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Moderating Government*

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Abstract

We consider a model where policy motivated citizens vote in two simultaneous elections, one for the President who is elected by majority rule, in a single national district, and one for the Congressmen, each of whom is elected by majority rule in a local district. The policy to be implemented depends not only on who is elected President but also on the composition of the Congress. We characterize the equilibria of the model using a conditional sincerity concept that takes into account the possibility that some voters may be simultaneously decisive in both elections. Such a concept emerges naturally in a model with trembles. A crucial feature of the solution is the moderation of Government.

1 Introduction

The present paper shows that Government moderation emerges naturally in a theoretical model where policy motivated citizens vote strategically in a Presidential and a Congressional election. Specifically, we consider a model with a finite number of voters who vote in two elections occurring at the same time, one for the President, who is elected by majority rule, in a single national district, and one for the Congressmen, each of whom is elected by majority rule in a local district. We assume that only one political issue is at stake and thus the policy space is unidimensional. There are two

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parties, each with a given position – one party is leftist, the other rightist- as regards the policy they would like to implement. The institutional architecture is such that the policy that will be implemented depends not only on who is elected President but also on the composition of the Congress. We conceive of final policy as being the outcome of negotiations of the President with the Congress and between majority and minority in Congress. Hence, for a given composition of the Congress, the policy implemented will be more leftist if the President is leftist than if the President is rightist; for a given elected President, the policy implemented will be increasingly more leftist for every extra leftist candidate who is elected to the Congress. Each voter has a preferred policy outcome and votes in each election for one of the two parties to try to obtain the implementation of a policy which is as close as possible to her preferred one.

We analyze the model using the tools of non-cooperative game theory. We restrict attention to the pure strategies of voters. If we were to adopt the Nash solution concept, many unpalatable pure strategy combinations would emerge as equilibria of the game and as a consequence a host of unpalatable equilibrium outcomes. Indeed, provided no voter is decisive – in the political economy jargon, pivotal- in either election, any strategy combination is a Nash equilibrium, since no voter alone can change the electoral result and thus the policy outcome. Some of these equilibria are unpalatable because voters could play (weakly) dominated strategies. Alternatively, we could require voters to vote for their preferred party - the closest to their preferred policy, thus making sure that their strategies are undominated. In voting models where the institutional set-up is such that the voters are deciding between two possible policy outcomes, as it would be the case here if there was only the Presidential race and then the elected President was free to implement his preferred policy, the solution concept prescribing that voters should use undominated strategies - i.e. vote sincerely- would be effective and would lead to the sharp prediction that the final policy outcome is the one preferred by the median voter. Unfortunately, due to our institutional set-up where several outcomes are possible depending on the composition of the Congress as well as the elected President, requiring sincere voting does not help. Suppose, instead, that voters behave in a "conditionally" sincere manner in each election as follows. Given the result of the Congressional election, they vote for their preferred Presidential candidate (Presidential sincerity); given the result of the Presidential election and the result of the Congressional election outside their district, they vote for their preferred candidate for the Congress (District sincerity). If voters behave in a Presidential and District sincere way, the situation seems much more promising, since voters, in effect, compare two outcomes in the Presidential election and two in the legislative one, behaving as if they were pivotal in the Presidential election and in their district for the Congressional election. The beauty of Presidential and District sincerity in our setting is that it narrows down the possible policy outcomes to two, those preferred by the median voter of the entire population and the median voter of a critical district, and carries with it naturally the idea of Government moderation. Indeed, for a given elected President, District sincerity pins down the distribution of seats in Congress uniquely. Since, for a given elected President, the policy outcome is more leftist the higher is the number of districts carried by the left party, and the districts can be ordered according to how leftist their median voter is, the allocation of seats for the two parties in Congress in a district sincere equilibrium is pinned down uniquely as the one favored by the median voter of a critical district. In turn, for a given distribution of seats in Congress, the policy outcome is more leftist if the President is leftist, which implies that if the President is leftist, in a district sincere equilibrium, the number of seats won in Congress by the left cannot be higher than the number of seats won if the President is rightist.

Unfortunately, this conditional sincerity concept is sometimes at odds with Nash equilibrium. This happens when some voters turn out to be pivotal in both elections at the same time. A voter who turns out to be simultaneously pivotal in the two elections should be allowed to compare outcomes across elections, but the use of two separate sincerity concepts, one for the President, another for the candidate to the Congress in her district, does not allow her to perform such a comparison. As a result, Presidential and District sincerity may sometimes imply that a voter who is simultaneously pivotal in the two elections does not play a best response.

We thus introduce the concept of "joint" sincerity, whereby a voter votes in a way that supports her most preferred policy outcome after conditioning on the result of the Congressional election in the other districts. This concept allows to make comparisons across the Presidential and Congressional elections. Our final solution concept - which we call Joint Conditional Sincerity- blends Presidential and District sincerity with Joint sincerity. Presidential and District sincerity will apply to the voters who are not simultaneously pivotal in both elections, while Joint sincerity will apply to the voters, if any, who happen to be simultaneously pivotal in both elections. The Joint Conditional Sincerity solution concept is not in conflict with Nash equilibrium, being in fact a refinement of Nash equilibrium. Since for pivotal voters Joint sincerity implies Presidential and District sincerity, Joint Conditional Sincerity has the same nice properties - specifically the moderation effect, in some cases even in the strong form of divided Government- of the previous concept without its shortcomings. Moreover, we can show that a pure strategy Joint Conditional Sincere equilibrium always exists.

Interestingly, thanks to the fact that in a pure Joint Conditional Sincere equilibrium the strategies of all voters are presidential and district sincere, we can provide a very simple and intuitive sufficient condition for Government to be divided. In a Joint Conditional Sincere equilibrium, Government is divided, if the median voter of the median district prefers a Congress where the party of the elected President has a one seat majority to a Congress where such a party has one less seat. Hence, the presence of a sufficiently moderate voter in a "central" district determines when Government is divided. This accords with intuition, since it is precisely moderate voters who should vote to counterpoise a President of one party with a Congress leaning towards the other party, in order to obtain a moderate policy.

The theoretical underpinnings of Joint Conditional Sincerity can be found in the appropriate notion of trembling-hand perfection for our framework. As we have seen, the Nash solution concept may include some unreasonable situations as equilibria, since it allows players to use weakly dominated strategies when they are not decisive. The idea of perfection consists in trying to eliminate unreasonable Nash Equilibria making sure that players are decisive with positive probability, assuming that players "tremble" and make mistakes with some probability when playing. Trembling-hand perfect equilibria, as defined by Selten (1975), are the limiting equilibria when the probability with which the players tremble and make mistakes vanishes. Joint Conditional Sincerity is related, since it requires voters to behave as if they were decisive. However, we need to exert some care when thinking about trembles here, since, in our voting game, a strategy of a voter consists of two actions, a vote for the Presidential election and a vote for the Congressional election. The material procedure of the vote in the American elections for the President and the Congress, where the voter is required to mark her vote on the ballot twice, once for the Presidential candidate, once for the congressional candidate, places some restrictions on the type of trembles that we should consider, implying, in particular, that the mistakes should be independent across the two actions. We call the appropriate perfection concept for our voting model with ballot trembles, b(allot)-perfection. We can show that a pure strategy combination is Joint Conditional Sincere if and only if it is b-perfect.

