_{\mathbb{R}^\ell_+\setminus\{0\}}\lambda y = (\lambda_1 y, \ldots, \lambda_\ell y) \text{ subject to } \ (U'(\bar{x}) - \lambda p^T)w \notin \mathbb{R}^\ell_+\setminus\{0\}, \forall w \in \mathbb{R}^N_+\setminus\{0\}. \tag{7.4}$$

This dual scheme was proposed by Isermann in [8], in the linear vector optimization case. We observe that problem $\overline{P}$ may have more solutions than problem P and, if $\overline{M}$, the set of optimal solutions of $\overline{P}$, is a singleton, then $\overline{M} = \{\bar{x}\}$.

What follows gives the concluding observations about the example developed in Section 2.

Example 7.1. Let us consider once more example 2.1. Applying the definition of the vector dual problem to these data, we obtain:

$$\mathrm{Min}_{\mathbb{R}^2_+\setminus\{0\}}(\lambda_1 y, \lambda_2 y) \text{ subject to } \begin{pmatrix} a - \lambda_1 p_1 & -\lambda_1 p_2 \\ -\lambda_2 p_1 & b - \lambda_2 p_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \notin \mathbb{R}^2_+ \setminus \{0\}, \forall w \in \mathbb{R}^2_+ \setminus \{0\}.$$

Therefore, in the feasible region, it may occur at least one of these cases:

i. $(a - \lambda_1 p_1)w_1 - \lambda_1 p_2 w_2 < 0$;

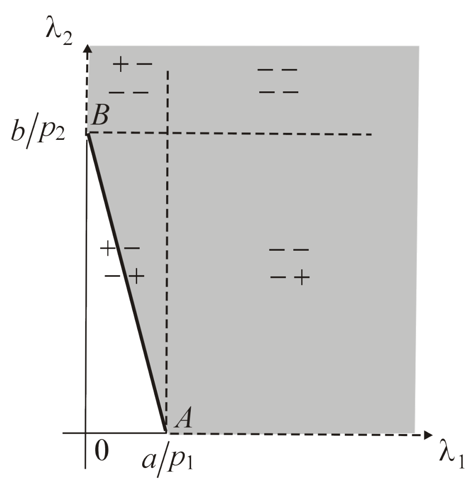
ii. $-\lambda_2 p_1 w_1 + (b - \lambda_2 p_2)w_2 < 0$;

iii. $(a - \lambda_1 p_1)w_1 - \lambda_1 p_2 w_2 = 0$ and $-\lambda_2 p_1 w_1 + (b - \lambda_2 p_2)w_2 = 0$.

To study these cases let us consider the entries of the matrix

$$\begin{pmatrix} a - \lambda_1 p_1 & -\lambda_1 p_2 \\ -\lambda_2 p_1 & b - \lambda_2 p_2 \end{pmatrix}$$

The entries $-\lambda_1 p_2$ and $-\lambda_2 p_1$ are always negative (with positive λ), while $a - \lambda_1 p_1 > 0$ if $\lambda_1 < a/p_1$ and $b - \lambda_2 p_2 > 0$ if $\lambda_2 < b/p_2$. Now, we plot a graph with the sign of the entries of this matrix and hence we study the feasible region of the vector dual problem.



The line from point A to point B is the efficient frontier, that is the set of all the optimal combinations of λ_1 and λ_2.

In this example, we obtain that the efficient frontier of the Lagrangian dual problem tells us the quantity of income that one agent withholds at the other agent's expense; in fact, if λ_1 increases, then λ_2 decreases, and viceversa. For a more illustrative treatment of linear vector optimization problems, see [11]. We also note that the efficient frontier is the set of points that satisfies condition iii.

The dual program can be thought as an alternative strategy. The ith household member, instead of maximizing her/his utility by purchasing goods, may save the necessary resources at the shadow rate λ_i. The primal and dual strategies are equivalent if, $\forall w \in \mathbb{R}^N_+ \setminus \{0\}$,

$\lambda_i p^T w > {U'}^i(\bar{x})w$ for at least one i, or $\lambda_i p^T w = {U'}^i(\bar{x})w$, $\forall i = 1, \ldots, \ell$, that is, for every bundle $w \in \mathbb{R}^N_+ \setminus \{0\}$ the value of goods $\lambda_i p^T w$ that consumer i saves, must be greater than the approximate utility given by that bundle, or, for each member, the condition is satisfied with the equality. The vector $\lambda \in \mathbb{R}^\ell_+ \setminus \{0\}$, with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, is such that $\lambda_j/\lambda_i = \frac{\partial U^j}{\partial U^i}(\bar{x}) = \gamma_{ij}$, see [12]. So, the second strategy looks for the set of marginal rates of utility with respect to total income that makes the individual evaluation of goods greater or equal than the utility of that bundle. The quantity λ_i allows to find out the minimal sharing rule $\phi_i = \lambda_i y$ that can be attributed to the ith member.

