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Eugenio Peluso, Alain Trannoy

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Preserving Dominance Relations Through Disaggregation: The Evil and the Saint*

Eugenio Peluso†

Department of Economics, University of Verona.

Alain Trannoy‡

EHESS, GREQAM-IDEP, Marseille.

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Abstract

Disaggregation arises when broad categories like households budget units are divided into elementary units as individual income recipients. We study the preservation of stochastic dominance for every order beyond two after disaggregation: If we observe a dominance relation among household income distributions, it is also true at the individual level. We find necessary and sufficient conditions satisfied by the common sharing rule adopted by households to divide the cake among individuals. The "sharing function", which maps the household income into the outcome of the disadvantaged individual, must have derivatives of the same sign as the utility function characterizing the stochastic order of interest. In addition, the household has to follow a compensating rule, meaning that at the margin the distribution should be in favour of the disadvantaged individual.

Key Words: Sharing rule, Stochastic dominance, Disaggregation.

JEL Codes: D31, D63, D81.

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†Via dell'Università 3, 37129 Verona (Italy). E-mail: eugenio.peluso@univr.it.

‡Vieille Charité, 3 rue de la Charité, 13002 Marseille (France). E-mail: alain.trannoy@eco.u-cergy.fr.

"*All inequality is a source of evil - the inferior loses more in the account of happiness than the superior is gained*"

J. Bentham,

First Principle preparatory to Constitutional Code, 1822.

1 Introduction

Maurice Salles is known for his constant and long dedication to the problem of aggregation. In very broad terms, aggregation means extracting information coming from a set of elements to form a coherent picture of the whole set, that is going from the individual sphere to the global one. Here, we focus on the reverse problem, that is, we try to infer information on the individual level from data obtained at the global level. This disaggregation issue does not seem easier than the standard aggregation problem and we have to be more specific to make it solvable.

We are interested in stochastic dominance orderings which are known to be important in assessing risk situations or coining inequality and welfare judgments. For instance the Second order Stochastic Dominance (SSD) induces a partial order among lotteries consistent with the choice of all risk-averse decision makers. In inequality measurement, SSD ranks income distributions consistently with any inequality averse social welfare functional and it was shown to be equivalent to the Generalized Lorenz test (Shorrocks 1983).

A natural problem of disaggregation involving stochastic dominance orderings occurs in several contexts. Our favorite example involves families. In many countries, we have information about the evolution of the inequality at the household level in terms of Lorenz curve statements. Let us suppose that the Lorenz curve at some terminal period dominates its counterpart at some initial date. The distribution of income among the members of a household is generally unobserved and is private information at least in usual data sets. The question that comes in is then whether the knowledge of a dominance relation at the aggregate level, here the household, is sufficient to deduce the existence of a dominance relation at the individual level. If it was the case in the particular example, we would be able to conclude that the Generalized Lorenz curve of the income distribution across individuals at the terminal date dominates its counterpart at the initial date. In other words, the dominance relation at the aggregated level would be a sufficient statistics of the dominance relation at the individual level. If by chance it is the case, we say that the Generalized Lorenz ordering is preserved through disaggregation.

The following sections will provide other examples where such an issue of disaggregation arises in risk or intertemporal configurations. To keep with inequality and well being statements, fiscal federalism provides another instructive example. Each state /region/city plays the role of an household and, this time, the elementary unit is represented by the household. It is crucial that some redistribution takes place at the state/regional/city level which is not completely known by the federal government. Otherwise the problem becomes trivial. In general terms, our setting considers a set of agents gathered in subsets which constitute a partition. Each subset constitutes a decision unit in terms of sharing resources. What is known is the evolution of inequality among subsets. What is unknown is the evolution of inequality among agents.

This type of problem is related to that of decomposition of inequality by population subgroups. The inequality among individual units according to some well-defined inequality index is the sum of a between-group term, a within-group term and some interaction term (that appears if the inequality index does not belong to the entropy class). We explore in this paper under which conditions the between-group term goes along with total inequality. In this case the within-group and interaction terms are not going in the opposite direction of the between-group term or are not sufficiently strong to counterbalance its weight.

Peluso and Trannoy (2007) investigated this kind of issue for the preservation of second stochastic dominance. They found the condition on the common sharing rule used by households to distribute resources among individuals. This necessary and sufficient condition is that the sharing rule must be concave, that is, the marginal share of additional resources devoted to the disadvantaged individual in the household must be decreasing. In other words, the disadvantaged individual in the household should receive more and more in the margin when the household becomes poorer.

