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Restricted Finite Time Dominance*

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Abstract

We investigate intertemporal dominance conditions for comparisons of finite unidimensional streams of outcomes in discrete time. We follow Ekern's [1981, Time dominance efficiency analysis. *Journal of Finance*, 36, pp.1023-1034] approach based on unanimous net present value comparisons for classes of discount factors representing temporal preferences. In order to overcome the problem of dictatorship of the present in intertemporal evaluations we restrict the class of discount factors, by imposing a limit on the decrease of the weight attached in the current evaluation, between the outcomes of two future adjacent periods.

The restricted time dominance theorems provide parametric dominance conditions that make explicit the policy maker's trade-off between current and future periods. We show that these conditions can be summarized by a single cutoff point that can be interpreted as maximal decrease in the discount factor that guarantees unanimous dominance for the class of intertemporal evaluations considered.

Keywords : Time Dominance, Discounting, Orderings, Sustainability.

JEL Classification : D81, D90, G11, O22.

1 Introduction

The temporal dimension of sustainable development evaluation requires to build connections between present and future when assessing public projects, environmental policies or investments activities whose effects will be spread out over a number of years. These analysis involve intertemporal evaluations for which the specification of appropriate discount structures is crucial and has raised a strong debate among

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the theoretical, empirical, psychological, behavioral and social choice scholars. Being confronted with intertemporal choices applied to multiple time periods and having to deal with trade offs between present costs and future benefits, we present necessary and sufficient dominance conditions to rank alternative prospects that give neither dictatorship to the present nor to the future and are able to respond whether a prospect is preferred to another by all policy makers that exhibit temporal preferences satisfying desirable properties. These conditions are an attempt to take a further step towards the goal of a more sustainable decision making process.

This work focuses on deriving dominance conditions for ordering finite streams of unidimensional cardinal outcomes, measured for instance as cash flows or net benefits, distributed in discrete time. The dominance conditions are obtained by considering unanimous ranking of projects for different classes of discount functions subject to various restrictions. The analysis relates to several strands of literature including the work by Ekern (1981) that focuses on deriving intertemporal evaluations that are robust to the use of different formulas and values for discounting future benefits and costs; Foster and Mitra (2003) on ranking investment projects in terms of net present value irrespective of the choice of the discount rate; Karcher et al. (1995), and Muller and Trannoy (2012) that investigate multidimensional stochastic dominance conditions with possible applications to intertemporal evaluations; Fishburn and Lavalley (1995) who work on special stochastic-dominance relations for probability distributions on a finite grid of evenly-spaced points, and De La Cal and Cárcamo (2010) and Chakravarty and Zoli (2012) that present an analogous counterpart for inverse stochastic dominance conditions. Moving away from the time dominance work of Bøhren and Hansen (1980) and Ekern (1981), we extend existing results and investigate the possibility of increasing the comparability induced by the dominance conditions by imposing a limit on the decrease between the outcomes of two future adjacent periods of the discount weight attached to them in the current evaluation.

Our main result will be implicitly characterized by the choice of two time thresholds: first, the horizon time T , that identifies the period after which the differences in the streams of outcomes are assumed to be negligible; second, a time threshold H , before which the discount structure gives enough weight to the outcomes, not allowing them to become negligible in the aggregate evaluation. We present two mechanisms that allow us to identify a temporal threshold, that indicates how long is the policy maker willing to postpone his myopic judgement in order for the future benefits to be able to compensate the initial costs of the project appraised. The obtained dominance conditions can also be summarized by a single cutoff point that can be interpreted as the maximal admissible decrease in the discount factor, between two adjacent periods, that guarantees unanimous dominance for the class of intertemporal evaluations considered. Following this procedure, given two intertemporal profiles, it will be possible to calculate a cutoff point, strictly related to the level of impatience of the policy maker, such that the profile that leads to higher total undiscounted net benefits will dominate the latter at a given order of dominance provided that the fall

in the discount factors applied between two periods is not larger than the calculated cutoff point.

The paper proceeds as follows. In the next section we present in more details the model and discuss the notion of time dominance, presenting some preliminary tools and findings. In Section 3 we motivate our approach and present the α -Restricted Time Dominance theorems, a novel toolkit for evaluating projects obtained by parametrically restricting the set of discount factors. Section 4 provides some concluding remarks.

2 Setting

We focus on models for implementing intertemporal comparisons that have as a central point the discount functions defined over streams of outcomes that are spread over a discrete and finite time span. The choice of *discrete time* is driven mainly by the fact that empirical intertemporal comparisons of costs and benefits are commonly made over time grids (i.e., over evenly distributed time periods). Meanwhile the choice of *finite time* is taken in order to avoid that the long future net benefits cumulate in an infinite advantage, giving in this way a dictatorship power of the future over the present irrespective of the presence of large negative values in the nearer future.

We build the analysis on a setting introduced by Bøhren and Hansen (1980) and Ekern (1981), that derives stochastic dominance conditions applied to outcomes ordered on the time dimension. The decision maker evaluates intertemporal prospects by ranking streams of unidimensional cardinal outcomes (cash flows, costs and benefits or utilities) arising from alternative projects/policies in terms of higher *Net Present Value (NPV)*. The ranking is made robust to all discount functions drawn from a particular class, which is defined by adding curvature restrictions on their derivatives with respect to time. For consistency of exposition we will focus on comparisons of projects leading to intertemporal distributions of net outcomes/benefits.

Consider two temporal profiles a and b represented by their return vectors $a = (a_0, a_1, \dots, a_T) \in \mathbb{R}^{T+1}$ and $b = (b_0, b_1, \dots, b_T) \in \mathbb{R}^{T+1}$, where time is discrete and finite, and $t = 0$ denotes the present, while T is the horizon of the most long-lived project. For simplicity of notation, all projects are represented as having the same finite horizon T ; if the projects were to have different horizons, the shorter vector could be augmented with zero entries so as to reconcile the size difference.

In discrete time, for a given set of discount functions v , the *NPV* of project x , written $NPV_v(x)$, is defined as

$$NPV_v(x) := \sum_{t=0}^T v_t \cdot x_t,$$

where the set $v = (v_0, v_1, \dots, v_T)$ represents the temporal preferences and w.l.o.g. v_t will be normalized such that $v_0 = 1$. When comparing two projects $a, b \in \mathbb{R}^{T+1}$, the

former project a will be (weakly) preferred to the latter one b if and only if $NPV_v(a) \geq NPV_v(b)$. The problem can be reformulated by considering the net project x , where

$$x = (x_0, x_1, \dots, x_T) := (a - b) = (a_0 - b_0, \dots, a_T - b_T),$$

and requiring the sign of its intertemporal evaluation to be non negative, that is $NPV_v(x) \geq 0$.

In the appraisal of long term projects the temporal structure has been a source of strong debate in the economic literature and in cost-benefit analysis, questioning mainly the use of the standard exponential discount function. A growing body of literature argues that, the conventional exponential discount factor is unsatisfactory when assessing sustainable development mainly because of its tendency to favor a myopic judgement of policies and, in the same time, for its inappropriateness to treat intergenerational issues [see, for instance Schelling, 1995; Lind, 1995]. Recently, the notion of *time declining discount rates* has gained considerable support from three main pillars of the economic literature. Experimental evidence suggests the use of hyperbolic discounting [Laibson, 1997; Loewenstein and Prelec, 1992] or its discrete time approximation, the quasi-hyperbolic discounting [Phelps and Pollak, 1968]. Uncertainties about the discount factors may motivate the use of the Gamma discounting [Weitzman, 2001]. While the concern for avoidance of tyranny of future or present evaluations have motivated alternative rules in the social choice literature [Chichilnisky, 1996; Li and Löfgren, 2000].

In the next sub section we will discuss the notion of time dominance and argue that these attitudes towards discounting can all be incorporated in it.

Aspects of Time Dominance. The time dominance (*TD*) approach applies the stochastic dominance (*SD*) methodology to a temporal context, where the consequences of a decision alternative are distributed over time, establishing an analogy between the multiperiod evaluation problem under certainty and the single period evaluation problem under uncertainty. While *SD* puts successively stronger restrictions on the utility function representing risk preferences, *TD* restricts, in a similar way, the discount functions representing temporal preferences, providing in this way dominance rules that lead to partial orders of temporal prospects.