This paper continues in the line of research began in the paper De Sinopoli, Ferraris and Iannantuoni (2010), where we studied a society electing candidates belonging to two parties to a national Parliament, with a policy outcome which is a function of the number of seats the two parties win in the election. There, we analyzed two electoral rules, multidistrict majority and single-district proportional, and we proved that under both systems there is a unique pure strategy perfect equilibrium outcome. We compared the outcomes under the two systems and found that the single-district proportional system tends to moderate the outcome for several - but, interestingly, not for all- possible distributions of voters.

In their work on divided Government, Alesina and Rosenthal (1995, 1996) have examined a similar voting model featuring a simultaneous election of the executive and legislature. Their model differs from ours in terms of modelling strategy and game theoretic analysis. In terms of modelling strategy, they assume that the legislature is elected by single district proportional rule rather than multidistrict majority rule. More importantly, in their model, the strength of a party in the Congress is given by the *share of votes* it obtains in the election, while, in our setting, it is the *number of seats* a party has in Congress that matters. As regards the analysis, their approach is cooperative - they assume a continuum of voters and use a coalition proof notion of equilibrium as a solution concept, rather than fully non-cooperative as in our paper. From our analysis, it emerges that the moderation of Government - and divided Government- arises naturally insisting on the non-cooperative nature of the situation and on the appropriate conditional sincerity concept. Our work is also related to an unpublished paper by Ingberman and Rosenthal (1997), who analyse a similar voting game with multiple contests using a conditional sincerity concept akin to ours. Unlike them, thanks to the higher tractability of our setting, we are able to provide a complete solution.

The rest of the paper is structured as follows. Section 2 presents the model. Section 3 makes some progress towards its solution, using Presidential and District Sincerity. Section 4 contains the complete solution, using Joint Conditional Sincerity, and discusses the relationship with perfection. Section 5 concludes.

2 The Model

Consider a society where voters simultaneously elect a President by plurality rule and a Congress of k members by multidistrict majority rule.

The policy space. The unidimensional policy space $\mathbb{X}$ is a closed interval of the real line. Without loss of generality we assume $\mathbb{X} = [0, 1]$.

Parties. There are two parties, Left and Right, indexed by $p \in P = \{L, R\}$. Each party p is characterized by a policy position $\theta_p \in \mathbb{X}$, such that $0 \leq \theta_L < \theta_R \leq 1$. We consider the policy position of party p and its presidential candidate's position as one and the same. We will thus refer to the position of a party and of its presidential candidate interchangeably.

Voters. There is a finite set of voters $N = \{1, 2, ..., n\}$. Each voter $i \in N$ has a most preferred policy (her bliss point) $\theta_i \in \mathbb{X}$. Voters' preferences are single peaked and symmetric. Each voter i casts her vote for one of the parties in both elections. Hence, a pure strategy of voter i is $s_i = (s_{i,1}, s_{i,2}) \in S_i = \{LL, LR, RL, RR\}$ where $s_{i,1} = \{L, R\}$ refers to the presidential election and $s_{i,2} = \{L, R\}$ to the congressional one. A pure strategy combination is $s \in S = S_1 \times S_2 \times ... \times S_n$.

Voters inhabit k districts, indexed by $d \in D = \{1, 2, ..., k\}$. Let N_d be the set of voters in district d, i.e. $N_1, N_2, ..., N_k$ is the partition of N in the k districts. With uppercase letters we denote sets, and with lowercase ones their cardinality: N_d is the set of voters in district d, and n_d the number of voters in such a district. Let m be the bliss point of median voter of the entire population and $m_d \in M = \{m_1, ..., m_k\}$ be the bliss point of the median voter in district d, and, without loss of generality, assume that $m_1 \leq m_2 \leq ... \leq m_k$.

Assumption 1. *The number of voters in district d, n_d, is odd for all d. The number of districts, k, is odd.*

The assumption that there is an odd number of voters both overall and in each district makes sure that the electoral result does not end in a tie.[1]

The electoral rule: the President. Voters elect a President by majority rule. Hence, party L wins the presidential race iff it obtains more votes, i.e. $\{\#i \in N \ s.t. \ s_{i,1} = L\} >$

[1] Alternatively, we could have used a deterministic tie-breaking rule and defined the median accordingly.

$\{\#i \in N \ s.t. \ s_{i,1} = R\}$. We define as $P(s)$ the elected president under the pure strategy combination s.

The electoral rule: the Congress. Voters cast their ballots to elect a Congress composed of k representatives elected by multidistrict majority rule, i.e. in each district voters elect a representative belonging to either party L or party R by majority rule. District d is carried by party L iff it obtains more votes in such a district, i.e. $\{\#i \in N_d \ s.t. \ s_{i,2} = L\} > \{\#i \in N_d \ s.t. \ s_{i,2} = R\}$. We define as $d^L(s)$ the number of districts carried by L under the pure strategy combination s.

The policy outcome. The policy outcome $X(P(s), d^L(s)) \in \mathbb{X}$ depends on who is elected President and on the composition of the Congress under the pure strategy combination s.

We make the following assumption on the policy outcome.

Assumption 2a. $X(R, j) > X(L, j)$ *for every* $j \in \{0, 1, ..., k\}$.

Assumption 2b. $X(P, j') > X(P, j)$ *for* $j > j\prime$ *and* $P = L, R$.

The first part of the assumption says that for any composition of the Congress, the policy outcome is more rightist when the President is rightist than in the case in which the President is leftist. The second part of the assumption says that given an elected President, the outcome is more rightist if the Congress has a higher number of rightist members.

Indifference Conditions. It is useful to define an indifference condition between a leftist or rightist President, given a composition of the Congress.

Definition 1 *Given* $j \in \{0, 1, ..., k\}$ *(i.e. given a composition of the Congress), define*

$$\alpha_j = \frac{X(R, j) + X(L, j)}{2}.$$

Analogously, we define the following indifference condition between a Congress with j or $j-1$ leftist members, given an elected President.

Definition 2 *Given $P \in \{L, R\}$ elected president, define for $j \in \{1, 2, ..., k\}$*

$$\alpha_j^P = \frac{X(P, j) + X(P, j-1)}{2}.$$

Finally, we define the following indifference condition between the outcomes when a Congress has j or $j-1$ leftist members and two different candidates are elected President.

Definition 3 *Define for $j \in \{1, 2, ..., k\}$ and $P, P' \in \{L, R\}$*

$$\alpha_j^{PP'} = \frac{X(P, j) + X(P', j-1)}{2}.$$

We make the assumption that no bliss point coincides with any of the indifference conditions in the three definitions above.

Assumption 3a. *There exists no voter i s.t. $\theta_i = \alpha_j$, for $j \in \{0, 1, ..., k\}$.*

Assumption 3b. *There exists no voter i s.t. $\theta_i = \alpha_j^P$, for $j \in \{1, 2, ..., k\}$ and $P \in \{L, R\}$.*

Assumption 3c. *There exists no voter i s.t. $\theta_i = \alpha_j^{PP'}$, for $j \in \{1, 2, ..., k\}$ and $P, P' \in \{L, R\}$.*

Hence, voters are never indifferent between any two outcomes.

3 Towards the Solution

3.1 An Example

Before proceeding to the solution of the model, an illustrative example is in order (see Figure 1).

