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Abstract

We consider the problem of measuring social exclusion using qualitative data. We suggest a class of social exclusion indicators deriving the partial orderings associated with dominace for these indicators. We characterize the set of transformations on the distribution of individual deprivation scores underlying the dominace conditions proposed.

Keywords: Social exclusion, dominance, measures.
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1 Introduction

Social exclusion refers to inability of individuals to participate in basic economic and social activities of the society in which they live. European Commission's Programme specification for targeted socioeconomic research describes it as 'disintegration and fragmentation of social relation and hence a loss of social cohesion'.

Social exclusion is not just a consequence of unemployment (Atkinson, 1998). It is certainly true that an unemployed person may not have sufficient income to maintain a subsistence standard of living and hence may become socially excluded. But an employed person may have to live in a polluted area, and this may make his living quite uncomfortable. Expansion of employment may increase wage inequality and hence may not end exclusion. Social exclusion can also generate from the operations of the market and the state. A firm's profit maximizing price may exclude many individuals from participation in a customary consumption activity. Targeted benefit programme of the state may benefit only some particular groups and deprive others.

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In view of growing concern about social exclusion, UNDP(2001) included it as a component of human poverty in industrialized countries.[1]

As social exclusion includes economic, social and political aspects of life, it is a multidimensional phenomenon. Since essential to achievement of human choices is building human capabilities, we can also interpret the issue in terms of (1) functionings, the various things an individual may value doing or being and (ii) capability, the ability to achieve (Sen, 1985). The valued functionings may vary from such elementary ones like life expectancy, adult literacy, adequate nourishment to complex activities or personal characteristics like participation in social gatherings and having self respect. The standard of living in this framework depends on the opportunity set of the basic capabilities to function. If social exclusion is defined as the inability to achieve valuable needs, then regarding it as capability failure makes considerable sense. Thus, social exclusion implies deprivation in a wide range of characteristics or functionings of living standard, which can be of quantitative as well as qualitative type.

Social exclusion is related to both multidimensional poverty or inequality, but should not be identified with either of them (Atkinson, 1998). Both multidimensional poverty and social exclusions are problems of capability failures, while in the former we view the failure in terms of shortfalls of the characteristics from respective thresholds, in the latter it is a problem of inability to participate. On the other hand, multidimensional inequality aggregates dispersions of different attributes across individuals. Atkinson (1998) further argued that social exclusion is a relative and dynamic concept, where relativity means that we cannot say whether a person is excluded or not without looking into the positions of the others and dynamic character comes from the fact that a person remains excluded if his deprivation continues or deepens over time.

In this paper we approximate the exclusion or deprivation score of a person by the number of characteristics from which the individual is excluded. This is a very simple way of calculating the deprivation score of a person because some characteristics may be more important than others and higher weights should be assigned to the failure of more important characteristics. But we make this assumption for simplicity of exposition and our analysis with the same weight (= 1) case can be easily extended to the variable weight case. Strictly speaking, this approach is quite similar to the view taken by Duffy (1995), Rowntree Foundation (1998), UK House of Commons (1999) and Paugam and Russell (2000), where social exclusion is considered as lack of participation in social institutions.[2] In our structure lack of participation is treated in terms of capability failures. We assume that a social exclusion measure is a real valued function of individual exclusion levels.

The objective of this paper is to rank different societies with respect to social exclusion. We demonstrate that several seemingly unrelated procedures for evaluating

[1] However,UNDP viewed social exclusion as a problem of unemployment only and measured it by the rate of long-term (12 months or more) unemployment rate.

[2] See Chakravarty and D'Ambrosio (2006), for an application of the variable weight approach using European Union data. For other implicit conceptualizations of social exclusions, see Room(1995), Akerlof (1997), Klassen(1998), Bradshaw et al.(2000) and Tsakloglou and Papadopoulos(2001).