This article explores the conditions allowing to preserve the dominance relation at higher orders. This exploration is worth it for at least two reasons. First, it is natural to refine the SSD test by resorting to higher order stochastic dominance when it reveals to be inconclusive. This practice is especially useful in poverty measurement for its interpretation in terms of poverty orderings pointed out by Foster and Shorrocks (1988). Second, from a mere theoretical perspective, it is well known that the second order represents a kind of "last frontier" for the equivalence between stochastic dominance and inverse stochastic dominance (see Muliere and Scarsini 1989). It is far from obvious, in view of these difficulties, that the result obtained for the second order extends to higher orders. The following conjecture was the natural impetus for the present study. The requirement of concavity for preserving the second order seems the straight extension of the requirement of monotonicity for preserving the first order. And then comes in the intuition that requiring the sharing function on top of that to be prudent (positive third derivative) will provide the right condition for preserving the third order. This paper proves that this conjecture is false. Finding the conditions become more intricate beyond second order. To overcome this difficulty we resort to the "Faà di Bruno formula"[1] which expresses the partial derivatives of a composite function in terms of the partial derivatives of the two initial functions. We are then able to express necessary and sufficient conditions able to preserve dominance relation through disaggregation at any finite order.

[1] Marquis Francesco Faà di Bruno (1825-1888) had been officer, Cauchy's student in Paris, then philanthropist. This excellent mathematicien became saint (but not full professor) in Italy, a biography that certainly Maurice will be fond of.

The paper is organized as follows: In the first section we present the set up, the concept of the sharing rule and the issue of preserving a stochastic order after disaggregation of the income distribution. We also discuss the large domain of applications. Previous results on second order stochastic dominance disaggregation are illustrated through new applications. Section 3 points out the nature of the difficulties arising with the extension of previous results to higher order stochastic dominance. It contains our main result, which is proved thanks to a lemma that has an autonomous interest in risk theory since it gives the transformations preserving stochastic dominance at any finite order m. Section 4 concludes the paper with some hints for further extensions.

2 The set up

We present the preservation of dominance relations through disaggregation using the example of a population composed of n couples homogeneous in all but income, indexed by $i = 1, ..., n$, with $n \geq 2$. Each household i is endowed with the same exogenous income k and a variable income y_i. Denoting $\mathbf{e}_n$ the unitary vector of dimension n, the feasible set of ordered households incomes is

$$\mathbb{D}_n = \left\{ \mathbf{y}+k\mathbf{e}_n \in \mathbb{R}_+^n \mid 0 \leq y_1 \leq y_2 ... \leq y_n \right\}.$$

Let $u^{(j)}$ be the j-th derivative of a real function u, differentiable as times as required. We designate by U_j the class of all real functions u such that $(-1)^j u^{(j)} < 0$, $\forall x \in \mathbb{R}$.

We will say that $\mathbf{y}$ dominates $\mathbf{y}'$ according to the j-th order stochastic dominance, and we will denote $\mathbf{y} \succsim_j \mathbf{y}'$, whenever:

$$\sum_{i=1}^{n} u(y_i) \geq \sum_{i=1}^{n} u\ (y_i'), \text{ for all } u \in U_j.$$

First order stochastic dominance $\succsim_1$ requires the utility to be increasing, second stochastic order $\succsim_2$ asks for the utility function to be increasing and concave, the third stochastic order $\succsim_3$ asks in addition to these two properties for the marginal utility to be convex and so on. It is well-known that first and second stochastic dominance may be expressed through elementary properties of the income distributions: $\mathbf{y} \succsim_1 \mathbf{y}'$ is equivalent to $y_i \geq y_i'$ for $i = 1, ..., n$, while SSD is equivalent to the Generalized Lorenz test pointed out by Shorrocks (1983). Similar interpretations cannot be provided for higher order stochastic dominance (see Le Breton and Peluso 2009).

2.1 The sharing function

We assume that each couple is composed of equally needy individuals, but a *weak* individual has an initial endowment k_w lower or equal than that of his *strong* partner, denoted k_s. We assume that these initial endowments are exogenous, and that $k_w + k_s = k$ is constant across the population. These initial endowments may figure out the net worth accumulated by each individual before the union. Each couple i pools the resources originated by additional incomes into y_i and then follows a well-defined sharing rule to split this additional income into $w(y_i)$, the amount received by the weak individual and $s(y_i) = y_i - w(y_i)$, the remaining amount of the strong individual. In what follows, we will focus on sharing functions that do not allow changes in the rank between weak and the strong individuals within each household. Formally:

Definition 1 *Let $\mathcal{F}$ be the set of continuous sharing functions $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:*

$$w(0) = 0$$

$$0 \leq w'(y) \leq 1 \text{ for all } y \in \mathbb{R}_+$$

$$w(y) + k_w \leq s(y) + k_s \quad \text{for all } y \in \mathbb{R}_+. \tag{1}$$

The two first assumptions are trivial, the third one imposes an explicit upper bound on the function w, which can be rewritten as $w(y) \leq \frac{y}{2} + \frac{k_s - k_w}{2}$ for all $y \in \mathbb{R}_+$. Applying these functions to all the elements of the initial vector $\mathbf{y}$ leads to disaggregate $\mathbf{y} \in \mathbb{R}_+^n$

into the vectors $\mathbf{w}(\mathbf{y}) = (w_1, .., w_n)$ (weak individuals) and $\mathbf{s}(\mathbf{y}) = (s_1, ..., s_j, ..., s_n)$ (strong individuals). Adding the initial endowments and ranging the resulting individual incomes into a unique ordered vector, we get $\mathbf{x}(\mathbf{y}; k_w, k_s) \in \mathbb{D}_{2n}$.