Example 1. Consider a society with $n = 9$ voters. The leftist candidate for the Presidency has a bliss point $\theta_L = 0.15$ and the rightist candidate $\theta_R = 0.85$. The Congress has three seats. The electorate is divided into three single-member districts. The first district has three voters with bliss point 0.21, the second district by three voters with bliss point 0.49, the third district by three voters with bliss point 0.79. Policy outcomes have been chosen to satisfy Assumption 2 (a, b). The outcomes are:

$$\begin{aligned}
0.15 &= X(L,3) < 0.25 = X(L,2) < 0.35 = X(L,1) \\
&< 0.45 = X(L,0) < 0.55 = X(R,3) < 0.65 = X(R,2) \\
&< 0.75 = X(R,1) < 0.85 = X(R,0),
\end{aligned}$$

which is more leftist when the President is leftist, for a given composition of the Congress, and more leftist when the Congress has one more leftist member, given an elected President. Moreover, in this example, the President is more powerful than the Congress, as it can be seen noticing that, say, the policy outcome when the President is leftist and the Congress has no leftist members is to the left of the policy outcome when the President is rightist and the Congress has three leftist members (0.45<0.55). The objective of the voters is to minimize the distance from their bliss point.

Nash Equilibrium. The strategy combination whereby all voters vote L in both elections is a Nash Equilibrium, with outcome $X(L,3) = 0.15$, since no voter can change the outcome by changing her strategy. Likewise, the strategy combination whereby all voters vote R in both elections is a Nash Equilibrium, with outcome $X(R,0) = 0.85$. In fact, any strategy combination whereby no voter can change the outcome alone is an equilibrium.

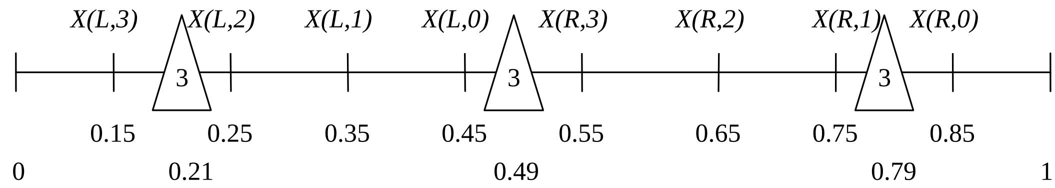

Figure 1: Example 1. Voters and Outcomes

The example shows that we should be careful when choosing the solution concept for our model. Not even the undominated equilibrium concept, i.e. sincere voting, which is often used in voting models with two parties to narrow down the multiplicity associated with the Nash solution concept, would be of much help in our context, since the policy outcome depends not only on who is elected President but also on how many seats each party gets in Congress. Next, we define an intuitive solution concept which will take us part of the way towards the solution of the model.

3.2 Presidential and District Sincerity

Although sincerity per se does not help, a "conditional" version of sincerity turns out to be key to advance towards the solution of our voting game. By conditional sincerity we mean that, in each election, holding the outcomes of other elections constant, each voter votes according to her preferred policy outcome. In our model, conditional sincerity implies two things. On the one hand, a voter casts her ballot for her preferred Presidential candidate in the Presidential election, taking as given the result of the Congressional election. On the other hand, a voter votes sincerely in the Congressional election casting her ballot for her preferred candidate in her district, taking as given the result of the Presidential election and the result in the Congressional election outside her district. We will refer to these two conditional sincerity concepts as Presidential sincerity and District sincerity respectively.

Presidential Sincerity. Presidential sincerity means that, in the Presidential election, a voter casts her ballot for her preferred President, given the composition of the Congress.

Definition 4 *Presidential sincerity. Given a strategy combination s, s_i is presidential sincere for voter i if, given the result in the Congressional election $d^L(s)$,*

$$\theta_i < \alpha_{d^L(s)} \textit{ iff } s_{i,1} = L.$$

A strategy combination s is presidential sincere (PS) if s_i is presidential sincere for all i.

Presidential sincerity implies that the crucial element to determine the result of the Presidential election is the position of the median voter of the entire population, m, relative to $\alpha_{d^L(s)}$ which marks the indifference between a leftist and a rightist President, given the result in the Congressional election $d^L(s)$.

District Sincerity. District sincerity means that, in the Congressional election, a voter who prefers L (resp. R) to win in her district casts her ballot for L (R), given the elected President and given the strategies of the voters in the other districts.

Given a strategy combination s and a district d, let $d^L_{-d}(s)$ the number of districts carried by L in the remaining D/d districts.

Definition 5 *District sincerity. Given a strategy combination s, s_i is district sincere for voter i in district d if, given the result in the Presidential election $P(s)$ and given the result in the Congressional election $d^L_{-d}(s)$ in the remaining D/d districts, the following holds*

$$\theta_i < \alpha^{P(s)}_{d^L_{-d}(s)+1} \textit{ iff } s_{i,2} = L.$$

A strategy combination s is district sincere (DS) if s_i is district sincere for all i.

District sincerity implies that the position of the median voter in a district d, m_d, relative to $\alpha^{P(s)}_{d^L_{-d}(s)+1}$ which marks the indifference between a Congress with $d^L_{-d}(s)+1$ seats for the left or one less, given the result in the Presidential election, $P(s)$, and the result in the Congressional election $d^L_{-d}(s)$ outside district d, will play an important role in the analysis. We, thus, introduce the following definition.

Definition 6 *Given the result in the President election, let us define*

$$\overline{d}_L = \begin{cases} 0 & if\ m_1 > \alpha_1^L \\ \max d\ s.t.\ m_d < \alpha_d^L & if\ m_1 < \alpha_1^L, \end{cases}$$

when L is elected president; and

$$\overline{d}_R = \begin{cases} 0 & if\ m_1 > \alpha_1^R \\ \max d\ s.t.\ m_d < \alpha_d^R & if\ m_1 < \alpha_1^R, \end{cases}$$

when R is elected president.

Each of these conditions characterizes the right-most district where, given a number of elected representatives for L outside the district and L (resp. R) is elected President, the median voter would like a Congress with one more L representative.

We define an outcome as a pure DS outcome if it is induced by a DS pure strategy combination and a pure PS and DS outcome if it is induced by a PS and DS pure strategy combination.

We show, in the context of Example 1, how presidential sincerity and district sincerity work.

Example 1 (Continued).

Presidential Sincerity. To understand which outcomes are presidential sincere, first suppose no leftist candidate has been elected member of Congress and notice that the median voter of the entire population has her bliss point in 0.49 which is smaller than $\alpha_0 = 0.65$; suppose one leftist candidate has been elected member of Congress and notice that the median voter of the entire population has her bliss point in 0.49 which is smaller than $\alpha_1 = 0.55$; suppose next that two leftist candidates have been elected to the Congress and notice that $0.49 > \alpha_2 = 0.45$; suppose finally that three leftist candidates have been elected to the Congress and notice that $0.49 > \alpha_3 = 0.35$. Hence, the possible PS pure outcomes are $X(L,0)$, $X(L,1)$, $X(R,2)$, $X(R,3)$.