The *TD* calls for curvature restrictions to classify discount functions. Let $v_t^0 := v_t$, then for any number k of restrictions imposed on the discount function, let

$$v_t^k := v_{t+1}^{k-1} - v_t^{k-1}.$$

Thus, v_t^k is obtained by differencing k times the function v_t . The widest class of discount functions is obtained for $k = 0$ and is denoted V_0 , it requires simply that, at any point in time more is preferred to less. Formally,

$$V_0 := \{v : v_t \geq 0, \text{ with } v_0 = 1 \text{ for all } t \in \{0, 1, \dots, T\}\}.$$

Downward sloping discount functions, representing time impatience, belong to V_1 and imply that a dollar at time t is (weakly) preferred to a dollar at time $t' > t$. The set V_2 exhibits decreasing time impatience being a subset of the functions from V_1 that are non increasing and convex in t [i.e. in the discrete time case, with non increasing differences]. Formally

$$\begin{aligned} V_1 &:= \{v : v \in V_0, \text{ and } v_t - v_{t+1} \geq 0 \text{ for all } t \in \{0, 1, \dots, T-1\}\}, \\ V_2 &:= \{v : v \in V_1, \text{ and } v_t - v_{t+1} \geq v_{t+1} - v_{t+2} \text{ for all } t \in \{0, 1, \dots, T-2\}\}. \end{aligned} \quad (1)$$

In general, by adding successive restrictions on v_t^k , subsets of discount functions can be recursively defined:

$$V_k := \{v : v \in V_{k-1}, \text{ and } (-1)^k v_t^k \geq 0\}.$$

Hence, v belongs to the class V_n if and only if v_t^k alternates in sign (starting with a positive sign), as k goes from 0 to n . In discrete time, the domain of v_t^k is the set $\{0, 1, 2, \dots, T-k\}$ supporting the fact that every time we add a condition on the discount functions v , we refer to a preceding period and therefore we loose a period from the finite set $\{0, 1, \dots, T\}$. Following this logic, the number n of restrictions on the discount functions could not overtake the horizon T . Before moving on, note that all the discount functions mentioned above, the exponential along with the time declining ones (the hyperbolic, the quasi-hyperbolic, the Gamma) have something in common: they are all positively valued, they all present time impatience and are all convex in time.

The dominance conditions based on these functions are defined as follows:

Definition 1 $NPV_v(a) \geq NPV_v(b)$ for all $v \in V_n$ is denoted as $a \succ_n b$.

The class of *TD* stochastic orders for $n \in \mathbb{N}$ (where \mathbb{N} denotes the set of natural numbers) has been investigated in Ekern (1981). We are going to focus on the cases where the discount functions present time impatience and decreasing time impatience with time, in other words on the 1st and 2nd order *TD*.

Given the restrictions on the discount function, what kind of mathematical properties must the distribution x of net benefits satisfy in order to ensure that prospect a is preferred to prospect b ? Similar to the *SD* literature or, more closely, to the Inverse Stochastic Dominance literature (*ISD*) [see Muliere and Scarsini, 1989], or to the majorization literature [see Marshall and Olkin, 1979], the *TD* ranking of alternatives will depend on comparisons of distributions of repeated summations of outcomes.¹

Rewriting the initial net stream $x_t = X_t^0$, calling it Stage 0, and using repeated summations we get

$$X_t^n := \sum_{s=0}^t X_s^{n-1}.$$

¹The main difference is that, while in the case of *ISD* or of majorization, the outcomes are ranked according to their magnitude, in the *TD* the outcomes are ordered according to the time dimension.

The Stage 1 of repeated summations at any point in time is nothing but the sum of the initial net benefits, starting from 0 up to that point in time. Recursively, the Stage n of repeated summations at any point in time is the sum of the previous level of repeated summations, up to that point in time, starting from 0.

Such repeated cumulations of consequences correspond to the repeated cumulations of probabilities in the stochastic dominance approach, although there are some important differences to be kept in mind. First of all, outcomes can be negative, unlike probabilities, and secondly, different projects' net benefits at the horizon do not necessarily coincide, hence X_T^1 may differ from 0. In fact, the TD conditions are more closely related with the ISD conditions [see De La Cal and Cárcamo, 2010; Muliere and Scarsini, 1989] as well as with the discrete version of Yaari's non expected utility model, where the discount function can be seen as a distortion across time and the outcomes are the respective payoffs [see Yaari, 1987].

Using a TD methodology, the information about the X_t^k values for $1 \leq k \leq n$ may suffice to conclude whether some decision alternatives are superior for all discount functions in the class V_n .

Definition 2 (n^{th} order Time Dominance) *Project a dominates b by the n^{th} order TD , denoted by $a \geq_n b$, if and only if for the net project $x = a - b$*

$$\begin{aligned} X_T^k &\geq 0 \text{ for all } k \in \{1, 2, \dots, n-1\} \\ X_t^n &\geq 0 \text{ for all } t \in \{0, 1, \dots, T-n+1\}. \end{aligned}$$

In order to illustrate the concept, consider the following example with $n = 2$, that will also be used to expound the novel criteria investigated in the next section.

Example 1 *Consider a net project x with time horizon $T = 8$. The 2^{nd} order TD conditions, visualized in the table below, are associated with the bold faced elements, that need to exhibit the same sign for a conclusive result*

t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4
	↓	↗↓	↗↓	↗↓	↗↓	↗↓	↗↓	↗↓	↗↓
X_t^1	-1	-3	0	3	1	5	8	6	2
X_t^2	-1	-4	-4	-1	0	5	13	19	

Note that the first column of the matrix consists of identical elements that are equal to the first net outcome x_0 . Secondly, each term in the matrix is the sum of the one before it (on the same row) and the one above it (from the same column). Third, for a given stage k of cumulation, the matrix elements participating in the time dominance conditions include all the elements of row k along with the terminal elements of the higher positioned rows, $k-1, k-2, \dots, 1$.

Ekern (1981) result relates the n^{th} order time (strict) dominance to the *NPV* (strict) superiority for discount functions in V_n .²

Theorem 1 (Ekern 1981) $a \succ_n b$ if and only if $a \geq_n b$.

Starting from the *TD* structure presented above, we further examine its applicability when facing sustainable intertemporal evaluations and we present the main results of the paper.

3 Results and Discussion

The classical *TD* approach makes some reasonable assumptions regarding the discount structure by agreeing on some properties of the temporal preferences and at a first glance may seem adequate for facing the problem of intertemporal decision making. However, from the sustainability point of view the *TD* criteria share a serious drawback: policies that give benefits for the generations in the distant future at the cost of those in the present are likely to be discarded even if benefits are substantial and current costs are minor.

In other words, the present yields a dictatorship over the future, in fact a negative net outcome $a_0 - b_0 < 0$ in the first period will always be sufficient for ruling out dominance of a over b regardless of the remaining outcomes of the vectors.

This limitation asks for a reconsideration of the type of restrictions to impose on the discount structure, as in our view when a stream of sufficiently large positive net outcomes succeeds some present negative ones, the socially desirable discount functions should react positively.

In fact, we restrict the classes of discount functions supporting the classical first and second order *TD* by imposing a further condition related to their maximum decline between two adjacent periods. Formally, we consider the following classes of discount functions that are subsets of V_1 and V_2 .

Definition 3 Let $n \in \{1, 2\}$, and $\alpha \in [0, 1)$, then

$$V_n^\alpha := \{v \in V_n \text{ s.t. } \Delta_t := v_t - v_{t+1} \leq \alpha \text{ for all } t \in \{0, 1, 2, \dots, T-1\}\}.$$

We call the associated dominance conditions α -restricted 1st *TD* and α -restricted 2nd *TD* respectively for $n = 1$ and $n = 2$, in short $\alpha - 1TD$ and $\alpha - 2TD$.

Definition 4 ($\alpha - TD$) Let $n \in \{1, 2\}$, and $\alpha \in [0, 1)$, then $NPV_v(a) \geq NPV_v(b)$ for all $v \in V_n^\alpha$ is denoted as $a \succ_n^\alpha b$.

²Strictly speaking Ekern's result concerns the equivalence between strict *TD* and strict *NPV* superiority for the subset of discount functions in V_n that satisfy the curvature conditions with strict inequality. The result presented here concerning weak inequalities can be derived analogously.

