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Abstract

In this working paper we study some properties of a particular mapping in $\mathbb{R}^n$ related to an optimization problem with one equality constraint. We motivate the definition of the relevant mapping starting from a portfolio selection problem, in which we minimize the risk of an investment (the variance of its return) with one equality constraint given by a fixed level of the return itself. The vector of the optimal portfolio is given by a particular mapping of the vector of returns and this mapping is taken into consideration. All the properties of this mapping may of course be considered in the more general context of an optimization problem with one equality contraint, but some of them may be reasonably extended in the further general case of more equality contraints. Although it has not been investigated in this work, some results may have a relevant meaning in explaining the relation between the vector of expected returns and the optimal portfolio.

Keywords. Portfolio selection, constrained optimization, linear mappings.

AMS Classification. 90C30, 90C46.

JEL Classification. C61.

1 Introduction and motivations

In section 2 we shall define a particular mapping in $\mathbb{R}^n$ and shall study some properties of it. To give some motivations to what we shall develop in the sequel, we consider a standard mean–variance approach to a portfolio selection problem. We recall first few basic statements about the problem ([1],[2]).

1.1 The portfolio selection problem

Suppose we want to invest an amount of money in purchasing some risky securities having a random return. We suppose that n securities are available in the market, we know the expected value of the return of these securities in the next time interval (it is not important if we talk of a year, a semester or whatever) and we know also the covariance matrix of the returns.

We are interested in finding how to invest our money in order to get an "optimal" in some sense portfolio of securities. Of course we can not take optimality in the sense of maximization of the expected return of the portfolio. This would lead us to the decision to invest the whole amount of money in the highest expected return security, and many investors would not consider this an optimal decision for sure. Some others could see as optimal the choice of minimizing the risk of the investment. If we use the variance of return as a measure of risk this would mean to invest all the money in the minimum variance security.

Let us give a formalization of the problem (following [6]).

Let $S_1, S_2, \ldots, S_n$ be the risky securities and we suppose to indicate with $s = (S_1, \ldots, S_n)'$ the vector of the random variables representing the returns of the securities. Let

$$\mu = (\mu_1, \ldots, \mu_n)' = (E(S_1), \ldots, E(S_n))'$$

be the vector of expected returns and Σ the covariance matrix of returns, where

$$\sigma_{ij} = \mathrm{cov}(S_i, S_j) = E[(S_i - \mu_i)(S_j - \mu_j)]$$

and

$$\sigma_{ii} = V_i = \mathrm{var}(S_i).$$

Suppose c is the available amount of money and c_i is the amount assigned to the investment on security S_i. We shall call *portfolio* the vector $x \in \mathbb{R}^n$ such that

$$x_i = \frac{c_i}{c} \quad i = 1, 2, \ldots, n.$$

Hence for each portfolio the following identity will hold

$$\sum_{i=1}^{n} x_i = 1,$$

or, in vector form,

$$u'x = 1,$$

where $u = (1, 1, \ldots, 1)'$. We recall that the model may or may not present restrictions on the signs of x_i ($x_i \geq 0$ for each i) dipending on the short selling policy.

We call return of the portfolio x the random variable which is the linear combination of security returns, that is $x's = \sum_{i=1}^{n} x_i S_i$. The expected value of the portfolio return is of course $E(x's) = x'\mu' = \sum_{i=1}^{n} x_i \mu_i$.

It is a general assumption of the mean–variance approach ([1],[2]) to take the variance of the return as a measure of risk. The variance of the portfolio return is

$$V(x's) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

or, in matrix notation,

$$V(x's) = x'\Sigma x.$$

In the following we shall consider the expected value of the portfolio return and the variance of the portfolio return as functions of the portfolio x. They will be indicated respectively by $E(x)$ and $V(x)$. Hence we have

$$E(x) = \mu' x \qquad \text{and} \qquad V(x) = x'\Sigma x.$$

We may say that the point $(\mu' x, x'\Sigma x)$ is a *feasible* (E, V) *combination* if x is a portfolio. By discarding for the moment the normality constraint, if we consider the mapping $\tau : \mathbb{R}^n \to \mathbb{R}^2$, with

$$\tau(x) = \begin{pmatrix} \mu' x \\ x'\Sigma x \end{pmatrix},$$

and we represent $\mathbb{R}^2$ on the plane (y, z), we have that the first component of $\tau(x)$ takes values on the entire y axis, while the second takes values on the nonnegative z axis, when x takes values in $\mathbb{R}^n$.