District Sincerity. To understand which outcomes are district sincere, first suppose L is elected President and notice that the median voter in district 1 has her bliss point in 0.21 which is smaller than $\alpha_1^L = 0.4$, while the median voter in district 2 has her bliss point in 0.49 which is bigger than $\alpha_2^L = 0.3$; suppose next R is elected President and notice that $\alpha_1^R = 0.8 > 0$, $\alpha_2^R = 0.7 > 0.49$, while the median voter of district 3 has her bliss point 0.79 which is bigger than $\alpha_3^R = 0.6$. Hence, the possible DS pure outcomes are $X(L,1)$, $X(R,2)$.

The next Proposition shows that district sincerity, in particular, narrows down the number of outcomes, which - as we noticed before- would otherwise be vast even if only equilibrium outcomes are taken into account, to two.

Proposition 1 *There are two district sincere pure outcomes*

$$(i) \quad X\left(L,\overline{d}_L\right) \text{ if } L \text{ is elected president;}$$
$$(ii) \quad X\left(R,\overline{d}_R\right) \text{ if } R \text{ is elected president.}$$

Proof. First we show that the two outcomes can be obtained by district sincere strategy combinations.

Case (i). Consider any strategy combination s where L is elected president while the legislative part of s is given by: every voter i in district $d \leq \overline{d}_L$ such that $\theta_i < \alpha_{\overline{d}_L}^L$ votes for L in the legislature, i.e. $s_{i,2} = L$; every voter i in district $d \leq \overline{d}_L$ such that $\theta_i > \alpha_{\overline{d}_L}^L$ votes for R in the legislature, i.e. $s_{i,2} = R$; every voter i in district $d > \overline{d}_L$ such that $\theta_i < \alpha_{\overline{d}_L+1}^L$ votes for L in the legislature, i.e. $s_{i,2} = L$; every voter i in district $d > \overline{d}_L$ such that $\theta_i > \alpha_{\overline{d}_L+1}^L$ votes for R in the legislature, i.e. $s_{i,2} = R$. All these strategy combinations give rise to the outcome $X\left(L,\overline{d}_L\right)$ because, L wins the president and exactly $\overline{d}_L$ districts (i.e., in the first $\overline{d}_L$ districts the median are to the left of $\alpha_{\overline{d}_L}^L$ while in the remaining to the right). Moreover, s is district sincere by construction. Case (ii). *Mutatis mutandis.*

We now prove that no other pure DS outcome exists.

Case (i). Suppose we have a pure DS where L is elected president and $\widehat{d}_L \neq \overline{d}_L$ districts are won by L. District-sincerity implies that in districts won by L, every voter i with $\theta_i < \alpha^L_{\widehat{d}_L}$ votes for L, and every voter i with $\theta_i > \alpha^L_{\widehat{d}_L}$ votes in favor of party R. Moreover, in districts in which R is getting the majority, voter i with $\theta_i < \alpha^L_{\widehat{d}_L+1}$ votes for L, and voter i with $\theta_i > \alpha^L_{\widehat{d}_L+1}$ votes for R. Suppose first that $\widehat{d}_L < \overline{d}_L$, then it must be that $\alpha^L_{\overline{d}_L} \leq \alpha^L_{\widehat{d}_L+1} < \alpha^L_{\widehat{d}_L}$ and hence district sincerity implies that party L gets at least $\overline{d}_L$ districts, which contradicts $X\left(L, \widehat{d}_L\right)$ being a district sincere outcome. Suppose next that $\widehat{d}_L > \overline{d}_L$ which implies $\alpha^L_{\widehat{d}_L+1} < \alpha^L_{\widehat{d}_L} \leq \alpha^L_{\overline{d}_L+1}$ and this with district sincerity and the fact that $\alpha^L_{\overline{d}_L+1} < m_{\overline{d}_L+1}$implies party R wins at least $\left(k - \overline{d}_L\right)$ districts, and, hence, party L wins at most $\overline{d}_L$ districts contradicting $X\left(L, \widehat{d}_L\right)$ being a district sincere outcome. Case (ii). *Mutatis mutandis*. ■

3.3 The Moderation of Government

The next Lemma, which follows immediately from our definitions, says, together with Proposition 1, that the number of members of Congress for the left when the elected President is leftist in a pure DS outcome cannot be higher than the number of members of Congress for the left when the president is rightist.

Lemma 2 $\overline{d}_L \leq \overline{d}_R$.

Proof. Since $\theta_L < \theta_R$, by definition 2 we have that for every d, $\alpha^L_d < \alpha^R_d$ and, by definition 6, $\overline{d}_L \leq \overline{d}_R$. ■

Hence, district sincerity tends to moderate the outcome: if the President is leftist, the composition of the Congress is less leftist than if the President is rightist. Interestingly, in our model the existence of a PS and DS pure outcome follows from this moderation result, as the next Proposition shows.

Proposition 3 *At least one PS and DS pure outcome exists.*

Proof. Consider all DS pure strategy combinations as in the first step, case (i) of Proposition 1. In a PS strategy combination all the voters i vote for L as president, i.e. $s_{i,1} = L$, iff $\theta_i < \alpha_{\overline{d}_L}$, hence $X\left(L, \overline{d}_L\right)$ is a DS and PS pure outcome if $m < \alpha_{\overline{d}_L}$. Similarly, $X\left(R, \overline{d}_R\right)$ is a DS and PS pure outcome if $m > \alpha_{\overline{d}_R}$. Lemma 2 implies that $\alpha_{\overline{d}_R} \leq \alpha_{\overline{d}_L}$, hence at least one of the two cases holds. ■

The moderation result of Lemma 2 implies that the party losing the Presidency gains a stronger Congressional force than in the case in which it wins the Presidency. In some cases, as in Example 1, a stronger result emerges, whereby the party losing the Presidency, gains the majority in Congress.

Example 1 (Concluded).

Two PS and DS outcomes. The PS and DS outcomes are $i)$ $X(L, 1) = 0.35$ and $ii)$ $X(R, 2) = 0.65$. The two outcomes can be supported by the following pure strategy combinations: $i)$ the three voters in district 1 vote for L in the legislative election and for L in the presidential election, the three voters in district 2 vote for R in the legislative election and for L in the presidential election, the three voters in district 3 vote for R in the legislative election and for R in the presidential election; $ii)$ the three voters in district 1 vote for L in the legislative election and for L in the presidential election, the three voters in district 2 vote for L in the legislative election and for R in the presidential election, the three voters in district 3 vote for R in the legislative election and for R in the presidential election.

Divided Government. In the first PS and DS pure outcome, L is elected President and R has the majority in Congress, in the other PS and DS pure outcome, R is elected President and L has the majority in Congress. This example produces divided government[2], in the sense that if the President is leftist the Congressional majority is rightist and if the President is rightist, the Congressional majority is leftist.

[2] We will provide a formal definition of divided Government later on.

3.4 A Stumbling Block

So far we have shown that a pure PS and DS outcome, i.e. an outcome induced by a district and presidential sincere pure strategy combination, exists as a consequence of the inherent moderation of our environment. We have not said anything yet about the relationship between PS and DS pure strategy combinations and Nash Equilibrium (henceforth, equilibrium), though. Example 1 seems to suggest that the former should be a refinement of the latter, in the sense that any PS and DS strategy combination is an equilibrium but the converse does not hold. However, the example has the special feature that no voter is pivotal, i.e. decisive, in both elections (in fact, no voter is pivotal in either election). This turns out to be crucial.