The parameter α can be interpreted as a magnitude restriction on the fall of the discount function that identifies an upper bound for $(v_t - v_{t+1})/v_0 = \Delta_t/v_0$ to be satisfied in any period. This normative condition allows for a compensation between present and future by taking into account the level of future outcomes and in the same time, as shown below, will implicitly identify a threshold in time that tells us how long the policy maker is willing to wait for the benefits to arrive before the more time impatient evaluations cancel their effect in the final evaluation. By construction, for a given $\alpha \in [0, 1)$ $a \succ_1^\alpha b$ implies $a \succ_2^\alpha b$.

Restricted *TD* and non dictatorship of the present.

We argue here that the $\alpha - TD$ restriction will allow to overcome the issue of *dictatorship of the present*. In line with Chichilnisky (1996), we say that dictatorship of the present occurs if, irrespective of future net positive benefits $\omega_t = \omega > 0$ for any $t \in \{h + 1, \dots, T\}$, there exists at least an admissible discount function v such that if for $h \in \{0, 1, \dots, T - 1\}$, $x_h < 0$ and $x_t = 0$ for all $t \in \{0, 1, \dots, h - 1\}$ then $NPV_v(x) < 0$ where $x = (0, 0, \dots, x_h, \omega, \omega, \dots, \omega)$. We consider a weaker version of this condition by requiring that $h < \min\{H, T\}$ for some time period $H \in \mathbb{N}$, where H denotes a temporal upper bound beyond which future outcomes can be considered irrelevant for the social evaluation.

The *Non Dictatorship of the Present (NDP)* condition postulates that this should not happen. It does not allow the negative sign of the first net outcome different from 0 to restrain future benefits from playing a part in determining the judgement of the policy as long as some of these benefits take place no later than the threshold period H . Considering a set of discount functions V_H , obtained conditioning on the value of H , the social evaluation $NPV_v(x)$ for $v \in V_H$ satisfies *NDP* if the following requirement holds.

Definition 5 (Non Dictatorship of the Present (NDP)) *Let $H \in \mathbb{N}$. For $x_h < 0$, where $h \in \mathbb{N}$, $h < \min\{H, T\}$, $x_t = 0$ for all $t \in \{0, 1, \dots, h - 1\}$ and for any $v \in V_H$ there exists $\omega > 0$ with $x_t = \omega$, for all $t \in \{h + 1, \dots, T\}$ s.t. $NPV_v(x) \geq 0$.*

The following is a direct implication of the axiom *NDP* on the ranking induced by $NPV_v(x)$ for $v \in V_H \subseteq V_1$. The case where $V_H \subseteq V_2$ is discussed further on.

Proposition 1 *$NPV_v(x)$ for $v \in V_H \subseteq V_1$ satisfies *NDP* if and only if there exists a sequence $\alpha_t \in [0, 1)$ for $t \in \{0, 1, \dots, T\}$ s.t. $\Delta_t := v_t - v_{t+1} \leq \alpha_t$ with $\sum_{t=0}^{\min\{H-1, T-1\}} \alpha_t < 1$.*

Proof. See the Appendix. ■

In this way, additional to the classical first order *TD* conditions where $v \in V_1$, *NDP* requires that the range of Δ_t should be limited in the interval $[0, \alpha_t]$, where $\alpha_t < 1$ for each t . Note that if $H < T$ the condition $\sum_{t=0}^{H-1} \alpha_t < 1$ implies that

$v_H = 1 - \sum_{t=0}^{H-1} \Delta_t \geq 1 - \sum_{t=0}^{H-1} \alpha_t > 0$. That is, the evaluation should assign a positive weight to all outcomes that are realized before period H .

As far as the class V_2 of discount functions is concerned, remember from (1) that the curvature restriction for the non decreasing and convex functions is obtained by differencing v_t twice, that is $v_t - v_{t+1} \geq v_{t+1} - v_{t+2}$. In other words, the differences between two adjacent periods decrease with time, $\Delta_t \geq \Delta_{t+1}$. Therefore, by simply imposing that $\Delta_0 \in [0, \alpha]$, where $\alpha < 1$, it will result that, for any $\alpha_t \leq \alpha < 1$, $\Delta_t \in [0, \alpha_t]$ for $t > 0$.

These considerations lead to next remark.

Remark 1 *Dominance in terms of NPV for all $v \in V_2$ that satisfy NDP coincides with dominance for all $v \in V_2^\alpha$ for some $\alpha \in [0, 1)$. While dominance in terms of NPV for all $v \in V_1^\alpha$ for some $\alpha \in [0, 1)$ is implied by dominance in terms of NPV for all $v \in V_1$ that satisfy NDP.*

Moreover, if $H < T$ and given that a necessary condition for *NDP* is that $v_H > 0$ for all admissible sets of discount functions, then the choice of α will implicitly allow to identify the more distant time threshold H . In fact, it should be that if $\Delta_t = \alpha$ for all $t \in \{0, 1, \dots, H-1\}$ we get $v_H = 1 - \sum_{t=0}^{H-1} \Delta_t = 1 - H \cdot \alpha > 0$, thus implying that $1/\alpha > H$. It follows that, for a given $\alpha \in [0, 1)$, the largest admissible value for H is obtained, for

$$H = \begin{cases} \text{Int}(1/\alpha), & \text{if } 1/\alpha \neq \text{Int}(1/\alpha) \\ 1/\alpha - 1, & \text{otherwise} \end{cases},$$

where the operator $\text{Int}(y)$ selects the integer component of y .

The time threshold $(1/\alpha - 1)$ will play a crucial role in our results and will be denoted by H_α .

α -restricted 1st TD criteria.

This section explicitly explores the first order restricted TD conditions, α -1TD, obtained considering the class V_1^α where the parameter $\alpha \in [0, 1)$. In order to derive them we make use of the curve $G_{X^{1*}}(t)$. The curve is obtained through the application of a double process of cumulation. First net benefits are cumulated across time using the exogenous order of time and obtaining the values X_t^1 for $t \in \{0, 1, \dots, T\}$. Then these values are censored at the value of X_T^1 thereby obtaining the distribution of elements $X_t^{1*} := \min\{X_t^1, X_T^1\}$. At the second stage the values of X_t^{1*} are ranked in a non decreasing order leading to the distribution of elements $X_{[t]}^{1*}$.

To conclude, these values are then cumulated, leading to

$$G_{X^{1*}}(t) := \sum_{\tau=0}^t X_{[\tau]}^{1*} \quad (2)$$

for $t \in \{0, 1, \dots, T\}$. The function can be extended to any value of $t > T$, by adding $(t - T)$ terms $X_T^1 = X_{[T]}^{1*}$, that is by evaluating it over a distribution of net benefits

obtained by expanding the original stream of net benefits x adding a sequence of 0's to all periods after T . Thus we obtain,

$$G_{X^{1*}}(t) := \sum_{\tau=0}^T X_{[\tau]}^{1*} + (t - T) \cdot X_T^1$$

for $t > T$, while in general the linear interpolation of the curve gives for $t^* = \text{Int}(\theta)$ the formula

$$G_{X^{1*}}(\theta) = (\theta - t^*) \cdot X_{t^*+1}^1 + G_{X^{1*}}(t^*). \quad (3)$$

In order to obtain a non negative net present value for all $v \in V_1^\alpha$ the value $G_{X^{1*}}(\frac{1}{\alpha} - 1)$ must be non negative.

Theorem 2 For $\alpha \in (0, 1)$ then $a \succcurlyeq_1^\alpha b$ if and only if $G_{X^{1*}}(\frac{1}{\alpha} - 1) \geq 0$.
If $\alpha = 0$ then $a \succcurlyeq_1^0 b$ if and only if $X_T^1 \geq 0$.

Proof. See the Appendix. ■

When $\alpha = 0$, that is when every period receives the same weight equal to 1, i.e., $v_0 = v_1 = \dots = v_T = 1$, the dominance requires a non negative sign of the cumulative undiscounted net outcomes at the horizon T , $X_T^1 \geq 0$.