Let $\operatorname{Im}\tau = \{(y, z) \in \mathbb{R}^2 : (y, z) = \tau(x),\ x \in \mathbb{R}^n\}$ be the image of τ. In the mean–variance approach the problem is

$$\begin{array}{ll} \min_x & x'\Sigma x \\ \text{sub} & \mu' x = y \end{array} \tag{1}$$

If y takes all the values in $\mathbb{R}$, the solution of the above problem gives us the points in $\mathbb{R}^n$ corresponding to the boundary of $\operatorname{Im}\tau$ (the inverse image of this boundary through the transformation τ).

Let us make the assumption that Σ is positive definite and $\mu \neq 0$.

By setting the necessary optimality conditions[1] on the Lagrangian

$$L(x, \lambda) = x'\Sigma x + \lambda(y - \mu' x) \qquad (\lambda \in \mathbb{R}),$$

that is

$$\begin{cases} \frac{\partial L}{\partial x} = 2\Sigma x - \lambda\mu = 0 \\ \mu' x = y, \end{cases}$$

we get

$$\begin{cases} x = \frac{\lambda}{2}\Sigma^{-1}\mu \\ \frac{\lambda}{2}\mu'\Sigma^{-1}\mu = y, \end{cases}$$

from which

$$\lambda = \frac{2y}{\mu'\Sigma^{-1}\mu} \qquad \text{and then} \qquad x = \frac{y}{\mu'\Sigma^{-1}\mu}\Sigma^{-1}\mu. \tag{2}$$

We get then

$$\begin{aligned} x'\Sigma x &= \frac{1}{(\mu'\Sigma^{-1}\mu)^2} \cdot \mu'\Sigma^{-1}\Sigma\Sigma^{-1}\mu \cdot y^2 \\ &= \frac{1}{\mu'\Sigma^{-1}\mu} y^2, \\ &= \frac{1}{\mu'\Sigma^{-1}\mu} y^2, \end{aligned} \tag{3}$$

which is the equation of a parabola in the plane (y, z), having vertex in the origin and axis of symmetry in the z axis.

We can observe that the vectors x which are mapped into the points of this parabola are proportional to vector $\Sigma^{-1}\mu$.

[1]Conditions are also sufficient for optimality as the problem is convex in the assumptions.

It is easy to see that the image of the mapping τ is the epigraph of the parabola defined in (3).

If we go back to (2) we see that in the solution to problem (1) the vector

$$\frac{\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\mu}$$

plays a particular role. In fact the solutions are proportional to it and the proportionality constant is the "level" y of the constraint. In other words, starting from the vector μ of the returns, the transformation $\frac{\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\mu}$ gives us the optimal portfolio,[2] that is the vector of the optimal portions of investment.

It is worthwhile to investigate on the algebraic and geometric properties of this transformation in a general context, that is to say as a mapping from $\mathbb{R}^n$ to itself. In the next section we study some of these properties.

2 The mapping

We consider the mapping $\varphi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by

$$\varphi(x) = \frac{Ax}{x'Ax}, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{4}$$

where A is a non singular matrix. We are actually interested in the case where A is a positive definite symmetric $n \times n$ matrix, so we shall make in the following that assumption.

It is trivial to observe that φ is not linear: vector $\varphi(x)$ is proportional to Ax, but the proportionality constant depends on x.

If x is any non null vector and α is a scalar, we have that

$$\varphi(\alpha x) = \frac{\alpha Ax}{\alpha^2 x'Ax} = \frac{1}{\alpha}\varphi(x) \quad \text{for every } \alpha \neq 0. \tag{5}$$

This gives the hint that, taking a vector x, the whole span of x is mapped into the span of Ax, in a non linear way which "interchanges what is outside x with what is inside x, with respect to the origin".

[2]The portfolio $\frac{\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\mu}$, being proportional to the optimal one, contains the relevant information, that is the proportions among the investments.