2.2 Preserving stochastic dominance through disaggregation

Preserving a dominance relation through disaggregation means to design a sharing rule $w : \mathbb{R}_+ \to \mathbb{R}_+$ such that the dominance relation established among two initial distributions $\mathbf{y}+k\mathbf{e}_n$ and $\mathbf{y}'+k\mathbf{e}_n$ of $\mathbb{D}_n$ is preserved among the two corresponding individual income distributions $\mathbf{x}(\mathbf{y}; k_w, k_s)$ and $\mathbf{x}(\mathbf{y}'; k_w, k_s)$ of $\mathbb{D}_{2n}$. Peluso and Trannoy (2007, 2009a) solved the problem in the case $k = 0$. Their main result is that whenever the sharing function w is assumed to be identical among all households, first and second stochastic dominance are preserved from households to individuals if and only if the sharing function w has derivatives of the same sign as the utility function characterizing the stochastic order of interest. We provide below the easy generalization of this theorem in the case with $0 < k_w < k_s$. We also give a new proof of the sufficiency part that will be useful to introduce later the general case.

Theorem 1 *(Peluso and Trannoy 2007, 2009a)*

a) $w \in \mathcal{F} \cap U_1 \iff \left[\mathbf{y}+k\mathbf{e}_n \succsim_{\mathbf{1}} \mathbf{y}'+k\mathbf{e}_n \implies \mathbf{x}(\mathbf{y}; k_w, k_s) \succsim_1 \mathbf{x}(\mathbf{y}'; k_w, k_s),\ \textit{for all } \mathbf{y}, \mathbf{y}' \in \mathbb{D}_n\right]$

b) $w \in \mathcal{F} \cap U_2 \iff \left[\mathbf{y}+k\mathbf{e}_n \succsim_{\mathbf{2}} \mathbf{y}'+k\mathbf{e}_n \implies \mathbf{x}(\mathbf{y}; k_w, k_s) \succsim_2 \mathbf{x}(\mathbf{y}'; k_w, k_s),\ \textit{for all } \mathbf{y}, \mathbf{y}' \in \mathbb{D}_n\right]$.

Proof. Let w be increasing and concave and assume $\mathbf{y} \succsim_2 \mathbf{y}'$. We will prove that under an increasing and concave sharing function

$$\sum_{i=1}^{n} \left[u(w(y_i) + k_w) + u(s(y_i) + k_s)\right] \geq \sum_{i=1}^{n} \left[u(w(y_i') + k_w) + u(s(y_i') + k_s)\right]$$

is increasing and concave for all increasing and concave u, which is equivalent to $\mathbf{x}(\mathbf{y}; k_w, k_s) \succsim_2 \mathbf{x}(\mathbf{y}'; k_w, k_s)$. We divide the proof into two steps.

Step 1 For a given individual utility function u, let V_u be the function defined by

$$V_u(y+k) = u(w(y)+k_w) + u(s(y)+k_s). \tag{2}$$

Under our assumptions, we show that $V_u'(y+k) > 0$ and $V_u''(y+k) < 0$, $\forall y \geq 0$.

$V_u'(y+k) = u'(w(y)+k_w)w'(y) + u'(s(y_i)+k_s)s'(y)$. This expression is positive and immediately guarantees the preservation of FSD.

$V_u''(y+k) = u''(w(y)+k_w)\left(w'(y)\right)^2 + u'(w(y)+k_w)w''(y) + u''(s(y_i)+k_s)\left(s'(y)\right)^2 + u'(s(y_i)+k_s)s''(y)$. Using the fact that $w''(y) = -s''(y)$ $\forall y \geq 0$, we get

$$V_u''(y;k) = w''(y)\left[u'(w(y)+k_w) - u'(s(y_i)+k_s)\right] + u''(w(y)+k_w)\left(w'(y)\right)^2 + u''(s(y_i)+k_s)\left(s'(y)\right)^2.$$

The concavity of u insures a negative sign for this expression.

Step 2 From $\mathbf{y}+k\mathbf{e}_n \succsim_{\mathbf{2}} \mathbf{y}'+k\mathbf{e}_n$, we get

$$\sum_{i=1}^{n} V_u(y_i+k) \geq \sum_{i=1}^{n} V_u(y_i'+k)$$

since V_u is increasing and concave and therefore

$$\sum_{i=1}^{n}\left[u(w(y)+k_w) + u(s(y_i)+k_s)\right] \geq \sum_{i=1}^{n}\left[u(w(y')+k_w) + u(s(y_i')+k_s)\right].$$

The reasoning is valid for all increasing and concave u, which implies $\mathbf{x}(\mathbf{y};k_w,k_s) \succsim_{\mathbf{2}} \mathbf{x}(\mathbf{y}';k_w,k_s)$ and the theorem is proved. ■