When $\alpha \rightarrow 1$, the period 0 will be the only one receiving importance since all the future periods starting from $t = 1$ will be discounted at values tending to zero. Therefore, $G_{X^{1*}}(0) \geq 0$ becomes a *necessary* condition, implying that $X_{[0]}^{1*} \geq 0$. Recall that by definition we have $0 \leq X_{[0]}^{1*} \leq X_t^{1*} \leq X_t^1$ for all t , and therefore $X_{[0]}^{1*} \geq 0$, by construction, implies $X_t^1 \geq 0$ for all t , condition also required by $a \succcurlyeq_1 b$. It follows that when $\alpha \rightarrow 1$ the $\alpha - 1TD$ coincides with classical $1TD$.

Furthermore, if two projects do not verify the classical conditions for presenting 1st order TD , a positive value for the cumulated undiscounted net benefits X_T^1 represents a *necessary* condition, indicating whether we can obtain $\alpha - 1TD$. In fact, by construction, if $X_T^1 \leq 0$ then the elements of the sequence $X_{[\tau]}^{1*}$ are non positive. Given that $1TD$ does not hold then some elements of the sequence should be negative, it will then follow that $G_{X^{1*}}$ is always negative.

Thus the $G_{X^{1*}}(t)$ function, produces a "verification criterion" for dominance of prospect a over b , in the sense that once α is specified, if the value of the function in period $H_\alpha := (\frac{1}{\alpha} - 1)$ is non negative, then so is the net present value $NPV_v(x)$. The next example clarifies the construction of the criterion.

Example 2 Consider the net project x in Example 1, then

t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4
X_t^1	-1	-3	0	3	1	5	8	6	2
$X_{[t]}^1$	-3	-1	0	1	2	3	5	6	8
$X_{[t]}^{1*}$	-3	-1	0	1	2	2	2	2	2
$G_{X^{1*}}(t)$	-3	-4	-4	-3	-1	1	3	5	7

Note that $G_{X^{1*}}(4.5) = 0$ thus $a \succ_1^\alpha b$ if and only if $\frac{1}{\alpha} - 1 \geq 4.5$, that is $\alpha \leq \frac{1}{5.5} = \alpha_1^*$. Thus $a \succ_1^\alpha b$ if and only if $\alpha \leq \alpha_1^* = \frac{1}{5.5} = 0.1818$.

Following from this example, whenever we are dealing with a value of α higher than the cutoff point α_1^* we cannot find conclusive dominance of project a over project b .

It is of immediate verification the fact that as α decreases the 1st restricted TD condition becomes less demanding.

There is another way of looking at the theorem above, other than concentrating on the fall of the discount function. Note from the Example 2 that the criterion divides the time stream into two sets: the periods where the violations of the standard 1TD are present T_1^- , and those where they are not T_1^+ , with

$$T_1^- := \{t : X_t^1 < 0\} \text{ and } T_1^+ := \{t : X_t^1 \geq 0\}.$$

The curve $G_{X^{1*}}(t)$ cumulates first the violations X_t^1 for $t \in T_1^-$, obtaining a negative value, and afterwards, it adds the non-violations X_t^1 for $t \in T_1^+$ ranked in non decreasing order and censored at the undiscounted aggregate net outcome level X_T^1 . The temporal threshold $H_{\alpha_1^*}$ is obtained when the sequence of values in T_1^+ manages to overcome the former sequence for $t \in T_1^-$ and leads to a positive value for $G_{X^{1*}}(t)$

The threshold value for time $H_{\alpha_1^*}$, and the cutoff value α_1^* in terms of maximal reduction of the discount factor between two adjacent periods, are obtained for

$$G_{X^{1*}}(H_{\alpha_1^*}) = G_{X^{1*}}\left(\frac{1}{\alpha_1^*} - 1\right) = 0. \quad (4)$$

In other words, the time threshold is obtained by adding to the cardinality of T_1^- , starting from period 0, the number of periods that it takes for the violations in T_1^- to be compensated by the non-violations in T_1^+ , censored at X_T^1 . The result depends not only on the total amount of the violations in T_1^- but also on their distribution in time. In fact, the more spread they are the higher the time threshold is. Moreover, also the distribution of the non-violations in T_1^+ and the undiscounted aggregate value of the projects do matter for the final result.

α -restricted 2nd TD criteria.

In this section we derive the second order restricted TD conditions $\alpha - 2TD$ obtained when considering the class V_2^α of the non increasing and convex in t discount functions, with the same magnitude restriction on their fall $\Delta_t := v_t - v_{t+1} \leq \alpha$. Our main result is summarized by the next theorem.

Theorem 3 For $\alpha \in (0, 1)$ and $H_\alpha := \left(\frac{1}{\alpha} - 1\right)$, $a \succ_2^\alpha b$ if and only if
(A) $X_T^1 \geq 0$, and

(B) $X_t^2 + X_T^1 \cdot [H_\alpha - t]_+ \geq 0$ for all $t \in \{0, 1, 2, \dots, T - 1\}$,
where $[x]_+ := \max\{0, x\}$.
If $\alpha = 0$ then $a \succ_2^0 b$ if and only if $X_T^1 \geq 0$.

Proof. See the Appendix. ■

Note that the condition (B) in the Theorem 3 can be rewritten as:

$$\begin{aligned} (B_1) \quad & X_t^2 \geq 0 \text{ for all } t \geq H_\alpha \\ (B_2) \quad & X_t^2 + X_T^1 \cdot [H_\alpha - t] \geq 0 \text{ for all } t < H_\alpha. \end{aligned}$$

A necessary condition for finding dominance of a over b is that the undiscounted cumulated sum of net benefits should be non negative. In fact, there is no point in considering restricted dominance conditions that limit the role of current outcomes if these are not somehow compensated by future net outcomes. Indeed, if $X_T^1 = 0$ then $\alpha - 2TD$ resume to the standard $2TD$. On the other hand if $X_T^1 > 0$ then for sufficiently low values of $\alpha \in (0, 1)$ the future positive net gains can compensate current losses. In fact according to the $\alpha - 2TD$ dominance test in (B), for a given $\alpha \in (0, 1)$ a time threshold H_α is identified and the dominance conditions require that (B₁) standard $2TD$ conditions holds for all $t \geq H_\alpha$, while (B₂) for the periods $t < H_\alpha$ negatives values of X_t^2 can be compensated by the consideration of the future undiscounted net benefits. Moreover, closer to the initial periods we are, the higher the weight ($H_\alpha - t$) received, in the compensation effect, by the undiscounted outcome $X_T^1 > 0$. Note for instance that for $t = 0$ condition (B₁) requires that $X_T^1 \geq \frac{\alpha}{1-\alpha} \cdot x_0$.

Consider the set of time periods where violations of the second order TD are presented $T_2^- := \{t : X_t^2 < 0\}$, and consider the proportion of each violation with respect to the undiscounted outcome by denoting it $-X_t^2/X_T^1 = p_t$. It follows from (B₂) that

$$H_\alpha \geq t + p_t.$$

This is to say that given the undiscounted outcome X_T^1 and the violation X_t^2 , in order for the future to be able to compensate the violation at time t , the threshold H_α must be sufficiently larger than the time period where the violation is verified.

Letting

$$H_{\alpha_2^*} = \max_{t \in T_2^-} \{t + p_t\} \tag{5}$$

and knowing that with $H_{\alpha_2^*} = \left(\frac{1}{\alpha_2^*} - 1\right)$, it results that, given X_T^1 and the violations X_t^2 for $t \in T_2^-$, the maximum fall in the discount function between two adjacent periods of time that assures a restricted TD of project a over b is

$$\alpha_2^* = \frac{1}{1 + \max_{t \in T_2^-} \left\{t - \frac{X_t^2}{X_T^1}\right\}}. \tag{6}$$

Similar to the $\alpha - 1TD$, the undiscounted net value of the project X_T^1 plays a crucial role in the construction of the criterion. Firstly, when no discount rate is used and the future is given the same weight as the present, the undiscounted outcome must be non negative in order to guarantee dominance. Alternatively, when a discount rate that follows the properties asked by the V_2^α is introduced and some violations of the classic $2TD$ occur, these violations can be compensated if X_T^1 is positive. The closer the violation is to the current period, the higher is the weight attached to X_T^1 . In other words, violations of $2TD$ are allowed but only if they do not occur after the time threshold H_α , and their value can be compensated by the undiscounted value of the entire project at every point in time where they appear. The verification of the classical conditions of $2TD$ can in this way be postponed to the periods after H_α .