Definition 7 *A voter is pivotal in the presidential (resp. legislative) election if changing her vote in the presidential (legislative) election changes the presidential (legislative) outcome.*

The next Proposition, shows that if no voter is decisive in both elections simultaneously, then the solution concept we have used so far does not suffer from major drawbacks, in the sense that every PS and DS pure strategy combination is an equilibrium.

Proposition 4 *If no voter is pivotal in both the presidential and legislative elections, a PS and DS pure strategy combination is an Equilibrium.*

Proof. Any strategy is a best reply for a voter who is not pivotal in either election. If a voter is pivotal just in the presidential (resp. legislative) election, her set of pure best replies coincides with the set of her presidential (resp. district) sincere strategies. Hence, the result. ■

On the other hand, as the next example shows, if a voter is pivotal in both elections, some PS and DS pure strategy combinations may not be equilibria.

Example 2. Consider the following variation of Example 1. In district 2, there is one voter with bliss point in 0.21, one voter with bliss point in 0.49 and one voter with bliss point in 0.79. Otherwise the example remains the same.

PS and DS outcomes. As in example 1, the PS and DS pure outcomes are $X(L,1) = 0.35$ and $X(R,2) = 0.65$, supported by strategy combinations whereby the four voters with bliss point in 0.21 vote for the left in both elections, the four voters with bliss point in 0.79 vote for the right in both elections and the remaining voter splits her vote between the left and the right, voting for a leftist President and a rightist member of Congress or vice versa.

A *PS and DS is not an equilibrium.* Notice that the outcome $X(L,1) = 0.35$ is closer to 0.49 than $X(R,2) = 0.65$. Hence, the voter with bliss point in 0.49, who - under the PS and DS pure strategy combinations- is pivotal in both elections, strictly prefers a leftist President and a Congress with a rightist majority to the opposite situation. The second PS and DS pure strategy combination is not an equilibrium.

There is a stumbling block on our way. However unlikely the event whereby a voter turns out to be pivotal in two simultaneous mass elections may be, it is still the case that when such an event occurs, the use of separate conditional sincerity concepts, one for the Presidential election and one for the Congressional election, is not appropriate. When voters are pivotal in at most one election, comparing the outcomes of one election holding the other fixed may be appropriate, but when a voter turns out to be pivotal in both elections, she should be allowed to compare outcomes across elections.

4 The Solution

4.1 Joint Conditional Sincerity

The conditional sincerity requirements we have been using so far, fail to produce an equilibrium when the situation calls for a comparison across elections, which is the case when some voters are pivotal simultaneously in the two elections. We address this difficulty, introducing a conditional sincerity concept we call "joint sincerity". Given a strategy combination s and a voter i in district d, let

$$\widetilde{X}_{d^L_{-d}(s)} = \left\{X(L, d^L_{-d}(s)+1), X(L, d^L_{-d}(s)), X(R, d^L_{-d}(s)+1), X(R, d^L_{-d}(s))\right\}.$$

Definition 8 *Joint sincerity. Voter i's pure strategy s_i is jointly sincere if:*

$s_i = LL$ when i's most preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$ is $X(L, d^L_{-d}(s)+1)$;

$s_i = LR$ when i's most preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$ is $X(L, d^L_{-d}(s))$;

$s_i = RL$ when i's most preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$ is $X(R, d^L_{-d}(s)+1)$;

$s_i = RR$ when i's most preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$ is $X(R, d^L_{-d}(s))$.

A strategy combination s, is jointly sincere (JS) if s_i is jointly sincere for every i.

That is, a voter's strategy is jointly sincere if she votes in a way that supports her most preferred policy outcome after conditioning on the result of the Congressional election in the other districts.

We will complement presidential and district sincerity with joint sincerity for those voters who are twice pivotal. We will use as a shorthand for our solution concept the label JCS, jointly conditional sincere.

Definition 9 *A pure strategy combination is JCS if the strategies of voters who are not twice pivotal are presidential and district sincere, and the strategies of twice pivotal voters are jointly sincere.*

The following Proposition shows that a JCS pure strategy combination is an equilibrium.

Proposition 5 *A JCS pure strategy combination is an Equilibrium.*

Proof. If a voter is pivotal in both elections, her unique best reply coincides with her jointly sincere strategy. The rest follows from Proposition 4. ■

On the other hand, not all Nash Equilibria are JCS.

Example 2 (Concluded)

JCS. We can now see that in a JCS pure strategy combination the voter with bliss point in 0.49, who is pivotal in both elections, will vote for a leftist President and a rightist member of Congress, i.e. in a jointly sincere manner, while the others still in a district and presidential sincere manner, inducing $X(L,1) = 0.35$ as the unique JCS outcome.

We can, moreover, prove that the moderation result derived in Lemma 2 when presidential and district sincerity applies, carries over to the situation in which JCS applies. As an intermediate step, the next Lemma proves that for a twice pivotal voter a jointly sincere strategy is presidential and district sincere.

Lemma 6 *For a twice pivotal voter, her jointly sincere strategy is presidential and district sincere.*

Proof. Consider a pure strategy combination s whereby a voter h is twice pivotal and uses her jointly sincere strategy. The outcome induced by s is $X(P(s), d^L(s))$. Since the voter h is twice pivotal, she must be part of the one vote majority electing the President and the one vote majority electing a member of Congress. The voter h must be voting for the winning $P(s)$ President and her preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$, must be $X(P(s), d^L(s))$. Since she prefers $X(P(s), d^L(s))$ over $X(P', d^L(s))$ for $P' \neq P(s)$, her strategy is presidential sincere. The result in the Congress $d^L(s)$ may be equal to either $d^L_{-d}(s) + 1$ or $d^L_{-d}(s)$. Consider, first, $d^L(s) = d^L_{-d}(s) + 1$. The voter h must be voting for the winning leftist candidate and her preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$, must be $X(P(s), d^L(s))$. Since she prefers $X(P(s), d^L_{-d}(s) + 1)$ over $X(P(s), d^L_{-d}(s))$, her strategy is district sincere. Suppose $d^L(s) = d^L_{-d}(s)$. The voter h must be voting for the winning rightist candidate and her preferred outcome in $\widetilde{X}_{d^L_{-d}(s)}$, must be $X(P(s), d^L(s))$. Since she prefers $X(P(s), d^L_{-d}(s))$ over $X(P(s), d^L_{-d}(s) + 1)$, her strategy is district sincere. ■

Since jointly sincere strategies of twice pivotal voters are presidential and district sincere, in a JCS pure strategy combination, the strategies of all voters are presidential and district sincere. This has two interesting consequences. First, Proposition 1 applies and thus there are at most two pure JCS outcomes[3]. Second, Government moderation immediately follows.

Theorem 7 *Let s and s' be two JCS pure strategy combinations and $P(s) = L$ and $P(s') = R$, then $d^L(s) \leq d^L(s')$.*

In other words, if there are two JCS equilibria, one with a leftist President and one with a rightist President, the number of seats obtained by the left in Congress cannot be higher in the former than in the latter equilibrium.

Next, we prove that a pure strategy equilibrium exists where voters who are not twice pivotal vote in a district and presidential sincere manner and those who are twice pivotal, if any such voter exists, vote in a jointly sincere manner.