alternative social exclusion profiles are equivalent. Our problem is closely related to the Atkinson (1970) result on the relation between Lorenz dominance and the principle of transfers, and the Rothschild and Stiglitz (1970) mean preserving spreads. However, differences from these papers arise because the support of a social exclusion distribution in this work is a unidimensional grid (see Fishburn and Lavalle, 1995) given that the deprivation scores are non negative integers, moreover our comparisons may involve also distribution with different aggregate deprivation score. It is shown that of two societies with a constant population size n, if one is at least as excluded as the other by all social exclusion measures that are normalized, monotonic, anonymous and have non-decreasing marginals, then the former social exclusion dominates the other. The converse is also true. Normalization of a social exclusion measure means that the level of social exclusion is zero if nobody is excluded. Monotonicity requires the social exclusion measure to increase if the deprivation score of a person increases. According to anonymity the social exclusion measure is symmetric, that is, it remains invariant under any permutation of individual deprivation scores. Marginal social exclusion is defined as the change in social exclusion if we increase the deprivation score of a person by one. Nondecreasingness of marginal social exclusion ensures that in aggregating the individual exclusion levels into an overall measure we attach higher weights to higher exclusions. This parallels an argument put forward by Sen (1976) in the context of poverty measurement. He argued that in the construction of the poverty index higher deprivation should be assigned higher weight, where deprivation of a poor person is given by his income shortfall from the poverty line representing the income necessary to maintain a subsistence standard of living. We say that one society social exclusion dominates another if the cumulative deprivation scores of the first k most excluded persons in the former is at least as large as that in the latter, where $k = 1, 2, ..., n$. These two conditions are also equivalent to the requirement that the weighted sum of individual deprivation scores in the former (dominant) society is at least as large as that in the latter (the dominated one), where the individual failures as well as nonnegative weights are arranged in non-increasing order. A fourth equivalent condition is that we can derive the dominant distribution from the dominated one by a sequence of transformations that correspond respectively to monotonicity, non-decreasingness of marginals and anonymity. We then extend our result to the variable population case using a population replication principle, which allows a cross population comparisons of exclusions.

Shorrocks (1983) showed that if u and v are two income distributions over a given population size, then u generalized Lorenz dominates v if and only if u is regarded as socially better than v by all increasing and S-concave social welfare functions. Generalized Lorenz domination of u over v is also equivalent to the condition that the former is obtained from the latter by successive applications of a finite number of T-transformations, assuming that the incomes are ordered non-decreasingly (Marshall and Olkin,1979, p.108). Therefore, our equivalence theorem can be regarded as the social exclusion counterpart to the Shorrocks (1983) and Marshall and Olkin (1979) result on generalized Lorenz ordering. In this respect the Social Exclusion curve characterized in this work is more directly related to the 'poverty counterpart' of Shorrocks (1983) generalized Lorenz curve, that is the *absolute poverty gap profile* *curve*, introduced by Spencer and Fisher (1992), Jenkins and Lambert (1997) and Shorrocks (1995, 1998).

2 Preliminaries and notation

Let $\mathbb{N}$ ($\mathbb{N}_0$) be the set of positive (non-negative) integers, and $\mathbb{R}$ the real line. For all $n \in \mathbb{N}$, D^n is the $n-$fold Cartesian product of $\mathbb{N}_0$ and 1^n is the $n-$coordinated vector of ones. For any society with a population size of $n \in \mathbb{N}$, there is a finite non-empty set of characteristics F relevant for social integration. We assume that F is the same for all societies under consideration so that cross population comparisons of social exclusion becomes possible in terms of the elements of F (see Atkinson et al., 2002).

The characteristics profile of individual i is represented by the $m-$dimensional vector $\mathbf{x}_{i,\cdot} = (x_{i,1}, x_{i,2}, ..., x_{i,j}, ..., x_{i,m})$ where $x_{i,j} \in \{0, 1\}$. We say that individual i is excluded in terms of characteristic j if $x_{i,j} = 1$, otherwise $x_{i,j} = 0$. It is assumed that the calculation of $x_{i,j}$ involves a longitudinal or dynamic aspect. Each individual's level of exclusion is represented by the deprivation (exclusion) score $x_i := \sum_{j=1}^{m} x_{i,j}$. That is, the personal level of exclusion of individual i is given by the sum of his/her exclusion indicators across all characteristics. Thus, in calculating the level of exclusion of a person we assume, for simplicity, that all characteristics are equally important. But some characteristics may be more important than others, and in this case different characteristic failures will get different weights. Our analysis with the same weight ($= 1$) for different characteristics can be easily extended to the unequal weight case.

This procedure of calculating the deprivation score of a person is similar to the Basu and Foster (1998) way of determination of illiterate persons in a household. Their procedure identifies an adult member of a household by the number 0 or 1 according as he/she is illiterate or literate. The total number of illiterates in the household is then given by the sum of zero's in the household.

The whole population exclusion profile is the vector $x = (x_1, x_2, ..., x_i, ..., x_n)$. D^n is the set of exclusion profiles for any $n-$person society, while $D := \cup_{n\in\mathbb{N}} D^n$, is the set of all possible exclusion profiles. Note that by construction $x_i \in \{0, 1, 2, ..., m\}$ i.e. the maximum level of individual social exclusion is m.