The solution of the dual program $\lambda(p, y, s)$ represents a measure of the power of the agents, and, in particular, $\lambda_i y$ gives the amount of income that agent i spends. Further, if it is a function, there are some properties that has to satisfies, so that the individual expenditure preserves the usual characteristics, as proved in [9].

We introduce now a numerical example to explain what happens with a non linear problem.

Example 7.2. Let us consider problem P: $\max_{\mathbb{R}^2_+ \setminus \{0\}} U(x_1, x_2) = (\sqrt{x_1 x_2}, \sqrt[3]{(x_1)^2 x_2})$, subject to $2x_1 + x_2 = 4$. The optimal solution of P is given by the points on the frontier such that $1 \leq x_1 \leq 4/3$. We construct problem $\overline{P}$ starting from the non optimal point $(1, 2)$, that is: $\max_{\mathbb{R}^2_+ \setminus \{0\}} U'(1, 2)x = ((1/\sqrt{2})x_1 + (1/2\sqrt{2})x_2, (2\sqrt[3]{2}/3)x_1 + (1/3\sqrt[3]{4})x_2)$, subject to $2x_1 + x_2 = 4$. The solution of $\overline{P}$ is given by $(2, 0)$, that is a point in which the functions are not differentiable. If we start from the optimum $(5/4, 3/2)$ we obtain that the optimal solution of $\overline{P}$ is all the frontier of the constraint. The dual, in this last case, is:

$$\mathrm{Min}_{\mathbb{R}^2_+ \setminus \{0\}} (4\lambda_1, 4\lambda_2) \text{ subject to}$$

$$\begin{pmatrix} \sqrt{3/10} - 2\lambda_1 & \sqrt{5/24} - \lambda_1 \\ (2\sqrt[3]{6/5}/3) - 2\lambda_2 & (1/3\sqrt[3]{36/25}) - \lambda_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \notin \mathbb{R}^2_+ \setminus \{0\}, \forall w \in \mathbb{R}^2_+ \setminus \{0\}.$$

Finally, we note that $U(5/4, 3/2) = (\sqrt{15/8}, \sqrt[3]{75/32})$, and thus an optimal multiplier is $\lambda = (\sqrt{15/8}, \sqrt[3]{75/32})/4$, which is the vector that equals the value functions of the primal and of the dual problem. In fact, this point belongs to the efficient frontier of the approximate problem, that is $\frac{3\lambda_2 - \sqrt[3]{25/36}}{\sqrt[3]{6/5} - \sqrt[3]{25/36}} = \frac{\lambda_1 - \sqrt{5/24}}{\sqrt{3/40} - \sqrt{5/24}}$. If we interpret the multipliers as the rate of total income to give to each subject, then we can also impose the constraint $\lambda_1 + \lambda_2 = 1$, but, in this case $\sqrt{15/8}/4 + \sqrt[3]{75/32}/4 < 1$ and we do not spend the entire budget.

8 Concluding Remarks

This paper studies the collective household program as a vector maximization problem, where all agents seek to maximize their utility functions, but they have to take into consideration all other members. In this setting, they make part of a household, and, hence, the economic constraint is the budget of the entire family. We have also analyzed other three kinds of optimization problems, in order to highlight the differences and the similarities among these programs.

There are different ways for scalarizing the primal problem, but all the approaches, presented in this work, point out a relationship among the Lagrangian dual variables. We observe that if we add structure to the problem, i.e., we scalarize in some way the vector problem, we lose the information given by the Pareto frontier, since we can get only one point of it. This means that the vector problem gives all the efficient points that maximize the utility functions of all agents at the same time; if we introduce an agreement among the agents or an outside decision maker, we get only one point among the optimal ones. Because the former problem generally is difficult to solve, we can refer to one of the other problems proposed, solve it, and take the needed information about the vector problem.

Moreover, from these models we can derive two measures of inequality within the household. The former is given by $\mu_i(p, y, s)$, which is used to share the power among the household members, while the latter is given by $\lambda_i(p, y, s)$. The multipliers $\lambda_i(p, y, s)$, $i = 1, \dots, \ell$, are the solutions of a vector dual program and they can be interpreted as the measures that redistribute the income within the household. The dual problem is given by a generalization of a linear scheme proposed by Isermann in [8].

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