Next example provides an illustration of the $\alpha - 2TD$ conditions.

Example 3 Consider the net project x in Examples 1 and 2, then

t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4
X_t^1	-1	-3	0	3	1	5	8	6	2
X_t^2	-1	-4	-4	-1	0	5	13	19	21

Note that condition A) is satisfied given that $X_T^1 = 2 > 0$.

Moreover, for $\alpha_2^* = 0.2$ we obtain $a \succ_{\alpha_2^*} b$. In fact the associated time threshold is $H_{\alpha_2^*} = \frac{1}{\alpha_2^*} - 1 = 4$. Condition B) is satisfied given that $X_t^2 \geq 0$ for $t \geq 4$, while for $t < 4$ we have:

$$\begin{aligned} \text{for } t = 3 : X_3^2 + X_T^1 \cdot [H_{\alpha_2^*} - 3] &= (-1) + 2 \cdot [4 - 3] = 1 \geq 0, \\ \text{for } t = 2 : X_2^2 + X_T^1 \cdot [H_{\alpha_2^*} - 2] &= (-4) + 2 \cdot [4 - 2] = 0, \\ \text{for } t = 1 \text{ and } t = 0 \text{ analogous conditions hold.} \end{aligned}$$

In fact, the threshold $H_{\alpha_2^*}$ in (5) is obtained for $t = 2$, for this reason the $\alpha - 2TD$ condition is satisfied with equality for $t = 2$.

Comparing the results in Examples 3 and 2, one can verify that when moving from $\alpha - 1TD$ to $\alpha - 2TD$ the range of values of α , s.t. prospect a dominates prospect b , is expanded. In fact $a \succ_{\alpha_1} b$ for all $\alpha \in (0, \alpha_1^* = \frac{1}{5.5}]$ while $a \succ_{\alpha_2} b$ for all $\alpha \in (0, \alpha_2^* = \frac{1}{5}]$.

Interesting properties of the $\alpha - TD$ conditions are presented in the next subsection.

Some corollaries.

From previous discussions or directly from the theorems 2 and 3, it is possible to derive the following corollaries.

Corollary 1 Let $\alpha' < \alpha$ then $a \succ_{\alpha} b \implies a \succ_{\alpha'} b$ for $n \in \{1, 2\}$.

This result can be verified by inspecting the dominance conditions in theorems 2 and 3. Furthermore it can be derived from the fact that when $\alpha' < \alpha$, then $V_n^{\alpha'} \subset V_n^\alpha$ for $n \in \{1, 2\}$.

Corollary 2 *Suppose that $a \succ_1^{\alpha'} b$ then $\exists \alpha \geq \alpha'$ such that $a \succ_2^\alpha b$.*

The result derives from the fact that $a \succ_1^{\alpha'} b \Rightarrow a \succ_2^{\alpha'} b$. Therefore, Corollary 2 holds for $\alpha = \alpha'$, but depending on the characteristics of the streams a and b may also hold for some $\alpha > \alpha'$, as is the case for the comparisons in Examples 3 and 2.

Corollary 3 *If $X_T^1 = 0$ then $a \succ_n^\alpha b \iff a \succ_n b \iff a \succcurlyeq_n b$ for $n \in \{1, 2\}$.*

That is, $\alpha - TD$ is more decisive when compared with standard TD only if $X_T^1 > 0$. If this is the case then the next corollary can be verified by inspecting the conditions for $\alpha - 1TD$ and $\alpha - 2TD$ and choosing a sufficiently small positive value for α in order to guarantee dominance. For $\alpha - 1TD$ one needs to expand the time span by adding to x a sufficiently long string of 0's, so that if $X_T^1 > 0$, then for some large t the function $G_{X^{1*}}(t)$ eventually becomes positive. For $\alpha - 2TD$ instead, by choosing a sufficiently small positive α one can increase the weight $[H_\alpha - t]_+$ attached to $X_T^1 > 0$ in condition (B) of Theorem 3 in order to guarantee dominance irrespective of the values of X_t^2 .

Corollary 4 *If $X_T^1 > 0$ then $\exists \alpha \in (0, 1)$ s.t. $a \succ_n^\alpha b$ for $n \in \{1, 2\}$.*

From Corollary 3 follows that when a and b cannot be ranked according to standard TD then $X_T^1 \neq 0$ is a necessary condition for ranking them according to $\alpha - TD$. Depending on the sign of X_T^1 one can get $\alpha - TD$ dominance for $\alpha \in (0, 1)$ as Corollary 4 states. Combining these findings with the result in Corollary 1 the next result follows.

Corollary 5 *Let $n \in \{1, 2\}$. If a and b cannot be ranked according to n^{th} order TD , i.e. neither $a \succcurlyeq_n b$ nor $b \succcurlyeq_n a$ hold, then*

- i) if $X_T^1 > 0$ then $\exists \alpha^* \in (0, 1)$ s.t. $a \succ_n^\alpha b$ for all $\alpha \in [0, \alpha^*]$,*
- ii) if $X_T^1 < 0$ then $\exists \alpha^* \in (0, 1)$ s.t. $b \succ_n^\alpha a$ for all $\alpha \in [0, \alpha^*]$.*

The procedures that can be applied in order to derive endogenously the α^* threshold for the $\alpha - 1TD$ and the $\alpha - 2TD$ are summarized in (4) and (6) respectively.

An implication of Corollary 5 is that one cannot arrive to a disagreement point according to the $\alpha - TD$ criterion, that is, it is not possible that there exists an α for which $a \succ_n^\alpha b$, and another α' for which $b \succ_n^{\alpha'} a$ for $n \in \{1, 2\}$. This feature will prove to be particularly relevant in next discussion.

Relation with existing literature.

The results presented above show clear links with the cost benefit literature on internal rates of return (*IRR*), in fact as shown in Corollary 5 also in the $\alpha - TD$ framework it is possible to derive values of α for which the net project has zero present value according to some sets of discount factors. This is the case for the thresholds α_1^* and α_2^* derived respectively in (4) and (6). However, one must keep in mind several important distinctions. In the *IRR* case, a rate of discount is obtained, that under some assumptions can be used as a cutoff between the range of discount rates that select one project and the range of rates that select the other, meanwhile in the case of $\alpha - TD$ criterion we get a magnitude restrictions on the fall of the discount function. The former criterion can result in multiple *IRR* (this may occur when the sign of the net cash flow changes more than once during the project's life) and therefore, the ranking can be reversed when choosing among them, whereas as shown in Corollary 5 the $\alpha - TD$ criterion gives an unique value for which one project dominates another. Finally, the parameter α can be found for a large class of discount functions, in our case for all those non-negative and non-increasing with time, whereas the *IRR* is often computed only for exponential discount functions.

The $\alpha - TD$ criterion has a common flavor with the Almost Stochastic Dominance (*ASD*) criteria, a relaxation of the *SD* concept that consider parametric restrictions of the set of utility functions based on the maximal ratio between marginal utilities at two different realizations within the domain [see Leshno and Levy, 2002; Huang, Shih and Tzeng, 2012] for the first order condition. Analogous ratio restrictions for the second derivatives are also applied to derive the 2nd order *ASD* conditions. The $\alpha - TD$ criteria consider instead a *unique* restriction $(v_t - v_{t+1})/v_0$ (for any order of dominance) on the *absolute magnitude* of the "marginal" change of the discount function between two adjacent periods. We can focus on absolute magnitudes because we apply a natural normalization of the discount function by setting $v_0 = 1$. Nevertheless, even if the two families of criteria follow a similar approach the set of restrictions considered are logically distinct.

Meyer (1977), developed a restricted form of the second order *SD* by adding to the assumption of risk aversion a restriction on the level of Arrow-Pratt coefficient $r(x) = -u''(x)/u'(x)$, eliminating in this way the utility functions that present extreme risk aversion. He suggested introducing a lower and upper bound on $r(x)$ but no closed form solution was expressed, in the sense that no specific result on the dominance conditions arose based on these upper and lower bounds. In contrast, in our case, explicit dominance tests are derived and the parametric restriction is characterized by the time threshold H and by the mathematical properties of the net benefits and by their distribution in time. Moreover, these parametric restrictions remains the same for both the first and second order of the restricted *TD* conditions developed herein.