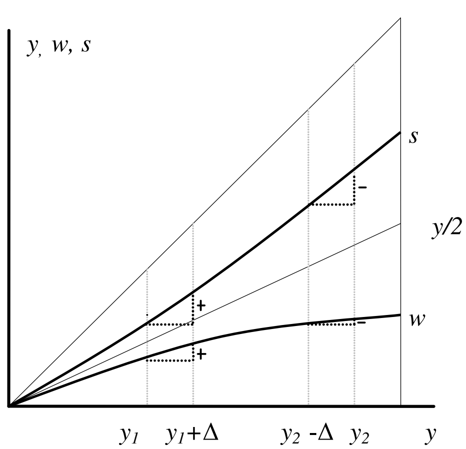
A concave sharing function always guarantees that a progressive transfer between households provides a 'double dividend' on social welfare valued at the individual level: a transfer from a richer family to a poorer one also represents a transfer from a less egalitarian household to a more egalitarian one as well. This situation is illustrated in Panel **a** of Figure 1, borrowed from Peluso and Trannoy (2007). Three essential properties of w and s emerge from this picture, where $k = 0$:

i) $w(y) \leq s(y)$ $\forall y \in \mathbb{R}_+$ by assumption.

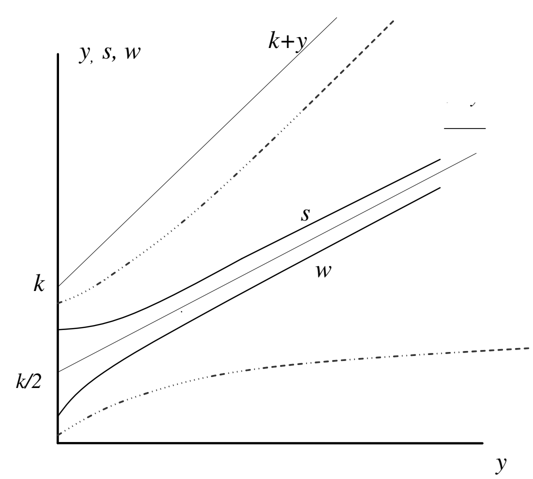
ii) $w'(y)$ is required to be decreasing by the theorem.

iii) $w'(y) \leq s'(y)\ \forall y \in \mathbb{R}_+$. This is a consequence of i), ii) and of the specific assumption $k = 0$.

Inspecting the proof above, it emerges that iii) is not essential to get the result. Moreover it imposes a strong constraint on the economic interpretation of the result. To illustrate this point, let us consider the situation depicted in Panel **b** of Figure 1, where the household is endowed with the "net worth" k. Theorem 1 and properties i) and ii) indicated above hold in two different cases. The first one, illustrated by the dashed lines, representing an intra-household behavior more and more unequal both in relative and marginal terms when y rises. This is an easy extension of the situation depicted in Figure 1-a. It is more interesting to examine the case represented by the solid lines in Figure 1-b. Even if properties i) and ii), are still satisfied, we get $w'(y) \geq s'(y)\ \forall y \in \mathbb{R}_+$. This case is perfectly in line with the preservation conditions stated in Theorem 1, but it degenerates into the perfectly egalitarian sharing rule if the result is required for all $k \geq 0$, that is over the domain retained by Peluso and Trannoy (2007). Keeping a strictly positive k, we allow the interesting case of households more and more unequal *at the margin*, but not in relative or absolute terms. Even if the sharing rule is concave (consistently with Theorem 1), the *initial* intra-household inequality decreases with y both in absolute and relative terms.



a- Concave sharing function



b- Concave sharing functions with an initial endowment.

In the following section, we show that the cases illustrated in the above figure behave in a different way in preserving dominance relations beyond the order two.

The result can be extended into two directions. First we can relax the assumption that the endowment is the same for all households.

Corollary 1 *Assume that households are increasingly ranked according to their endowments.* $k_1 \leq k_2 ... \leq k_n$ *and that the weak individual is all the more well treated than the household has a high endowment, that is* k_w/k_s *is increasing in* k*. Then*

$$w \in U_2 \cap \mathcal{F} \iff \left[\begin{array}{c} \mathbf{y}+\mathbf{k} \succsim_{\mathbf{2}} \mathbf{y}'+\mathbf{k} \implies \mathbf{x}(\mathbf{y};\mathbf{k}_w,\mathbf{k}_s) \succsim_2 \mathbf{x}(\mathbf{y}';\mathbf{k}_w,\mathbf{k}_s) \text{ such that} \\ k_{i+1} \geq k_{i+1} \Leftrightarrow \ y_{i+1} \geq y_{i+1} \ \Leftrightarrow \ y'_{i+1} \geq y'_{i+1} \end{array} \right].$$

We can cope with the heterogeneity of endowments, provided that the distribution of income and endowment are comonotone and that the endowments are less unequally distributed in rich households.