A measure of social exclusion is a function $E : D \to \mathbb{R}$, with $E^n : D^n \to \mathbb{R}$ representing the restriction of the measure over the set of distributions of the same population size n. For a fixed population size n and number of exclusion characteristics m, $D^n = \otimes_{i=1}^{n} \{0, 1, 2, ..., m\}_i$. For any $n \in \mathbb{N}$, $x \in D^n$, $E^n(x)$ is an indicator of the degree of exclusion suffered by the persons in the society.

We write $\bar{x}$ to denote the non-increasingly ordered permutation of x, i.e. $\bar{x}_1 \geq \bar{x}_2 \geq ... \geq \bar{x}_i \geq ... \geq \bar{x}_{n-1} \geq \bar{x}_n$.

The following properties, considered by Chakravarty and D'Ambrosio (2006), specify the behavior of the social exclusion indicators $E^n(.)$.

Axiom 1 ((NOM) Normalization) *For all* $n \in \mathbb{N}$, $E^n(0 \cdot 1^n) = 0$.

Axiom 2 ((ANY) Anonymity) *For all* $n \in \mathbb{N}$, $x \in D^n, E^n(x) = E^n(xP)$, *where* P *is an* $n \times n$ *permutation matrix.*

Axiom 3 ((MON) Monotonicity) *For all $n \in \mathbb{N}$, $x \in D^n$, all $i \in \{1, 2, ..n\}$ such that $x_i < m$*

$$E^n(x_1, x_2, .., x_i + 1, .., x_n) \geq E^n(x_1, x_2, .., x_i, ..., x_n).$$

Axiom 4 ((NMS) Non-Decreasingness of Marginal Social Exclusion) *For all $n \in \mathbb{N}$, $x \in D^n$, all $i, j \in \{1, 2, ..n\}$, if $m > x_i \geq x_j$ then*

$$E^n(x_1, x_2, .., x_i + 1, ..., x_j, .., x_n) \geq E^n(x_1, x_2, .., x_i, ..., x_j + 1, .., x_n).$$

To make comparisons between distributions of social exclusion indicators with different population size we consider a replication of the original population. Let x^r denote the $r-$times replica of the vector x i.e. $x^r := (x, x, x, ..., x, x)$ with x repeated r times for $r \in \mathbb{N}$.

Axiom 5 ((POP) Population Principle) *For all $n, r \in \mathbb{N}$, $x \in D^n$, $E^{rn}(x^r) = E^n(x)$.*

NOM, which is a cardinality property, says that if nobody in the society is excluded from any characteristic, then the level of social exclusion is zero.

The remaining set of axioms identifies the impact on the evaluation of social exclusion of some transformations of the vectors x. We now make these transformations explicit.

Axiom ANY requires that social exclusion does not depend on the identities of the individuals but only on their values of the Social Exclusion profiles. Therefore permuting the identities of the individuals does not affect the level of social exclusion. We introduce first the following transformation

Definition 1 (T_A transformation) *Let $x, y \in D^n$. x is obtained from y through a "T_A transformation" i.e. $x = T_A(y)$ if and only if: there exists a permutation function $\pi : \{1, 2, ..n\} \to \{1, 2, ..., n\}$ such that $x_{\pi(i)} = y_i$ for all $i \in \{1, 2, .., n\}$.*

Let $\mathcal{T}_A$ denote the set of all T_A transformations associated with all permutation functions π. Therefore ANY is equivalent to stating that for all $T_A \in \mathcal{T}_A$, if $x = T_A(y)$ then $E^n(x) = E^n(y)$.

The following transformation is associated with the MON axiom, it requires that x is obtained from y through an increase in social exclusion of one individual in one characteristic.

Definition 2 (T_M transformation) *Let $x, y \in D^n$. x is obtained from y through a "T_M transformation" i.e. $x = T_M(y)$ if and only if: there exist $i \in \{1, 2, .., n\}$ such that $x_i = y_i + 1 \leq m$, and $x_h = y_h$ for all $h \in \{1, 2, .., n\} \setminus i$.*

MON says that under T_M transformation the final distribution cannot show less social exclusion than the original one.

Let $\mathcal{T}_M$ denote the set of all T_M transformations associated with all $i \in \{1, 2, .., n\}$ then MON is equivalent to stating that for all $T_M \in \mathcal{T}_M$, if $x = T_M(y)$, then $E^n(x) \geq E^n(y)$.

Next transformation requires that an increase in an individual's exclusion score has an higher impact on social exclusion the higher is the individual's exclusion score.