4 Conclusions

This paper tries to respond to whether the *TD* approach is suitable when facing sustainable intertemporal evaluations. We underline the fact that without any further (parametric) restrictions to the class of discount function, the *TD* framework continues to give prevalent weight to the present in ranking streams of net outcomes in terms of higher *NPV*, calling in this way for a method to control for the problem of the dictatorship of the present.

We have suggested the α -restricted *TD* approach focussing on discount functions that exhibit time impatience and on those that exhibit also decreasing time impatience. The parameter α can be interpreted as a magnitude restrictions on the fall of the intertemporal weight. It allows for a compensation between present and future by taking into account the level of future outcomes. Moreover, it will implicitly identify a threshold in time that tells us how long the policy maker is willing to wait for the benefits to arrive. This approach gives two intuitive, easily testable verification criteria for evaluating long run projects, that make explicit the trade-off between current and future periods. Moreover, the dominance conditions related to each of the two orders of restricted TD can be expressed in terms of a limit upper value for α which guarantees that dominance holds.

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A Appendix

A.1 Proof of Proposition 1

According to NDP $v_h x_h + (\sum_{t=h+1}^T v_t) \cdot \omega \geq 0$ should hold for all $v \in V_H \subseteq V_1$. If $v_h > 0$, this is the case only if $\sum_{t=h+1}^T v_t > 0$ where $h < H$. Recall that $V_H \subseteq V_1$ thus all v_t are non increasing. Even if $v_t = 0$ for all $t > H$ the condition of $v_H > 0$ turns out to be necessary, but it is not sufficient to derive the result. This aspect can be verified because if $h = H - 1$, when $v_t = 0$ for all $t > H$ the NDP condition requires that $v_{H-1} x_{H-1} + v_H \omega \geq 0$, however, for given values $v_{H-1} > 0$, $x_{H-1} < 0$ and $\omega > 0$ for any $v_H \in (0, -v_{H-1} x_{H-1} / \omega)$ we get that $NPV_v(x) < 0$. Thus it is not sufficient to have a positive values of v_H but it has also to be bounded above a positive level.

In fact, for a given $H \in \{1, \dots, T\}$ a necessary and sufficient condition for NDP to hold is that there exists a value $\beta_H > 0$ such that $v_H \geq \beta_H > 0$. Given that we consider $v \in V_H \subseteq V_1$ then there exists a sequence of values $\beta_t > 0$ with $\beta_t \geq \beta_{t+1}$ for $t \in \{0, 1, \dots, H\}$ such that $v_t \geq \beta_t > 0$ for all t . Recalling that $v_0 = 1$, and that $\Delta_t = v_t - v_{t+1} \geq 0$ for all t , the condition can be rephrased as $\alpha_t \in [0, 1)$ for $t \in \{0, 1, \dots, T\}$ s.t. $\Delta_t \leq \alpha_t$ and $\sum_{t=0}^{\min\{H-1, T-1\}} \alpha_t < 1$. ■

A.2 Proof of Theorem 2

We first derive an intermediate lemma and then we will move to the direct proof of the necessary and sufficient conditions of the theorem.

Recall that $a \succ_1^\alpha b$ requires that $NPV_v(x) = \sum_{t=0}^T v_t \cdot x_t \geq 0$ for all $v \in V_1^\alpha$. These discount functions satisfy the conditions (i) $\Delta_t := v_t - v_{t+1} \geq 0$ for all $t \in \{0, 1, \dots, T\}$ with $v_{T+1} = 0$, and (ii) $\Delta_t \leq \alpha$. Writing $v_T = \Delta_T$, $v_{T-1} = \Delta_T + \Delta_{T-1}$ and in general $v_t = \sum_{k=t}^T \Delta_k$ we can therefore rewrite

$$\begin{aligned} NPV_v(x) &= \sum_{t=0}^T \left[\sum_{k=t}^T \Delta_k \right] \cdot x_t \\ &= \Delta_T \cdot \sum_{t=0}^T x_t + \Delta_{T-1} \cdot \sum_{t=0}^{T-1} x_t + \dots + \Delta_1 \cdot \sum_{t=0}^1 x_t + \Delta_0 \cdot x_0. \end{aligned}$$

Denoting $X_t^1 := \sum_{h=0}^t x_h$ for all $t \in \{0, 1, \dots, T\}$ and substituting, we obtain

$$\begin{aligned} NPV_v(x) &= \Delta_T \cdot X_T^1 + \Delta_{T-1} \cdot X_{T-1}^1 + \Delta_{T-2} \cdot X_{T-2}^1 + \dots + \Delta_0 \cdot X_0^1 \\ &= \sum_{t=0}^T \Delta_t \cdot X_t^1. \end{aligned}$$

The result is formalized in next lemma.

Lemma 1 *Let $\alpha \in (0, 1]$ then $a \succ_1^\alpha b$ if and only if*

$$\sum_{t=0}^T \Delta_t \cdot X_t^1 \geq 0 \tag{7}$$

for all Δ_t such that

$$\begin{cases} \Delta_t \geq 0 \text{ for all } t \in \{0, 1, \dots, T\}, \\ \Delta_t \leq \alpha \text{ for all } t \in \{0, 1, \dots, T-1\} \\ v_0 = \Delta_T + \Delta_{T-1} + \Delta_{T-2} + \dots + \Delta_1 + \Delta_0 = 1. \end{cases} \quad (8)$$

Note that X_t^1 is obtained cumulating net benefits, thus can also be negative.

In order to derive the necessary conditions for Theorem 2 we investigate for what values of the weights in (8) the formula in (7) is minimized and we impose that its value is non-negative.

Necessity part. Rank the values of X_t^1 's in non decreasing order, leading to the distribution $X_{[t]}^1$ for $t \in \{0, 1, \dots, T\}$ with

$$X_{[0]}^1 \leq X_{[1]}^1 \leq X_{[2]}^1 \leq \dots \leq X_{[T]}^1. \quad (9)$$

Note that if $\Delta_t = 0$ for all $t \in \{0, 1, \dots, T-1\}$ then every period receives the same weight equal to 1, thus $v_0 = v_1 = \dots = v_T = 1$, it follows that the *necessary* conditions for having dominance is that the cumulated sum of the initial net outcome at the horizon, must be non negative, that is $X_T^1 \geq 0$. This necessary condition holds irrespective of the value of α for any set V_1^α .

Assigning positive weights to the cumulated sums with higher values than X_T^1 increases the value of the expression in (7). However, such configurations would not lead to the worst case scenario for the minimization of the expression in (7) that we need to check for deriving necessary conditions. In fact, extending the time horizon by a string of 0's of appropriate length, will coincide with expanding the sequence in (9) with terms of values X_T^1 without affecting the *NPV* of the net project. Thus, for the derivation of necessary conditions all values of X_t^1 larger than X_T^1 should not be considered. Henceforth, the next step is to censor all the reordered cumulative values at the value of X_T^1 , thereby obtaining $X_{[t]}^{1*} = \min\{X_{[t]}^1, X_T^1\}$. In this way, in the sequence (9) the X_T^1 receives all the weight that remains from the value of 1 after deducting the weights of the $X_{[t]}^1$ located before it.

Now we need to check when the *NPV* of these censored, rank dependent cumulative net outcomes is non negative. For this purpose, the reordered values are cumulated again, introducing the following function

$$G_{X^{1*}}(t) := \sum_{\tau=0}^t X_{[\tau]}^{1*}$$

for $t \in \{0, 1, \dots, T\}$.

We are interested in the configuration leading to the lowest possible *NPV*, thus the smallest $X_{[t]}^{1*}$'s in the sequence have to receive the highest possible weight α . These weights have to satisfy condition (8), therefore if the first $t^* + 1$ elements in the sequence receive the maximum weight α , the next one gets the remaining weight $[1 - \alpha(t^* + 1)]$.