Proposition 8 *A JCS pure strategy combination exists.*

Proof. We show that a PS and DS pure strategy combination can always be constructed so that if a voter is twice pivotal under such a strategy combination, her strategy is already jointly sincere. Under presidential and district sincerity there are two possible pure outcomes, $X\left(L, \overline{d}_L^M\right)$ and $X\left(R, \overline{d}_R^M\right)$. We will distinguish three cases: *i*) when only the former PS and DS pure outcome arises, *ii*) when both arise, *iii*) when only the latter arises.

i) Suppose $m < \alpha_{\overline{d}_R^M}$, then the only PS and DS pure outcome is $X\left(L, \overline{d}_L^M\right)$. A PS and DS strategy combination inducing such an outcome will be as follows. In districts $d \leq \overline{d}_L$, if $\theta_i < \alpha^L_{\overline{d}_L}$ and $\theta_i < \alpha_{\overline{d}_L}$ then $s_i = LL$; if $\theta_i < \alpha^L_{\overline{d}_L}$ and $\theta_i > \alpha_{\overline{d}_L}$ then $s_i = RL$; if $\theta_i > \alpha^L_{\overline{d}_L}$ and $\theta_i < \alpha_{\overline{d}_L}$ then $s_i = LR$; if $\theta_i > \alpha^L_{\overline{d}_L}$ and $\theta_i > \alpha_{\overline{d}_L}$ then $s_i = RR$. In districts $d > \overline{d}_L$, if $\theta_i < \alpha^L_{\overline{d}_L+1}$ and $\theta_i < \alpha_{\overline{d}_L}$ then $s_i = LL$; if $\theta_i < \alpha^L_{\overline{d}_L+1}$ and $\theta_i > \alpha_{\overline{d}_L}$ then $s_i = RL$; if $\theta_i > \alpha^L_{\overline{d}_L+1}$ and $\theta_i < \alpha_{\overline{d}_L}$ then $s_i = LR$; if $\theta_i > \alpha^L_{\overline{d}_L+1}$

[3]By district sincerity, there can be at most two pure JCS outcomes, one with a leftist and one with a rightist President. The case with two outcomes can happen only if there exists a median voter of the entire population which is moderate enough, $\alpha_k < m < \alpha_1$.

and $\theta_i > \alpha_{\overline{d}_L}$ then $s_i = RR$. If no voter is pivotal in both elections under s, we are done. If a voter h is pivotal in the presidential election and in the legislative election in her district, then it has to be that she is part of the one vote majority which is voting for L as President and PS implies $\theta_h \leq m < \alpha_{\overline{d}_R}$. If such a twice pivotal voter is in a district $d \leq \overline{d}_L$ then in order for her to be twice pivotal she has to be part of the one vote majority in her district which is voting for a leftist candidate and DS implies that $\theta_h \leq m_d < \alpha_d^L$ for $d \leq \overline{d}_L$. The pure strategy of such a voter is already jointly sincere since her most preferred outcome in $\widetilde{X}_{\overline{d}_L - 1}$ is $X\left(L, \overline{d}_L\right)$. If such a voter is in a district $d > \overline{d}_L$ then to be twice pivotal it has to be that she is part of the one vote majority which is voting for L as a President and of the one vote majority in her district which is voting for R as a member of Congress, and PS and DS imply $\alpha_d^L < m_d \leq \theta_h \leq m < \alpha_{\overline{d}_R}$ for $d > \overline{d}_L$. Notice that if $\overline{d}_L < \overline{d}_R$, then $\alpha_{\overline{d}_R} < \alpha_{\overline{d}_L+1}^{RL}$ and thus $\alpha_{\overline{d}_L+1}^{RL} > \theta_h$ implying that the strategy of voter h is jointly sincere. Suppose $\overline{d}_L = \overline{d}_R$. For this to happen it has to be that in a district $d > \overline{d}_L$, $m_d > \alpha_d^R$. Hence, such a voter will have to have $\theta_h > \alpha_{\overline{d}_L+1}^R$, but $\alpha_{\overline{d}_L+1}^R > \alpha_{\overline{d}_L}$ which implies that she cannot be twice pivotal. By Lemma 2, $\overline{d}_L > \overline{d}_R$ never holds.

ii) Suppose $\alpha_{\overline{d}_R} < m < \alpha_{\overline{d}_L}$, then there are two PS and DS pure outcomes $X\left(L, \overline{d}_L\right)$ and $X\left(R, \overline{d}_R\right)$. Notice that for this case to be possible it has to be that $\overline{d}_L < \overline{d}_R$. The PS and DS pure strategy combination supporting the former outcome is as in case *i*) above, while the PS and DS pure strategy combination supporting the latter outcome is as follows. In districts $d \leq \overline{d}_R$, if $\theta_i < \alpha_{\overline{d}_R}^R$ and $\theta_i < \alpha_{\overline{d}_R}$ then $s_i' = LL$; if $\theta_i < \alpha_{\overline{d}_R}^R$ and $\theta_i > \alpha_{\overline{d}_R}$ then $s_i' = RL$; if $\theta_i > \alpha_{\overline{d}_R}^R$ and $\theta_i < \alpha_{\overline{d}_R}$ then $s_i' = LR$; if $\theta_i > \alpha_{\overline{d}_R}^R$ and $\theta_i > \alpha_{\overline{d}_R}$ then $s_i' = RR$. In districts $d > \overline{d}_R$, if $\theta_i < \alpha_{\overline{d}_R+1}^R$ and $\theta_i < \alpha_{\overline{d}_R}$ then $s_i' = LL$; if $\theta_i < \alpha_{\overline{d}_R+1}^R$ and $\theta_i > \alpha_{\overline{d}_R}$ then $s_i' = RL$; if $\theta_i > \alpha_{\overline{d}_R+1}^R$ and $\theta_i < \alpha_{\overline{d}_R}$ then $s_i' = LR$; if $\theta_i > \alpha_{\overline{d}_R+1}^R$ and $\theta_i > \alpha_{\overline{d}_R}$ then $s_i' = RR$. If no voter is pivotal in both elections under either strategy combination, we are done. If some voters are twice pivotal, as we have seen above under pure strategy combination s, the only voters who might be twice pivotal are those in districts $d > \overline{d}_L$ with $\alpha_d^L < m_d \leq \theta_h \leq m < \alpha_{\overline{d}_L}$.

Analogously, under strategy combination s' the only voters who might be twice pivotal are those in districts $d \leq \overline{d}_R$ with $\alpha_{\overline{d}_R} < m \leq \theta_l \leq m_d < \alpha_d^R$. To check for joint sincerity, we need to distinguish two cases for the outcome $X\left(L, \overline{d}_L\right)$ and two for the outcome $X\left(R, \overline{d}_R\right)$. If a. $X\left(L, \overline{d}_L\right) < X\left(R, \overline{d}_L + 1\right)$, then the pure strategy of voter h is jointly sincere if $\theta_h < \alpha_{\overline{d}_L+1}^{RL}$; if b. $X\left(L, \overline{d}_L\right) > X\left(R, \overline{d}_L + 1\right)$, then the pure strategy of voter h is jointly sincere if $\theta_h > \alpha_{\overline{d}_L+1}^{RL}$. If A. $X\left(L, \overline{d}_R - 1\right) < X\left(R, \overline{d}_R\right)$, then the pure strategy of voter l is jointly sincere if $\theta_l > \alpha_{\overline{d}_R}^{RL}$; if B. $X\left(L, \overline{d}_R - 1\right) > X\left(R, \overline{d}_R\right)$, then the pure strategy of voter h is jointly sincere if $\theta_l < \alpha_{\overline{d}_R}^{RL}$.