Definition 3 (**T_{NMS} transformation**) *Let $x, y \in D^n$. x is obtained from y through a "T_{NMS} transformation" i.e. $x = T_{NMS}(y)$ if and only if: there exist $i, j \in \{1, 2, .., n\}$ such that $m > x_i \geq x_j + 1$, $x_h = y_h$ for all $h \neq i, j$, $x_i = y_i + 1$ and $x_j + 1 = y_j$.*

Let $\mathcal{T}_{NMS}$ denote the set of all T_{NMS} transformations associated with all $i, j \in \{1, 2, .., n\}$ such that $x_i \geq x_j + 1$. Axiom NMS is therefore equivalent to stating that for all $T_{NMS} \in \mathcal{T}_{NMS}$, if $x = T_{NMS}(y)$ then $E^n(x) \geq E^n(y)$. Clearly, NMS captures the relativity aspect of social exclusion.

Finally, we need a transformation that will allow us to link distributions with different population sizes.

Definition 4 (**T_{PR} transformation**) *Let $x, y \in D^n$. x is obtained from y through a "T_{PR} transformation" i.e. $x = T_{NMS}(y)$ if and only if: there exist $r \in \mathbb{N}$ s.t. $x = y^r$.*

Let $\mathcal{T}_{PR}$ denote the set of all T_{PR} transformations associated with all $r \in \mathbb{N}$. Axiom POP is therefore equivalent to stating that for all $T_{PR} \in \mathcal{T}_{PR}$, if $x = T_{PR}(y)$ then $E^{rn}(x) = E^n(y)$.

3 The results

The following theorem identifies the set of partial orderings defined over distributions in D^n consistent with the measures satisfying the previous axioms. A rank-dependent class of social exclusion indices is also presented. It can be considered the social exclusion equivalent of the generalized Gini social evaluation function for profile of opportunity sets characterized in Weymark (2003).

According to condition 1of Theorem 1, the weighted sum of individual exclusion levels in profile $\bar{x}$ is at least as high as that in the profile $\bar{y}$, where the non-negative weights are arranged non-increasingly. Given that $\bar{x}$ and $\bar{y}$ are arranged in non-increasing order, condition 2 says that the cumulative sum of the exclusion levels of the first k persons in $\bar{x}$ is at least as large as that in $\bar{y}$, where $k = 1, 2, ..., n$, that is, x dominates y by the social exclusion criterion. Theorem 1 shows that these two conditions are equivalent to the requirement that y does not have more social exclusion than x for all social exclusion indices that fulfil NOM, ANY, MON and NMS.

Theorem 1 *Let $x, y \in D^n$. The following statements are equivalent:*

(1)$\sum_{i=1}^{n} v_i^n \cdot \bar{x}_i \geq \sum_{i=1}^{n} v_i^n \cdot \bar{y}_i$ for all $v_i^n \geq v_{i+1}^n \geq 0$.

(2) $\sum_{i=1}^{k} \bar{x}_i \geq \sum_{i=1}^{k} \bar{y}_i$ for all $k = 1, 2, ...n$..

(3) x can be obtained from y through a finite sequence of T_A, T_M and T_{NMS} transformations.

(4) $E^n(x) \geq E^n(y)$ for all indices $E^n(.)$ satisfying NOM, ANY, MON and NMS.

Proof of Theorem 1. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$:

Let $\delta_i := \bar{x}_i - \bar{y}_i$. We prove that $\sum_{i=1}^{n} v_i^n \cdot \delta_i \geq 0$ for all $v_i^n \geq v_{i+1}^n \geq 0$ implies $\sum_{i=1}^{k} \delta_i \geq 0$ for all $k = 1, 2, ...n$. Without loss of generality we define $v_i^n = \sum_{k=i}^{n} \alpha_k^n$. If $\alpha_k^n \geq 0$ for all $k = 1, 2, ..., n$ then $v_i^n \geq v_{i+1}^n \geq 0$. As a result we can rewrite

$$\sum_{i=1}^{n} v_i^n \cdot \delta_i = \sum_{i=1}^{n} \sum_{k=i}^{n} \alpha_k^n \cdot \delta_i = \sum_{i=1}^{n} \alpha_k^n \sum_{i=1}^{k} \delta_i.$$

Condition (1) is therefore equivalent to $\sum_{i=1}^{n} \alpha_k^n \left(\sum_{i=1}^{k} \bar{x}_i - \bar{y}_i \right) \geq 0$ for all $\alpha_k^n \geq 0$. Clearly condition (2) is sufficient as well as necessary for (1). Suppose that $\sum_{i=1}^{k^*} \bar{x}_i < \sum_{i=1}^{k^*} \bar{y}_i$ for a given k^*, letting $\alpha_k^n = 0$ for all $k \neq k^*$ and $\alpha_{k^*}^n > 0$, then condition (1) is violated.