Now, if $\alpha T \geq 1$, then by denoting with $t^* = \text{Int}(\theta)$, where $\theta = (\frac{1}{\alpha} - 1)$, for $t^* \in \mathbb{N}$ then the condition in (7) becomes

$$\begin{aligned} \alpha \cdot (X_{[0]}^{1*} + X_{[1]}^{1*} + \dots + X_{[t^*]}^{1*}) + [1 - \alpha(t^* + 1)] \cdot X_{[t^*+1]}^{1*} &\geq 0 \\ \alpha \cdot \left[X_{[0]}^{1*} + X_{[1]}^{1*} + \dots + X_{[t^*]}^{1*} + \left(\frac{1}{\alpha} - t^* - 1\right) \cdot X_{[t^*+1]}^{1*} \right] &\geq 0. \end{aligned}$$

We get therefore

$$NPV_v(x) = \alpha \cdot G_{X^{1*}}(\theta) \quad (10)$$

where

$$G_{X^{1*}}(\theta) = (\theta - t^*) \cdot X_{t^*+1}^1 + G_{X^{1*}}(t^*).$$

If $\alpha \cdot T < 1$ then the *necessary* condition requires that

$$\alpha \cdot (X_{[0]}^{1*} + X_{[1]}^{1*} + \dots + X_{[T-1]}^{1*}) + (1 - \alpha T) \cdot X_{[T]}^{1*} \geq 0$$

where by construction $X_{[T]}^{1*} = X_{[T]}^1$. The condition can be rewritten by adding further elements $X_{[T]}^{1*}$ to the sequence of cumulated values, i.e. adding 0's to the original string of net values, thereby re-obtaining

$$\alpha \cdot \left[X_{[0]}^{1*} + X_{[1]}^{1*} + \dots + X_{[t^*]}^{1*} + \left(\frac{1}{\alpha} - t^* - 1\right) \cdot X_{[t^*+1]}^{1*} \right] \geq 0.$$

The necessary condition thus requires that $G_{X^{1*}}(\frac{1}{\alpha} - 1) \geq 0$.

Sufficiency part. If $\alpha \in (0, 1)$ then by construction the *NPV* in (10), is the lowest among all possible *NPV* for all $v \in V_1^\alpha$. For a given distribution of values of X_t^1 for $t \in \{0, 1, \dots, T\}$, any other admissible distribution of values of Δ_t won't decrease the *NPV*. It follows that if $G_{X^{1*}}(\frac{1}{\alpha} - 1) \geq 0$ then $NPV_v(x) \geq 0$ for all $v \in V_1^\alpha$.

If $\alpha = 0$ then $NPV_v(x) = X_T^1$. ■

A.3 Proof of Theorem 3

To prove the result we first make a sequence of transformations that will allow us to express the dominance condition in a more transparent way.

Consider the result in Lemma 1.

Let $\beta_t := \Delta_t - \Delta_{t+1} \geq 0$ for all $t \in \{0, 1, \dots, T-2\}$, then $\Delta_t := \sum_{h=t}^{T-2} \beta_h + \Delta_{T-1}$ for $t \in \{0, 1, \dots, T-2\}$. It then follows that

$$\begin{aligned} NPV_v(x) &= \Delta_T \cdot X_T^1 + \sum_{t=0}^{T-1} X_t^1 \cdot \Delta_t = \Delta_T \cdot X_T^1 + \sum_{t=0}^{T-2} \left(\sum_{h=t}^{T-2} \beta_h \right) \cdot X_t^1 + \Delta_{T-1} \cdot \sum_{t=0}^{T-1} X_t^1 \\ &= \Delta_T \cdot X_T^1 + \Delta_{T-1} \cdot \sum_{t=0}^{T-1} X_t^1 + \sum_{t=0}^{T-2} \left(\sum_{h=0}^t X_h^1 \right) \beta_t \\ &= \Delta_T \cdot X_T^1 + \Delta_{T-1} \cdot X_{T-1}^2 + \sum_{t=0}^{T-2} \beta_t \cdot X_t^2 \end{aligned}$$

where $X_t^2 := \sum_{h=0}^t X_h^1$ for $t \in \{0, 1, \dots, T-1\}$. The weights are such that $\Delta_T \geq 0$, $\Delta_{T-1} \geq 0$, $\beta_t \geq 0$ for $t \in \{0, 1, \dots, T-2\}$, and

$$\begin{aligned} \sum_{h=0}^T \Delta_h &= \Delta_T + \Delta_{T-1} + (\Delta_{T-1} + \beta_{T-2}) + \\ &\quad + (\Delta_{T-1} + \beta_{T-2} + \beta_{T-3}) + \dots + (\Delta_{T-1} + \beta_{T-2} + \beta_{T-3} + \dots + \beta_0) \\ &= \Delta_T + T \cdot \Delta_{T-1} + (T-1)\beta_{T-2} + \dots + 2\beta_1 + \beta_0 = 1. \end{aligned}$$

Moreover, recall that the restriction based on α requires that $\Delta_t \leq \alpha$ for all $t \in \{0, 1, \dots, T-1\}$. Given that Δ_t is non increasing with respect to t , then this latter condition boils down to let $\Delta_0 \leq \alpha$. Recalling that $\Delta_t := \sum_{h=t}^{T-2} \beta_h + \Delta_{T-1}$ it follows

that the magnitude restriction $\Delta_0 \leq \alpha$ can be expressed as $\Delta_0 = \sum_{h=0}^{T-2} \beta_h + \Delta_{T-1} \leq \alpha$.

In order to simplify the exposition we make the following change of notation by renaming the weights letting $\omega_{t+1} := \beta_t$ for $t \in \{0, 1, \dots, T-2\}$, $\omega_T := \Delta_{T-1}$ and recalling that $\Delta_T = v_T$.

It then follows that

$$NPV_v(x) = v_T \cdot X_T^1 + \sum_{t=0}^{T-1} \omega_{t+1} \cdot X_t^2$$

where $v_T \geq 0$, $\omega_t \geq 0$ for $t \in \{1, \dots, T\}$, $\Delta_T + T \cdot \omega_T + (T-1) \cdot \omega_{T-1} + \dots + 2\omega_2 + \omega_1 = v_T + \sum_{t=1}^T t \cdot \omega_t = 1$ and $\sum_{t=1}^T \omega_t \leq \alpha$.

Thus, the dominance condition requires that

$$v_T \cdot X_T^1 + \sum_{t=1}^T \omega_t \cdot X_{t-1}^2 \geq 0 \quad (11)$$

subject to the following constraints on the set of weights

$$\sum_{t=1}^T \omega_t \leq \alpha, \quad (12)$$

$$v_T + \sum_{t=1}^T t \cdot \omega_t = 1, \quad (13)$$

$$v_T \geq 0 \text{ and } \omega_t \geq 0 \text{ for } t \in \{1, \dots, T\}. \quad (14)$$

Substituting for $v_T = 1 - \sum_{t=1}^T t \cdot \omega_t$ from the second constraint into the dominance condition we obtain the final set of conditions that can be summarized in the following lemma.

Lemma 2 Let $\alpha \in (0, 1]$ then $a \succ_2^\alpha b$ if and only if

$$X_T^1 - X_T^1 \cdot \sum_{t=1}^T t \cdot \omega_t + \sum_{t=1}^T \omega_t \cdot X_{t-1}^2 \geq 0, \quad (15)$$

for all weights ω_t such that

$$\sum_{t=1}^T \omega_t \leq \alpha, \quad (16)$$

$$\sum_{t=1}^T t \cdot \omega_t \leq 1, \quad (17)$$

$$\omega_t \geq 0 \text{ for } t \in \{1, \dots, T\}. \quad (18)$$

We now move to the proof of the theorem.

Note that condition (B) in the theorem can be rewritten as:

$$\begin{aligned} (B_1) \quad & X_t^2 \geq 0 \text{ for all } t \geq H_\alpha, \\ (B_2) \quad & X_t^2 + X_T^1 \cdot [H_\alpha - t] \geq 0 \text{ for all } t < H_\alpha, \end{aligned}$$

where $t \in \{0, 1, \dots, T-1\}$.

Necessity Part:

Consider (15), let $\omega_t = 0$ for all $t \in \{1, \dots, T\}$. This restriction satisfies (16), (17) and (18), it then follows that $X_T^1 \geq 0$ is a necessary condition for (15) to hold, as required in condition A of the theorem. Consider now the sequence of weights $\bar{\omega}$ that satisfy (16), (17) and (18), s.t. $\bar{\omega}_t = 0$ for all $t \in \{1, \dots, T\}$ s.t. $t \neq t^*$ and $\bar{\omega}_{t^*} > 0$. From this it follows that, based on the set of constraints, we can have two cases: either (i) $t^* \geq \frac{1}{\alpha}$ or (ii) $t^* < \frac{1}{\alpha}$, for $t^* \in \{1, 2, \dots, T\}$. Constraint (18) is satisfied and constraints (16) and (17) reduce respectively to

$$\begin{aligned} \bar{\omega}_{t^*} &\leq \alpha \text{ and} \\ t^* \cdot \bar{\omega}_{t^*} &\leq 1 \Leftrightarrow \bar{\omega}_{t^*} \leq \frac{1}{t^*}. \end{aligned}$$

Case (i).