The following properties trivially hold:

$$\text{if } \alpha \leq 1 \text{ then } \|\varphi(\alpha x)\| \geq \|\varphi(x)\| \qquad \text{and} \qquad \text{if } \alpha \geq 1 \text{ then } \|\varphi(\alpha x)\| \leq \|\varphi(x)\|.$$

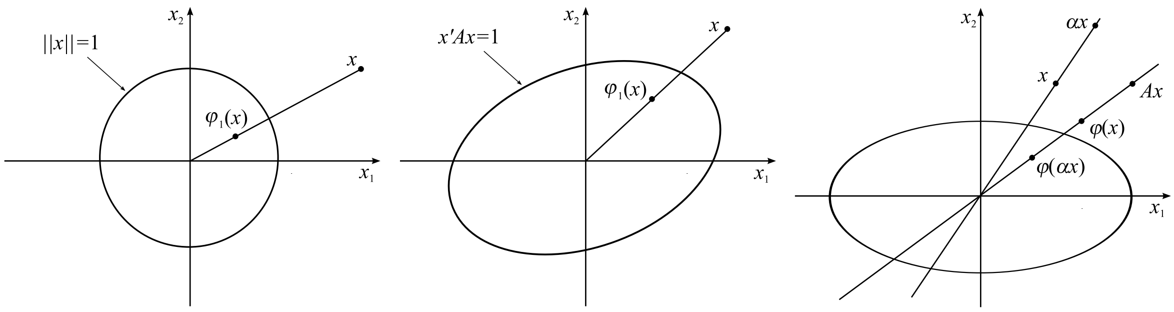
Remark. The mapping φ may be seen as the composition of

$$\varphi_1(x) = \frac{x}{x'Ax} \qquad \text{and} \qquad \varphi_2(y) = Ay,$$

as the identity $\varphi(x) = \varphi_2(\varphi_1(x))$ holds.

We can observe that in $\mathbb{R}^2$ if A is the identity matrix the mapping φ_1 is a "reflection with respect to the unit circle" (figure on the left) and in the general case, with A positive definite, is a "reflection with respect to an ellipse" (figure in the middle). The figure on the right shows the composed behavior of the mapping φ as a scalar transformation of the linear mapping Ax.

FIGURE 1. The mapping φ as a composed mapping



If A is positive definite, the points of the (hyper)surface of equation $x'Ax = 1$ are fixed points for φ_1.

Proposition 1 *If A is positive definite, then φ is a one to one correspondence in $\mathbb{R}^n \backslash \{0\}$.*

Proof. We prove that the mapping

$$\psi(y) = \frac{A^{-1}y}{y'A^{-1}y}$$

is the inverse of φ, that is $\psi = \varphi^{-1}$. We have

$$\psi(\varphi(x)) = x'Ax\ \psi(Ax) = x'Ax\ \frac{A^{-1}Ax}{(Ax)'A^{-1}(Ax)} = x'Ax\ \frac{x}{x'A'A^{-1}Ax} = x'Ax\ \frac{x}{x'Ax} = x.$$

We have used the simmetry of A at the end.

In the same way we can prove that $\varphi(\psi(y)) = y$. $\square$

Remark. The simmetry and non singularity of A are sufficient for the thesis.

Proposition 2 *If A is positive definite, then $\|\varphi(x)\| \geq 1/\|x\|$ for each $x \in \mathbb{R}^n \setminus \{0\}$.*

Proof. If $x \neq 0$

$$x'\varphi(x) = x'\frac{Ax}{x'Ax} = 1.$$

Then from the Cauchy–Schwartz inequality we get

$$1 = |x'\varphi(x)| \leq \|x\| \cdot \|\varphi(x)\|$$

and the thesis follows. □

Remark. Here non singularity of A is sufficient for the thesis.

Corollary 1 *If $x \in \mathbb{R}^n \setminus \{0\}$ and $\|x\| \leq t$, then $\|\varphi(x)\| \geq 1/t$.*

Proof. It follows directly from Proposition 2. □

In particular, taking $t = 1$, we have that if $\|x\| \leq 1$, then $\|\varphi(x)\| \geq 1$.