Consider the combination a. and A. Suppose $m < \alpha_{\overline{d}_L+1}^{RL}$, then $\alpha_{\overline{d}_L+1}^{RL} > \theta_h$ and the strategy of voter h is jointly sincere. Suppose $\alpha_{\overline{d}_R}^{RL} < m$, then $\theta_l > \alpha_{\overline{d}_R}^{RL}$ and the strategy of voter l is jointly sincere. Since $\overline{d}_L < \overline{d}_R$, it follows that $\alpha_{\overline{d}_R}^{RL} \leq \alpha_{\overline{d}_L+1}^{RL}$and thus at least one of the two cases holds.

Consider the combination b. and A. Notice that this case is possible only if $\overline{d}_R \geq \overline{d}_L + 2$. Observe that in this case $X\left(L, \overline{d}_L\right) > X\left(R, \overline{d}_L + 1\right) > X\left(R, \overline{d}_R\right)$ and $X\left(L, \overline{d}_L + 1\right) \geq X\left(L, \overline{d}_R - 1\right)$, hence $\alpha_{\overline{d}_L+1}^{L} > \alpha_{\overline{d}_R}^{RL}$. Since $\alpha_{\overline{d}_L+1}^{L} < m_{\overline{d}_L+1} \leq m_d \leq m \leq \theta_l$ (where $d > \overline{d}_L$), the pure strategy of l is jointly sincere.

Consider the combination a. and B. Notice that this case is possible only if $\overline{d}_R \geq \overline{d}_L + 2$. Observe that in this case $X\left(L, \overline{d}_L\right) > X\left(L, \overline{d}_R - 1\right) > X\left(R, \overline{d}_R\right)$ and $X\left(R, \overline{d}_L + 1\right) \geq X\left(R, \overline{d}_R - 1\right)$, hence $\alpha_{\overline{d}_R}^{R} < \alpha_{\overline{d}_L+1}^{RL}$. Since $\theta_h \leq m \leq m_d \leq m_{\overline{d}_R} < \alpha_{\overline{d}_R}^{R}$ (where $d \leq \overline{d}_R$), the pure strategy of h is jointly sincere.

Consider the combination b. and B. Here it has to be that $\alpha_{\overline{d}_L+1}^{L} < m < \alpha_{\overline{d}_R}^{R}$. However, in this case $\alpha_{\overline{d}_R}^{R} < \alpha_{\overline{d}_L+1}^{L}$.

Hence, there cannot exist twice pivotal voters whose PS and DS pure strategies are not jointly sincere in both scenarios: if there exist voters who are twice pivotal under one of the two PS and DS pure strategy combinations and their strategies are not jointly sincere, then voters with such characteristics cannot exist under the other PS and DS pure strategy combination. Thus, we can always compute the two PS and DS pure outcomes, construct the corresponding PS and DS pure strategy combinations and pick the right one.

iii) Finally suppose $m > \alpha_{\overline{d}_L}$, then the only PS and DS pure outcome is $X\left(R, \overline{d}_R\right)$. The proof for this case mirrors the proof of part *i*). This exhausts all the possible cases. Hence, a pure strategy combination whereby the voters who are not twice pivotal vote in a district and presidential sincere manner and voters who are twice pivotal, if they exist, vote in a jointly sincere manner can always be constructed. ■

4.2 Divided Government

Thanks to the fact that in a pure JCS equilibrium the strategies of all voters are presidential and district sincere, a very intuitive sufficient condition for Government to be divided holds for our model. For simplicity, suppose the Congress has an odd number of seats, so that there cannot be a hung Congress. To determine if Government is divided in a JCS equilibrium, we only need to check whether the median voter of the median district prefers a Congress where the party of the elected President has a one seat majority to a Congress where such a party has one less seat.[4] Next, we define divided Government.

Definition 10 *Given a pure strategy combination s, Government is divided if:*

a) when $P(s) = L$, $d^L(s) < \frac{k+1}{2}$; or,

b) when $P(s) = R$, $d^L(s) \geq \frac{k+1}{2}$.

Notice, incidentally, that Theorem (7) implies that, if there are two JCS outcomes, then divided Government will happen in at least one of them.

Proposition 9 *Let s and s' be two JCS pure strategy combinations with $P(s) = L$ and $P(s') = R$, then either $d^L(s) < \frac{k+1}{2}$ or $d^L(s') \geq \frac{k+1}{2}$, or both.*

[4] A similar result holds when k is even, with the proviso that the condition will involve the two median voters of the two central districts, since there is no single median district, when k is even.

Proof. If $d^L(s) < \frac{k+1}{2}$, the proof is complete. If $d^L(s) \geq \frac{k+1}{2}$, by Theorem (7) $d^L(s') \geq d^L(s)$ and the proof is complete. ■

More interestingly, thanks to district sincerity, we can provide a simple sufficient condition for the Government to be divided.

Theorem 10 *In a JCS pure strategy equilibrium Government is divided if*

$$\alpha^L_{\frac{k+1}{2}} < m_{\frac{k+1}{2}} < \alpha^R_{\frac{k+1}{2}}. \tag{1}$$

Proof. Suppose the JCS pure strategy combination s is such that $P(s) = L$. Since $m_{\frac{k+1}{2}} > \alpha^L_{\frac{k+1}{2}}$, by district sincerity L obtains at most $\frac{k+1}{2} - 1$ seats and Government is divided. Suppose the JCS pure strategy combination s is such that $P(s) = R$. Since $m_{\frac{k+1}{2}} < \alpha^R_{\frac{k+1}{2}}$, by district sincerity L obtains at least $\frac{k+1}{2}$ seats and Government is divided. ■

In other words, the presence of divided Government in a JCS equilibrium hinges on having a sufficiently moderate median voter in the median district. This is fairly intuitive, since it is precisely moderate voters who should be interested in counter-balancing a President of one party with a Congress leaning towards the other party in order to obtain a more centrist policy.

4.3 B-Perfection

The theoretically minded reader may wonder what are the underpinnings of the JCS solution concept. In this section, we show that JCS is related to perfection. The Nash solution concept may include some unreasonable situations as equilibria, since it does not necessarily prescribe maximizing behavior in parts of the game that are not reached under the equilibrium strategies. The idea of perfection consists in trying to eliminate unreasonable Nash Equilibria making sure that every part of the game is reached with positive probability, assuming that players "tremble" and make mistakes with some probability when playing. Trembling-hand perfect equilibria, as defined by Selten (1975), are the limiting equilibria when the probability with which the players tremble and make mistakes vanishes. We need to exert some care, though, when applying the idea of the trembles to our framework. In our voting game, a strategy of a voter consists of two actions, a vote for the Presidential election and a vote for the Congressional election. The material procedure of the vote - for instance in the American elections for the President and the Congress occurring at the same time, where the voter is required to mark her vote on the ballot twice, once for the Presidential candidate, once for the congressional candidate- places some restrictions on the type of trembles that we should consider and leads naturally to the idea that the mistakes should be independent across the two actions. We are going to refer to the corresponding solution concept as b(allot)-perfection. To define b-perfection, we need to introduce some new pieces of notation. Next, we define b-strategies.