If $t^* \geq \frac{1}{\alpha}$ then $\alpha \geq \frac{1}{t^*}$ and therefore a value $\bar{\omega}_{t^*} = \frac{1}{t^*} \leq \alpha$, would satisfy the two constraints above. By substituting the considered sequence of weights $\bar{\omega}$ into (15) we get

$$\frac{X_{t^*-1}^2}{t^*} \geq 0 \Rightarrow X_{t^*-1}^2 \geq 0 \text{ for any } t^* \geq \frac{1}{\alpha}.$$

To summarize, the above condition shows that $X_t^2 \geq 0$ for any $t \geq \frac{1}{\alpha} - 1$, where $t \in \{1, 2, \dots, T\}$

Case (ii).

If $t^* < \frac{1}{\alpha}$ then $\alpha < \frac{1}{t^*}$, it follows that the constraints in (16) and (17) are satisfied if $\bar{\omega}_{t^*} = \alpha < \frac{1}{t^*}$. By substituting this sequence of weights into (15) we get

$$X_T^1 - X_T^1 \cdot t^* \cdot \alpha + \alpha \cdot X_{t^*-1}^2 \geq 0.$$

This can be rewritten as $X_{t^*-1}^2 + X_T^1 \left[\frac{1}{\alpha} - t^* \right] \geq 0$ for any $t^* < \frac{1}{\alpha}$ where $t^* \in \{1, \dots, T\}$ and where by construction $\left[\frac{1}{t^*\alpha} - 1 \right] > 0$. That is

$$X_t^2 + X_T^1 \cdot \left[\frac{1}{\alpha} - 1 - t \right] \geq 0$$

for all $t \in \{0, 1, \dots, T-1\}$ s.t. $t < \frac{1}{\alpha} - 1$.

Sufficiency Part:

We need to show that the conditions (A), (B₁), and (B₂) in the theorem are sufficient in order for expression (15) to hold when the sequence of weights satisfy the constraints (16), (17) and (18).

To do so, we show that there is no sequence of weights ω^0 satisfying (16), (17) and (18) s.t. the inequality in (15) is reversed and at the same time, the conditions of the theorem are satisfied.

Consider first condition (A), if $X_T^1 = 0$ then (B₁) and (B₂) require that the sufficient condition in order for (15) to hold is that $X_t^2 \geq 0$ for all $t \in \{0, 1, \dots, T-1\}$. This is indeed the case because when $X_t^2 \geq 0$ the condition in (15) is satisfied for any $\omega_t \geq 0$.

Here we prove that this is the case also for $X_T^1 > 0$.

Consider (B₁), without loss of generality let $X_t^2 = 0$ for all $t \geq H_\alpha := \left(\frac{1}{\alpha} - 1 \right)$. In fact, if the sufficiency condition holds in this case it will also hold for $X_t^2 \geq 0$ for all $t \geq H_\alpha := \frac{1}{\alpha} - 1$ by recalling that the constraint (18) should hold.

If $X_t^2 = 0$ for all $t \geq \frac{1}{\alpha} - 1$, then (15) becomes

$$X_T^1 \cdot \left[1 - \sum_{t=1}^T t \cdot \omega_t \right] + \sum_{t=1}^{Int(\frac{1}{\alpha}-1)} \omega_t \cdot X_{t-1}^2 \geq 0, \quad (19)$$

where $Int(x)$ denotes the integer part of x .

Now, given that it has been assumed that $X_T^1 > 0$ and (17) holds, the first term in (19) is always non negative. In order to get a negative value for (19), we would need to attach a positive weight ω_t^0 to some negative X_{t-1}^2 where $t \leq \frac{1}{\alpha}$ for $t \in \{1, \dots, T\}$.

If we set a $\omega_{t_0}^0 > 0$ for only a time period $t_0 \leq \left(\frac{1}{\alpha} - 1 \right)$, with $\omega_t^0 = 0$, for all $t \neq t_0$, then we obtain condition (B₂), as argued in the related necessity part of the proof. We need to show that the value of (19) is not reduced if at least two positive weights, $\omega_{t_0}^0 > 0$ and $\omega_{t_1}^0 > 0$, for the time periods $t_0, t_1 < \frac{1}{\alpha}$, are assigned to two negative outcomes $X_{t_0-1}^2 < 0$ and $X_{t_1-1}^2 < 0$. If this is the case, the previous line of argument can be generalized to any number of periods $t_i < \frac{1}{\alpha}$.

Going back to the set of weights ω^0 , with $\omega_{t_0}^0 = \alpha$ and $\omega_t^0 = 0$ for all $t \neq t_0$, suppose that ω^0 is set such that the value in (19) is minimized. We have just seen that condition (B₂) guarantees that (19) is non negative.

If, in exchange, we move from ω^0 to $\omega^{0'}$ where $\omega_{t_0}^0$ is reduced by $\varepsilon \in (0, \alpha)$ and simultaneously is set $\omega_{t_1}^{0'} = \varepsilon$ where $X_{t_1-1}^2 < 0$, then we obtain the following change in expression (19)

$$-\varepsilon \cdot \left[-X_T^1 \cdot t_0 + X_{t_0-1}^2 \right] + \varepsilon \cdot \left[-X_T^1 \cdot t_1 + X_{t_1-1}^2 \right] \quad (20)$$

First of all, we need to show that this "transfer of weight" from $\omega_{t_0}^0$ to $\omega_{t_1}^0$ is admissible according to the constraints (16) and (17). Recall that $t_0, t_1 < \frac{1}{\alpha}$ and therefore $\alpha < \frac{1}{t_0}, \alpha < \frac{1}{t_1}$. When $\omega_{t_0}^0 = \alpha$ it results also that $\omega_{t_0}^0 = \alpha < \frac{1}{t_0}$, therefore $\omega_{t_0}^0 t_0 < 1$, thereby satisfying (17).

Recall that $\omega_{t_0}^{0'} = (\omega_{t_0}^0 - \varepsilon)$ and $\omega_{t_1}^{0'} = \varepsilon$, thus $\omega_{t_0}^{0'} + \omega_{t_1}^{0'} = \alpha$. We know that both $t_0 < \frac{1}{\alpha}$ and $t_1 < \frac{1}{\alpha}$. Rearranging them in an increasing order we get

$$\begin{aligned} \min\{t_0, t_1\} &< \max\{t_0, t_1\} < \frac{1}{\alpha} \\ \alpha \min\{t_0, t_1\} &< \alpha \max\{t_0, t_1\} < 1. \end{aligned}$$

This helps us conclude that any convex combination between a general $\omega_{t_0}^0$ and $\omega_{t_1}^0$ satisfying (16), s.t. $\omega_{t_0}^{0'} + \omega_{t_1}^{0'} \leq \alpha$, will lead to a value for $\sum_{t=1}^T t \cdot \omega_t^{0'}$ included in the interval $[\alpha \min\{t_0, t_1\}, \alpha \max\{t_0, t_1\}]$ and is by definition < 1 , which verifies (17) when (16) is binding.

Going back to (20), in order to further reduce the value of the expression in (19) it is necessary that

$$-X_T^1 \cdot t_0 + X_{t_0-1}^2 < -X_T^1 \cdot t_1 + X_{t_1-1}^2.$$

However, if this were true, then $\omega_{t_0}^0 = \alpha$ would not minimize (19) because it will be possible to set $\omega_{t_1}^0 = \alpha$ and $\omega_{t_0}^0 = 0$ and then reduce the value of (19). This statement creates a contradiction, thereby violating the initial assumption and in this way confirming that (B_2) is also a sufficient condition in order for (15) to hold.

To conclude we consider explicitly the case $\alpha = 0$, if this is the case then $NPV_v(x) = X_T^1$. ■