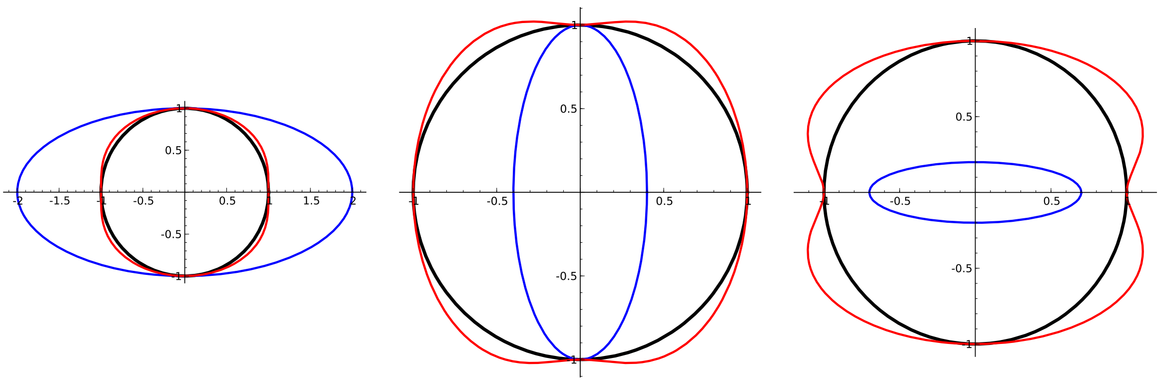
Remark. The meaning of the Corollary is significant for small values of t, roughly saying that if x is in a small neighborhood of the origin then its image is far from it. We can obviously observe that a result such as "if $\|x\| \geq t$, then $\|\varphi(x)\| \leq 1/t$" can not be derived from the previous one, and in fact is not true.[3]

The result of Corollary 1 is less significant with large values of t.

2.1 Some examples

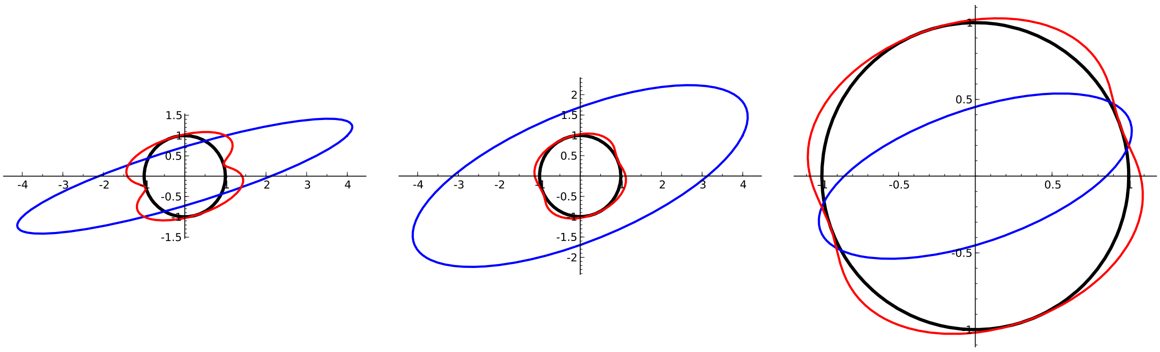
We now consider some examples related to particular mappings φ in $\mathbb{R}^2$. Both the cases with diagonal and non diagonal matrix are represented.

FIGURE 2. Some examples with a diagonal matrix



[3]By taking for example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $x = \frac{101}{100\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have that both $\|x\|$ and $\|\varphi(x)\|$ are greater than 1.

FIGURE 3. Some examples with a non diagonal matrix



Examples in Figure 2 are relative to mappings with a diagonal matrix A and different values for the eigenvalues.[4] Examples in Figure 3 are relative to mappings with a non diagonal matrix A.[5]

In all the examples the black line indicates the unit vectors (vectors of norm one), the blue line gives the image of the unit vectors through the linear mapping associated to the matrix A and the red line gives the image of the unit vectors through the mapping φ. We see in all the examples that for the unit vectors their transformed vector has a norm not less than 1, no matter the eigenvalues are.

3 A particular case

In order to deeper investigate on the geometric properties of the mapping φ, we start considering a particular case of (4), the case in which tre matrix is a diagonal matrix D, with elements d_i in the main diagonal. As we are interested in the positive definite case, elements d_i are supposed to be positive. We are going to derive some properties, that are equivalent to the result of Corollary 1, in this particular case first and then in the general case.

The first property we want to consider is the existence of fixed points for the mapping

$$\varphi(x) = \frac{Dx}{x'Dx}.$$

It is easy to realize that some fixed points exist. If we call e^i the fundamental vectors of

[4]Eigenvalues are: $\lambda_1 = 2, \lambda_2 = 1$ on the left; $\lambda_1 = 0.4, \lambda_2 = 1$ in the middle; $\lambda_1 = 0.7, \lambda_2 = 0.2$ on the right.