The preservation of the relative Lorenz test can also be obtained as a corollary of the theorem but we have to distinguish the compensation case from the discrimination one.[2]

Definition 2 *The discriminating case. Let* $\mathcal{F}_d \subset \mathcal{F}$ *be the set of sharing functions with* $0 \leq w'(y) \leq \frac{1}{2}$ *for all* y*, that is the poor always receives less than than the rich at the margin.*

Definition 3 *The compensating case. Let* $\mathcal{F}_c$ *be the set of continuous sharing functions with* $\frac{1}{2} \leq w'(y) \leq 1$*, that is the poor always receives more than the rich at the margin.*

In the first case, the dominant distribution must have a lower mean to preserve the relative Lorenz ordering, in the second case a higher mean. More precisely we get the following extension.

Corollary 2 $w \in U_2 \cap \mathcal{F}_d$ *if and only if*

[2] We recall that the relative Lorenz criterion $\succsim_{\mathbf{RL}}$is equivalent to SSD applied to the income vectors divided by the mean of the distribution.

$$\left[\mathbf{y}+\mathbf{k} \succsim_{\mathbf{RL}} \mathbf{y}'{+}\mathbf{k} \implies \mathbf{x}(\mathbf{y};\mathbf{k}_w,\mathbf{k}_s) \succsim_{\mathbf{RL}} \mathbf{x}(\mathbf{y}';\mathbf{k}_w,\mathbf{k}_s), \text{ for all } \mathbf{y},\mathbf{y}' \in \mathbb{D}_n \text{ with } \mu_y \leq \mu_{y'}\right]$$

Corollary 3 *$w \in U_2 \cap \mathcal{F}_c$ if and only if*

$$\left[\mathbf{y}+\mathbf{k} \succsim_{\mathbf{RL}} \mathbf{y}'{+}\mathbf{k} \implies \mathbf{x}(\mathbf{y};\mathbf{k}_w,\mathbf{k}_s) \succsim_{\mathbf{RL}} \mathbf{x}(\mathbf{y}';\mathbf{k}_w,\mathbf{k}_s), \text{ for all } \mathbf{y},\mathbf{y}' \in \mathbb{D}_n \text{with } \mu_y \geq \mu_{y'}\right].$$

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2.3 Further applications to decision making

Several interpretations of the model and results can be offered in addition to those already provided. The two more relevant cases that come in mind are those of individual choice under risk and intertemporal consumption choice.

Risk configuration

Let $\mathbf{y}$ and $\mathbf{y}'$ represent a list of outcomes available in different states of the world. The problem of income disaggregation can be easily translated into that of replacing each of these outcomes y_i by a lottery X_i such that $E(X_i) = y_i$ and taking two possible values, associated to two further states of the world: the "bad state" $w(y_i)$ and the "good state" $s(y_i)$. Our results gives us the condition to be satisfied for stochastic dominance of $\mathbf{y}$ over $\mathbf{y}'$ to be preserved after introducing this new risk. The values taken by the bad risk w must be concave with respect to the expected value y. It implies that the variance of this new risk is positively correlated with the expected outcome, but it is not sufficient. Notice that the result does not depend on the probability associated to each new state of the world. Any other case where a list of certain outcomes is replaced by a list of elementary lotteries with a "good" and a"bad" outcome can be treated in a similar way. A less immediate application arises in the case of intertemporal choice.

Life-cycle model

The model fits in with the life-cycle problem of a generation of n individuals living two periods 1, 2. Let $\mathbf{y}$ be the wealth vector received in the first period. The individual can allocate her wealth to consumption across the two periods. Let us also introduce two further endowments received in each period, k_1 and k_2, with $k_1 \leq k_2$, which cannot be moved from a period to another and are equally distributed across people. We can think of earning capacities which are related to hours of work. The agent wishes to smooth consumption over the two periods, denoted x_1 and x_2. She has an intertemporal separable utility function, where the subjective discount utility factor is $\beta < 1$. A risk-free asset brings an interest r. The resulting maximization program is:

$$\max_{x_1,x_2} v(k_1 + x_1) + \beta v(k_2 + x_2)$$
$$\text{s.t.} \quad x_1 + \frac{1}{1+r} x_2 = y$$

From the first order conditions of the above program, it follows that

$$\tfrac{v'(k_1+x_1^*)}{v'(k_2+x_2^*)} = \frac{\beta}{(1+r)}.$$

We impose that the subjective discount factor β is higher than the market discount factor $\frac{1}{(1+r)}$, which ensures a lower consumption in the first period. In this framework, a concave consumption function allows to preserve a SSD relation between $\mathbf{y}$ and $\mathbf{y}'$ (wealth distributions) to $\mathbf{x}(\mathbf{y};k_1, k_2)$ and $\mathbf{x}(\mathbf{y}';k_1, k_2)$ (snapshot consumption distributions).[3]

We now discuss the case of stochastic dominance of any finite order m.