Definition 11 *A b-strategy for voter i is a function $b_i : \{1, 2\} \to [0, 1]$, representing the probability that voter i votes for party L in each election. A b-strategy is completely mixed if it takes values strictly in between 0 and 1. A b-strategy is pure if it takes values 0 and 1 only. A b-strategy combination is a vector $b = (b_1, ..., b_n)$.*

We denote with $b(s)$ the pure b-strategy combination corresponding to the pure strategy combination s. We are now ready to provide a definition of b-perfection.

Definition 12 *A strategy combination s is a b-Perfect Equilibrium (bPE) if there exists a sequence of completely mixed b-strategy combinations converging to $b(s)$ such that the original strategy combination is a best reply against the entire sequence for all players.*[5]

Presidential and district sincerity is equivalent to b-perfection when no voter is pivotal in both elections.

[5] Clearly, b-perfection is a refinement of trembling-hand perfection. In the appendix we show why this type of refinement is appropriate here.

Proposition 11 *If no voter is pivotal in both elections, every pure strategy combination s is a bPE iff it is PS and DS.*

Proof. We show that i) if a pure strategy combination s is b-perfect then it is PS and DS, ii) if s is PS and DS then it is b-perfect. i) We show first that s is b-perfect, then that it is PS and DS. Consider a completely mixed b-strategy combination converging to $b(s)$. Close enough to $b(s)$, the probability of being pivotal for voter i in district d being the outcome outside the district $d^L_{-d}(s)$ and the president $P(s)$ is infinitely greater than the probability of being pivotal in both elections and than the probability of being pivotal with a different outcome outside the district d and/or in the Presidential election. Hence, every best reply needs to be DS. Analogously, close enough to $b(s)$, the probability of being pivotal for voter j in the presidential election being the legislative outcome $d^L(s)$ is infinitely greater than the probability of being pivotal in both elections and than the probability of being pivotal in the presidential election with a different legislative outcome. Hence, every best reply needs to be PS. ii) To establish the converse it's enough to observe that the considerations above also imply that a district and presidential sincere pure strategy remains a best reply against b-strategies close to $b(s)$. This holds for all voters, hence a PS and DS pure strategy combination s is b-perfect. ■

On the other hand, when some voters are pivotal in both elections, we require joint sincerity for the "twice pivotal" voters and presidential and district sincerity for the others.

Proposition 12 *A pure strategy combination is a bPE, iff it is JCS.*

Proof. If a voter is pivotal in both elections, she has a unique best reply which coincides with her jointly sincere strategy. For close-by strategies of the opponents and a fortiori for b-strategies close to $b(s)$, it is still a best reply. The rest follows from the proof of Proposition 11. ■

Since by Proposition 8 a JCS pure strategy combination exists, the existence of a pure strategy b-perfect equilibrium follows immediately.

Corollary 13 *A pure strategy bPE exists.*

5 Conclusion

We have considered a model where policy motivated citizens vote in two simultaneous elections, one for the President, elected by majority rule, in a single national district, and one for the members of Congress, each of whom elected by majority rule in a local district. Two political parties are present with fixed policy positions on the unidimensional policy space. The policy to be implemented depends not only on the elected President but also on the composition of the Congress. Each voter has a preferred policy outcome and votes in each election for one of the two parties to obtain a policy as close as possible to her preferred one. We have characterized the equilibria of the model using a Jointly Conditional Sincerity concept that takes into account the possibility for some voters to be simultaneously pivotal in both elections. We have shown that this implies the moderation of Government and, in some cases, divided Government. We have also shown that Joint Conditional Sincerity coincides with b-perfection, the appropriate version of trembling-hand perfection for our model.

References

[1] Alesina A., Rosenthal H. (1995), *Partisan politics, divided government, and the economy*, Cambridge University Press, Cambridge

[2] Alesina A., Rosenthal H. (1996), A theory of divided government, *Econometrica,* 64: 1311-1341.

[3] De Sinopoli F., Ferraris L., Iannantuoni G. (2010), Electing a Parliament, mimeo

[4] Ingberman D. E., Rosenthal H. (1997), Median Voter Theorems for Divisible Governments, mimeo

[5] Selten R. (1975), Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory,* 4: 25-55

[6] van Damme E. (1991), *Stability and perfection of Nash equilibria*, Springer-Verlag, Berlin

6 Appendix

In this Appendix we relate b-perfection to trembling-hand perfection. A bPE is trembling-hand perfect (PE), in the sense of Selten (1975), but the converse is not true. The next example shows what may go wrong if we were to ignore the material procedure of the vote and adopt trembling-hand perfection as a solution concept instead.

Example 3. Consider a society with $n = 9$ voters. The leftist candidate for the Presidency has a bliss point $\theta_L = 0$ and the rightist candidate $\theta_R = 1$. The Congress has three seats, hence there are three districts. The policy outcomes are:

$$\begin{aligned} 0.15 &= X(L,3) < 0.25 = X(L,2) < 0.35 = X(L,1) \\ &< 0.45 = X(L,0) < 0.55 = X(R,3) < 0.65 = X(R,2) \\ &< 0.75 = X(R,1) < 0.85 = X(R,0). \end{aligned}$$

The first district is inhabited by two voters with bliss point in 0.36 and one voter in 1; the second district by two voters with bliss point in 0.66 and one voter in 0; the third district by two voters with bliss point in 0.84 and one voter in 0.

JCS and PE. There is one JCS pure strategy combination, in which the first two voters in district one vote for a leftist President and a leftist member of Congress, the remaining voter in district one votes for a rightist President and a rightist member of Congress; the first two voters in district two vote for a rightist President and a leftist member of Congress, the remaining voter in district two votes for a leftist President and a leftist member of Congress; the first two voters in district three vote for a rightist President and a rightist member of Congress, the remaining voter in district one votes for a leftist President and a leftist member of Congress. The first two voters in district three are twice pivotal under this strategy combination and their strategy is jointly sincere. The strategy combination is perfect. The outcome is $X(R,2) = 0.65$.

PE but not JCS. The pure strategy combination which is identical to the previous one except for the strategies of the first two voters of district one who vote for a leftist President and a rightist member of Congress is perfect but not JCS, since the strategies of the first two voters in district one are not district sincere. To see that this strategy combination is perfect, it is enough to construct a perturbation of the strategies of the voters whereby the third voter in district one makes a mistake in only one of the two elections with a probability which is of smaller order than a mistake simultaneously in both elections and the other voters are making mistakes with a probability of smaller order. In this case, as the trembles vanish, the first - analogously the second- voter in district one faces a higher probability of being twice pivotal than pivotal in a single election and her unique best reply is her jointly sincere strategy, which entails voting for a leftist President and a rightist member of Congress. The outcome is $X(R,1) = 0.85$.

The example shows two things. First, there exists a trembling-hand perfect equilibrium which does not satisfy our sincerity requirements. Second, the outcome generated by such a perfect equilibrium cannot be obtained as a JCS equilibrium outcome. The reason why the only-if part of Propositions 11 and 12 would not hold with trembling-hand perfection is that district and presidential sincerity contains an idea of independence between the Presidential and Congressional vote, while under trembling-hand perfection the trembles are not necessarily independent across the two votes.