[5]Eigenvalues are: $\lambda_1 = 0.71, \lambda_2 = 4.30$ on the left; $\lambda_1 = 1.59, \lambda_2 = 4.14$ in the middle; $\lambda_1 = 0.43, \lambda_2 = 1.07$ on the right.

$\mathbb{R}^n$, it holds that

$$\varphi(e^i) = \frac{De^i}{(e^i)'De^i} = \frac{d_i e^i}{d_i} = e^i.$$

The same holds for the opposite of the fundamental vectors.

Proposition 3 *The eigenvectors of norm one of D are the fixed points of φ.*

Proof. Starting from equation

$$\frac{Dx}{x'Dx} = x,$$

by taking the inner product with x we get

$$x'\frac{Dx}{x'Dx} = x'x \qquad \text{and then} \qquad \|x\| = 1.$$

Now we can observe that, in the hypothesis of positive definiteness of D and $x \neq 0$, the equation

$$\frac{Dx}{x'Dx} = x \qquad \text{is equivalent to} \qquad Dx = (x'Dx)x. \tag{6}$$

Hence x is a fixed point of φ if and only if $x'Dx$ is an eigenvalue of D and x is an eigenvector associated to it. □

We can specify more in detail what are the fixed points of φ.

Corollary 2 *If D has distinct eigenvalues, equation (6) has no solutions apart from the fundamental vectors and their opposite.*

Proof. The proof may be obtained directly fron Proposition 3, as in the case D has distinct eigenvalues we have n eigenspaces each one spanned by a fundamental vector. In the ith eigenspace the only vectors with norm one are e^i and $-e^i$.

Alternatively we can observe that equation (6) means that

$$d_i x_i = (x'Dx)x_i \quad \forall i.$$

This implicates that all the x_i's except one must be zero. In fact, if we suppose that $x_j, x_k \neq 0$, the corresponding equations take to

$$d_j = (x'Dx) \quad \text{and} \quad d_k = (x'Dx),$$

which can not be true as we have distinct eigenvalues. Let us assume x_j is the only non zero component in x. Then the corresponding equation

$$d_j = (x'Dx) \quad \text{gives} \quad d_j = d_j x_j^2 \quad \text{and this gives} \quad x_j^2 = 1.$$

This implicates that $x = \pm e^j$. $\square$

We can have actually other solutions for equation (6) if D has multiple eigenvalues.

Corollary 3 *If D has a multiple eigenvalue, then the set of the norm one vectors in the associated eigenspace are fixed points associated to that eigenvalue.*

Proof. The result may be directly obtained fron Proposition 3. In the case D has a multiple eigenvalue the eigenspace associated to it is spanned by the corresponding fundamental vectors. Hence the fixed points are the norm one vectors in this eigenspace. Alternatively, assuming D has a multiple eigenvalue d, then we may say that $d_{\nu_i} = d$ for $i = 1, \ldots, k$. Suppose x is a linear combination of the e^{ν_i}'s, that is $x = \sum_{i=1}^k a_i e^{\nu_i}$. We have then

$$Dx = \textstyle\sum_{i=1}^k a_i D e^{\nu_i} = \sum_{i=1}^k a_i d_{\nu_i} e^{\nu_i} = d \sum_{i=1}^k a_i e^{\nu_i} = dx.$$

We also have

$$x'Dx = x'(dx) = dx'x = d \sum_{i=1}^k a_i^2.$$

The equation (6) may then be written as

$$dx = d \sum_{i=1}^k a_i^2 x.$$

This takes to $\sum_{i=1}^k a_i^2 = 1$ and then, being $x \neq 0$, $\|x\| = 1$. Hence, when the matrix D has multiple eigenvalues, every linear combination of norm one of the fundamental vectors associated to the multiple eigenvalues is a fixed point for the mapping φ. $\square$

Remark. To summarize we see that, in the same way the linear mapping represented by D has eigenspaces associated to each eigenvalue, the mapping φ has a set of fixed points corresponding to each eigenvalue of D, and these are given by the norm one vectors in the eigenspace.

If we are interested in vectors which are proportionally transformed by φ, that is to say vectors x such that $\varphi(x) = \lambda x$ for some real constant λ, we can see, with the same arguments as before and holding property (5), that these vectors are the ones proportional to the fixed points.