3 Result beyond order two

If we try to extend the sufficiency condition of Theorem 1 in the case of third-degree stochastic dominance a difficulty arises: let us consider the function V introduced in expression

[3] The reader can refer to Peluso and Trannoy (2009b) for further applications of this simple model in individual and group decision making.

(2) above. Its third derivative, using $w'' = -s''$ and $w''' = -s'''$, can be written as:

$$\begin{aligned} V'''(y) &= u'''(w(y)+k_w)\left(w'(y)\right)^3 + u'''(s(y)+k_s)\left(s'(y)\right)^3 + \\ & 3w''(y)\left[u''(w(y)+k_w)w'(y) - u''(s(y)+k_s)s'(y)\right] + \\ & w'''(y)\left[u'(w(y)+k_w) - u'(s(y)+k_s)\right]. \end{aligned}$$

It is clear that if $w'(y) > s'(y)$ (Compensating case) then $V'''(y)$ is positive, while in the opposite case we cannot exclude

$$u''(w(y)+k_w)w'(y) - u''(s(y)+k_s)s'(y) > 0$$

and the sign of $V_u'''(y)$ will depend in general on the specification of u and w. This example hints at the compensation case. The same general counter-example can be derived for higher order derivatives of V. The following theorem shows that the compensating sharing functions are the relevant class.

Theorem 2 *For any finite* $m \geq 3$,

$w \in U_m \cap \mathcal{F}_c \iff \mathbf{y}+k\mathbf{e}_n \succsim_m \mathbf{y}'+k\mathbf{e}_n \implies \mathbf{x}(\mathbf{y};k_w,k_s) \succsim_m \mathbf{x}(\mathbf{y}';k_w,k_s)$, *for all* $\mathbf{y},\mathbf{y}' \in \mathbb{D}_n$

This theorem is proved through two lemma based on the Faà di Bruno formula (see Johnson 2002).

Faà di Bruno formula If f and g are C^m-real functions, then

$$\frac{d^m}{dx^m}g(f(t)) = \sum_{\mathbf{b}\in\mathbb{B}} \frac{m!}{b_1!b_2!..b_m!} g^{(s_\mathbf{b})}(f(t)) \left(\frac{f^{(1)}(t)}{1!}\right)^{b_1} \left(\frac{f^{(2)}(t)}{2!}\right)^{b_2} .... \left(\frac{f^{(m)}(t)}{m!}\right)^{b_m} \tag{3}$$

where $b_i \in \mathbb{N}^+$ for $i = 1,.,m$,

$$\mathbb{B} = \left\{\mathbf{b} \subset \mathbb{N}^{m+} | \sum_{i=1}^m ib_i = m\right\} \text{ and } s_\mathbf{b} = \sum_{i=1}^m b_i.$$

Using this formula we prove the following.

Lemma 1 *If f and g belong to U_m, then $h = g(f(t))$ belongs to U_m.*

Proof. Starting from (3), we decompose $s_{\mathbf{b}}$ into the sum of its odd and even elements, $s_{\mathbf{b}} = s_{\mathbf{be}} + s_{\mathbf{bo}}$.

Claim: If $m = \sum_{i=1}^{m} ib_i$ is even, then $s_{\mathbf{bo}}$ is even. We consider the case where the sum such that i and b_i are odd since if either i or b_i is even, their product is even and it can be subtracted from m without changing its parity. Using the Abel decomposition, we obtain

$$\sum_{i \in O} ib_i = -2 \sum_{h=1}^{m-1} \sum_{j=1}^{h} b_j + (m-1)s_{\mathbf{b0}}.$$

For the LHS to be even, we need $s_{\mathbf{b0}}$ to be even. Similarly we can prove that if $m = \sum_{i=1}^{m} ib_i$ is odd, then $s_{\mathbf{bo}}$ is odd.

Now we show that each term of the sum (3) corresponding to a specific b has the same sign as the sign of the m^{th} derivative of the functions of the class U_m.

Case 1:m even. Then we have to prove $h^{(m)} < 0$. We do not know the parity of $s_{\mathbf{b}}$.

Let us suppose first that $s_{\mathbf{b}}$ is even. Then $g^{(s_{\mathbf{b}})} < 0$. Then we need

$$\left(\frac{f'(t)}{1!}\right)^{b_1} \left(\frac{f''(t)}{2!}\right)^{b_2} .... \left(\frac{f^{(m)}(t)}{m!}\right)^{b_m} > 0.$$

In this product, we divide the elements for which b_i is even from the remaining. Indeed, the odd b_i are associated with positive derivatives. The sign of the product of negative derivatives depends on the parity of the sum of even b_i. By the claim we know that $s_{\mathbf{b0}}$ is even too. Then from $s_{\mathbf{b}} = s_{\mathbf{be}} + s_{\mathbf{bo}}$ we deduce that $s_{\mathbf{be}}$ must be even or equal to zero and the proof is done for this configuration, since a negative number powered to a even number is positive.