Let us now call $d = (d_1, \ldots, d_n)$ the vector of the main diagonal of a positive definite diagonal matrix D. The following statement can be proved.

Proposition 4 *If D is a positive definite diagonal matrix and $\sum d_i \leq 1$ then*

$$x'Dx \geq (d'x)^2 \quad \textit{for each } x \in \mathbb{R}^n. \tag{7}$$

Proof. The thesis may be written as

$$x'Dx \geq x'dd'x \quad \text{for each } x$$

and then

$$x'(D - dd')x \geq 0 \quad \text{for each } x.$$

In other words we want to prove that the matrix $D - dd'$ is positive semidefinite. We have

$$D - dd' = \begin{pmatrix} d_1 - d_1^2 & -d_1d_2 & \cdots & -d_1d_n \\ -d_1d_2 & d_2 - d_2^2 & \cdots & -d_2d_n \\ \vdots & \vdots & \ddots & \vdots \\ -d_1d_n & -d_2d_n & \cdots & d_n - d_n^2 \end{pmatrix}.$$

It is easy to see that the matrix is singular. In fact if we add the elements in the ith row we get

$$d_i - d_i^2 - \sum_{j \neq i} d_id_j = d_i(1 - d_i) - d_i \sum_{j \neq i} d_j = d_i \sum_{j \neq i} d_j - d_i \sum_{j \neq i} d_j = 0.$$

Then columns (and rows) are linearly dipendent. We are going to prove now that the principal minors of the matrix are nonnegative. A $k \times k$ principal minor is the determinant of a submatrix of the form

$$\begin{pmatrix} d_{\nu_1} - d_{\nu_1}^2 & -d_{\nu_1}d_{\nu_2} & \cdots & -d_{\nu_1}d_{\nu_k} \\ -d_{\nu_1}d_{\nu_2} & d_{\nu_2} - d_{\nu_2}^2 & \cdots & -d_{\nu_2}d_{\nu_k} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{\nu_1}d_{\nu_k} & -d_{\nu_2}d_{\nu_k} & \cdots & d_{\nu_k} - d_{\nu_k}^2 \end{pmatrix}.$$

In the assumption that $\sum d_i \leq 1$ this matrix is diagonally dominant. In fact, if we sum up in the ith row, using the fact that $\sum_{j=1}^{k} d_{\nu_j} < 1$ and then $1 - d_{\nu_i} > \sum_{j \neq i} d_{\nu_j}$, we get

$$\begin{aligned} d_{\nu_i} - d_{\nu_i}^2 - \sum_{j \neq i} d_{\nu_i} d_{\nu_j} &= d_{\nu_i}(1 - d_{\nu_i}) - d_{\nu_i} \sum_{j \neq i} d_{\nu_j} \\ &> d_{\nu_i} \sum_{j \neq i} d_{\nu_j} - d_{\nu_i} \sum_{j \neq i} d_{\nu_j} \\ &= 0. \end{aligned}$$

This shows that the matrix is diagonally dominant and then all the $k \times k$ principal minors with $k < n$ are positive. In conclusion the matrix $D - dd'$, being singular, is positive semidefinite and the thesis follows. $\square$

Remark. There may be the feeling that a sufficient property for (7) could be that all the eigenvalues of D are less or equal to one (that is $d_i \leq 1$ for every d_i), which is a necessary condition for the assumption of Proposition 4. It is not true, and as a counterexample we may take $D = \epsilon I$, the identity matrix, and $x = 1 = (1, \ldots, 1)'$, which give the inequality

$$\epsilon 1' I 1 \geq \epsilon^2 (1'1)^2, \quad \text{equivalent to} \quad \epsilon n \geq \epsilon^2 n,$$

which is false for example with $\epsilon = \frac{3}{4}$ and $n = 2$. Then the condition "eigenvalues less or equal to one" is not sufficient for having property (7). The condition then is just necessary.[6]

Remark. The thesis of Proposition 4 may be false if we do not assume that $\sum d_i \leq 1$. In fact, with the same example as before, $D = \epsilon I$ and $x = 1$, the inequality (7) is false with $\epsilon > \frac{1}{n}$.

Remark. The thesis of Proposition 4 may be strengthened if we assume that $\sum d_i < 1$. By repeating the proof with this stronger assumption, it is straightforward to prove that the matrix $D - dd'$ is positive definite. Hence we can state that

Proposition 5 *If D is a positive definite diagonal matrix and $\sum d_i < 1$ then*

$$x' D x > (d'x)^2 \quad \textit{for each } x \neq 0.$$

From Proposition 4 the following interesting inequality in $\mathbb{R}^n$ may be derived.

[6] Actually, in the assumption D is positive definite, the necessary condition is $d_i < 1$ for every d_i.

Corollary 4 *If $x \in \mathbb{R}^n$, then*

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum_{i=1}^{n} x_i^2.$$

Proof. The proof follows from (7) taking $d = (\frac{1}{n}, \ldots, \frac{1}{n})$. We get

$$\sum_{i=1}^{n} \frac{1}{n} x_i^2 \geq \left(\sum_{i=1}^{n} \frac{1}{n} x_i\right)^2, \qquad \text{then} \qquad \frac{1}{n}\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2,$$

then the result follows. □

Again from Proposition 4 the following inequality, more interesting in our context, may be derived.

Corollary 5 *If D is a positive definite diagonal matrix and $\|x\| \leq 1$, then*

$$\|Dx\| \geq x'Dx.$$

Proof. Proposition 4 says that if $\sum d_i \leq 1$ then

$$\sum d_i x_i^2 \geq \left(\sum d_i x_i\right)^2 \qquad \text{for each } x \in \mathbb{R}^n. \tag{8}$$

The assumption $\|x\| \leq 1$ may be written as $\sum x_i^2 \leq 1$. Then, if in (8) x_i^2 plays the role of d_i and d_i the role of x_i, we get

$$\sum x_i^2 d_i^2 \geq \left(\sum x_i^2 d_i\right)^2$$

and then

$$\left(\sum d_i^2 x_i^2\right)^{1/2} \geq \sum d_i x_i^2,$$

which means

$$\|Dx\| \geq x'Dx. \qquad \square$$

4 A general result

The result of Corollary 5 is true in the general case of a positive definite matrix.

Proposition 6 *If A is a positive definite matrix and $\|x\| \leq 1$, then*

$$\|Ax\| \geq x'Ax. \tag{9}$$

Proof. By means of the spectral theorem ([4],[5]) we have that the positive definite matrix A may be written as $A = U'DU$, where U is an orthogonal matrix and D is the diagonal matrix of the positive eigenvalues of A. Then, by applying Corollary 5,

$$\|Ax\| = \|U'DUx\| = \|DUx\| \geq (Ux)'D(Ux) = x'U'DUx = x'Ax. \text{ [7]} \qquad \square$$

Remark. Proposition 6 gives us again for the mapping φ what had been previously found by means of the Cauchy–Schwartz inequality, that is the property:

$$\text{if } \|x\| \leq 1 \text{ then } \|\varphi(x)\| \geq 1.$$

5 Concluding remarks

What has been considered and found since now has just the meaning of an investigation on the underlined mapping that plays a role in a simple minimization problem with one linear constraint.

In order to see if some results are of some importance for the problem which has given a motivation to this working study a few remarks have to be presented.

The first remark is that if we consider the mapping φ in the context of a portfolio selection problem, the matrix we are dealing with is the inverse of the covariance matrix of the returns. This matrix may reasonably be taken as positive definite. Moreover we have to consider that what plays the role of vector x is the vector of returns and finally the image $\varphi(x)$ is the portfolio, that is the vector of the optimal investments. A sort of normalization on the vector of returns, in order to get a vector belonging to the region with norm less than one, does not substantially modify the structure of the constraint and consequently the mapping. Considering the general properties we have been dealing with in this work, it can be argued that properties concerning the norms of vectors are less significant in our context than properties concerning the ratios among components of these vectors, as while the norms give a sort of collected information, much more interesting could be a knowledge on how the proportions among components are modified. In fact a valuable result could be to know the role of the mapping on how the relative values of the returns

[7] We use the property that $\|Uy\| = \|y\|$ if U is orthogonal.

map into the relative values of the components of the investments. This can be the topic for a further investigation.

It could finally be observed that on the solution vector the additional normality constraint is usually taken into account, that is the components of the solution must sum up to one.[8] The additional constraint on the sum of components in the initial problem does not deeply modify the structure of the constraints and consequently the structure of the solution ([6]). Anyway this seems to be the first necessary step to be considered in a further development.

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[8] Sometimes a non negativity restriction is also introduced, but this is not always the case.