Let us consider the subcase where $s_{\mathbf{b}}$ is odd. Then $g^{(s_{\mathbf{b}})} > 0$. Then we need

$$\left(\frac{f'(t)}{1!}\right)^{b_1} \left(\frac{f''(t)}{2!}\right)^{b_2} .... \left(\frac{f^{(m)}(t)}{m!}\right)^{b_m} < 0.$$

By the same argument, we focus on the even b_i. By the claim we know that $s_{\mathbf{b0}}$ is even, from $s_{\mathbf{b}} = s_{\mathbf{be}} + s_{\mathbf{bo}}$ we deduce that $s_{\mathbf{b0}}$ must be odd and the proof of case 1 is now complete.

Case 2:m even can be proved along the same lines. ■

The previous lemma has a general interest in risk and stochastic dominance theory. It pins down the kind of transformation that preserves stochastic dominance at any finite order. We use it to prove a further lemma, which immediately demonstrates the sufficiency part of Theorem 2. Necessity of theorem for $m = 3$ is already provided by the above example. The writing of the necessity part in the general case is tedious and hence avoided.

Lemma 2 *Let $u \in U_m$ and $w \in U_m \cap \mathcal{F}_c$.*

Then the function $V(x) = u(w(y) + k_w) + u(s(y) + k_s)$ belongs to U_m.

Proof. By definition of s we get:

$$s^{(i)}(y) - w^{(i)}(y) = 0 \; for \; i = 2, ..m. \tag{4}$$

and it is also true that

$$(-1)^r \left[u^{(r)}(w(y) + k_w) - u^{(r)}(s(y) + k_s)\right] \leq 0, \text{for } r = 0, .., m. \tag{5}$$

Indeed, suppose that r is odd. Then $u^{(r)} > 0$ Then both $u^{(r)}(w(y) + k_w)$and $u^{(r)}(s(y) + k_s) > 0$ Then because of (1) and $u^{(r+1)} < 0$. then

$$\left[u^{(r)}(w(y) + k_w) - u^{(r)}(s(y) + k_s)\right] > 0$$

and we got the formula (5).

The proof is similar for r even.

Using (3) we get:

$$\begin{aligned} V^{(m)}(t) &= \sum_{\mathbf{b} \in \mathbb{B}} \frac{m!}{b_1!..b_m!} u^{(k)}(w(t)) \left(\frac{w'(t)}{1!}\right)^{b_1} ... \left(\frac{w^{(m)}(t)}{m!}\right)^{b_m} + \\ & \sum_{\mathbf{b} \in \mathbb{B}} \frac{m!}{b_1!..b_m!} u^{(k)}(s(t)) \left(\frac{s'(t)}{1!}\right)^{b_1} ... \left(\frac{s^{(m)}(t)}{m!}\right)^{b_m} \end{aligned} \tag{6}$$

As shown for the third stochastic dominance case, the difficulty comes from the terms involving the marginal utility of income. It will be useful to separate the sum $\sum$ defined in (3) (that is over all $\mathbf{b} \in \mathbb{B}$) into two terms: the first one, denoted $\sum_{\mathbb{B}_{1E}}$contains all the terms with b_1 even or equal to 0, while $\sum_{\mathbb{B}_{1O}}$collects all the terms with b_1 odd. More precisely:

$$\begin{aligned} \mathbb{B}_{1E} &= \left\{\mathbf{b} \in \mathbb{B} | (-1)^{b_1} = 1\right\} \\ \mathbb{B}_{1O} &= \left\{\mathbf{b} \in \mathbb{B} | (-1)^{b_1} = -1\right\} \end{aligned}$$

Let us focus on m even (the proof is quite similar if m odd). As in Lemma 1, we distinguish the cases with $s_{\mathbf{b}}$ even or odd.

Case 1- $s_{\mathbf{b}}$ is even. From Lemma 1 we know that both $s_{\mathbf{be}}$ and $s_{\mathbf{bO}}$ must be even. Then:

- Using (4 and the fact that $s_{\mathbf{be}}$ and $s_{\mathbf{bO}} - b_1$ are both even, the addenda of (6) with b_1 even or equal to 0 may be grouped as follows:

$$\sum_{\mathbb{B}_{1E}} \frac{m!}{b_2!..b_m!} \left[u^{(s_{\mathbf{b}})}(w(t)) \left(\frac{w'(t)}{1!}\right)^{b_1} + u^{(s_{\mathbf{b}})}(s(t)) \left(\frac{s'(t)}{1!}\right)^{b_1}\right] \left(\frac{w''(t)}{2!}\right)^{b_2} ... \left(\frac{w^{(m)}(t)}{m!}\right)^{b_m} \tag{7}$$

Since, b_1 is even or equal to 0 and $s_{\mathbf{b}}$ even, the term in square brackets is negative. The product term is positive thanks to the evenness of $s_{\mathbf{be}}$. Then each term of the sum is negative.

- Using (4 and the fact that $s_{\mathbf{be}}$ is even and $s_{\mathbf{bO}} - b_1$ is odd, the addenda of (6) with b_1 odd may be grouped as follows

$$\sum_{\mathbb{B}_{1O}} \frac{m!}{b_2!..b_m!} \left[u^{(s_{\mathbf{b}})}(w(t)) \left(\frac{w'(t)}{1!}\right)^{b_1} - u^{(s_{\mathbf{b}})}(s(t)) \left(\frac{s'(t)}{1!}\right)^{b_1}\right] \left(\frac{w''(t)}{2!}\right)^{b_2} ... \left(\frac{w^{(m)}(t)}{m!}\right)^{b_m}$$

-Using the decisive condition ii) and $s_{\mathbf{b}}$ even and (5) the term in square brackets is negative. The product term is positive thanks to the evenness of $s_{\mathbf{be}}$. Then, each term of the sum is negative.

Case 2- $s_{\mathbf{b}}$ is odd. From Lemma 1 we know that $s_{\mathbf{be}}$ must be odd and $s_{\mathbf{bO}}$ must be even. Then:

- The addenda of (6) with b_1 even or equal to 0, using (4 and given that $s_{\mathbf{be}}$ is odd and $s_{\mathbf{bO}} - b_1$ remains even, may be grouped as follows:

$$\sum_{\mathbb{B}_{1E}} \frac{m!}{b_2!..b_m!} \left[u^{(s_{\mathbf{b}})}(s(t)) \left(\frac{s'(t)}{1!} \right)^{b_1} - u^{(s_{\mathbf{b}})}(w(t)) \left(\frac{w'(t)}{1!} \right)^{b_1} \right] \left(\frac{w''(t)}{2!} \right)^{b_2} ... \left(\frac{w^{(m)}(t)}{m!} \right)^{b_m}$$

From ii) and the oddness of $s_{\mathbf{b}}$ and (5) it follows that the term in square brackets is positive. The product term is negative because $s_{\mathbf{be}}$ is odd

- The addenda of (6) with b_1 odd, using (4 and given that $s_{\mathbf{be}}$ is odd and $s_{\mathbf{bO}} - b_1$ is odd, may be grouped as follows:

$$\sum_{\mathbb{B}_{1O}} \frac{m!}{b_2!..b_m!} \left[u^{(s_{\mathbf{b}})}(w(t)) \left(\frac{w'(t)}{1!} \right)^{b_1} + u^{(s_{\mathbf{b}})}(s(t)) \left(\frac{s'(t)}{1!} \right)^{b_1} \right] \left(\frac{w''(t)}{2!} \right)^{b_2} ... \left(\frac{w^{(m)}(t)}{m!} \right)^{b_m} \tag{8}$$

From the oddness of $s_{\mathbf{b}}$ it follows that the term in square brackets is positive. The product term is negative because $s_{\mathbf{be}}$ is odd. ■

The theorem brings conditions that are useful in a variety of contexts. For instance suppose that we are interested in poverty relations à la Foster-Shorrocks (1988). Theorem 1 gives us conditions that are sufficient to preserve head-count ratio and poverty gap dominance. While these orderings are sufficient in a number of applications, it may desirable or necessary to go beyond and consider the case of quasi-orderings where the poverty gap is raised to power two, three and so on. The class of sharing functions should have derivatives which alternative in sign up to the desired order and in addition we have to be in the compensation case.

4 Conclusive remarks

This paper investigates the preservation of stochastic dominance of any finite order m, when each element of the compared income distributions is disaggregated into some sub-elements according to a well-defined sharing rule. We present our results in the simplest configuration where income disaggregation arises when elementary units (as individual income recipients) are gathered in subsets (let us say families) that generate a less fine partition of the initial set. Our aim is to characterize the class of sharing rules that allow to convey dominance relations established at the household level to the individual one. Albeit we have chosen the case of households versus individuals to illustrate the results of this paper, the model has several applications in different fields: a simple example is that of a geographical partition with several federal States. Is it possible to infer something about inequality among regions belonging to different States, just knowing the inequality among States and the sharing rule within each State? This paper provides useful answers for this type of problems. We also illustrate further applications in terms of individual decision-making under risk and time.

The general theorem that we are able to prove shows that two conditions must be respected by the uniform sharing function for stochastic dominance statements being preserved. The sharing function must have derivatives of the same sign as the utility function characterizing the stochastic order of interest. This condition generalizes the condition obtained previously by Peluso and Trannoy (2007). But the important message here is that this condition is not sufficient to guarantee the preservation of higher order stochastic dominance. A more subtle condition has to be added: the household has to follow a *compensating rule* towards within-household inequality: given an unequal initial endowment of resources within the group, additional incomes has to be shared in favour of the disadvantaged individual *and* this compensating attitude has to be less and less strong in marginal terms when household income increase.

A mathematical extension could be to study what is going on about the preservation property when the m order of stochastic dominance goes to infinity. In that case, the underlying class of utility functions corresponds to the "mixed risk averse" class studied in economics by Fisburn and Willig (1984) and Caballé and Pomansky (1996). Adressing this problem represents an avenue for further research.

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