$\mathbf{(2)} \implies \mathbf{(3)}$:

Note that because of T_A transformations we can restrict attention to non decreasingly ordered vectors $\bar{x}, \bar{y}$. We show that if (2) is satisfied it is possible to decompose the vector of elements δ_i into a finite sequence of changes associated with T_M and/or T_{NMS} transformations. The decomposition process is divided into two parts. (A) First we identify the T_M transformations δ_i^* s.t. $\sum_{i=1}^{n} \bar{x}_i = \sum_{i=1}^{n} (\bar{y}_i + \delta_i^*)$, i.e. starting from $\bar{y}$ we get a distribution with the same total exclusion score as $\bar{x}$. (B) Then we compare the two distributions $\bar{x}$, $\bar{y} + \delta^*$ with same total exclusion levels identifying the set of T_{NMS} transformations leading to $\bar{x}$ starting from $\bar{y} + \delta^*$.

Part (A):

Let $\delta_i := \bar{x}_i - \bar{y}_i$ and $\Delta_i := \bar{y}_{i-1} - \bar{y}_i$ for $i = 2, 3, ..., n$. Denote $X = \sum_{i=1}^{n} \bar{x}_i$, and $Y = \sum_{i=1}^{n} \bar{y}_i$.

Suppose that $X > Y$. Let $i^* = \max \{i : \delta_i > 0, \ \Delta_i > 0 \text{ if } i > 1\}$. Then we identify the sequence of T_M transformations making use of a sequence of increases $\delta_i^{(t)} \in \mathbb{N}$, where t denotes the index of the element in the sequence and i is the position of the individual experiencing the increase in social exclusion. Each of these increases corresponds to $\delta_i^{(t)}$ type T_M transformations and will lead to a sequence of distributions $\bar{y}^{(t)}$ starting from $\bar{y}^{(0)} := \bar{y}$.

Consider now $\delta_{i^*}^{(1)}$ i.e. the first element of the sequence.

More precisely, $\delta_{i^*}^{(1)} := \min \{\delta_{i^*}, \Delta_{i^*}, X - Y\}$. By construction, the new distribution $\bar{y}^{(1)}$ is obtained letting $\bar{y}_i^{(1)} := \bar{y}_i$ for all $i \neq i^*$ and $\bar{y}_{i^*}^{(1)} := \bar{y}_{i^*} + \delta_{i^*}^{(1)}$. According to the definition of $\delta_{i^*}^{(1)}$, the ranking in $\bar{y}^{(1)}$ is preserved (because $\Delta_{i^*} > 0$ is considered in the definition of $\delta_{i^*}^{(1)}$), furthermore $\bar{y}_{i^*} < \bar{y}_{i^*}^{(1)} \leq \bar{x}_{i^*}$ (given that $\delta_{i^*} > 0$ is considered in the definition of $\delta_{i^*}^{(1)}$) and $X \geq Y^{(1)} > Y$ (because $X - Y > 0$ is considered in the definition of $\delta_{i^*}^{(1)}$).

Having derived the first stage of the algorithm we can move to its general specification. The decomposition algorithm can be constructed letting $\bar{y}_i^{(t)} := \bar{y}_i^{(t-1)}$ for all $i \neq i^*$ and $\bar{y}_{i^*}^{(t)} := \bar{y}_{i^*}^{(t-1)} + \delta_{i^*}^{(t)}$, where $\bar{y}^{(0)} := \bar{y}$. We then denote $\delta_{i,t} := \bar{x}_i - \bar{y}_i^{(t-1)}$ and $\Delta_{i,t} := \bar{y}_{i-1}^{(t-1)} - \bar{y}_i^{(t-1)}$ for $i = 2, 3, ..., n$, and $Y^{(t)} = \sum_{i=1}^{n} \bar{y}_i^{(t)}$.

The algorithm is obtained by identifying

$$i^* = \max \{i : \delta_{i,t} > 0, \ \Delta_{i,t} > 0 \text{ if } i > 1\}$$

and letting

$$\delta_{i^*}^{(t)} = \min \left\{\delta_{i^*,t}; \Delta_{i^*,t}; X - Y^{(t-1)}\right\}.$$

The procedure followed by the algorithm stops after T stages when $\delta_{i^*}^{(T)} = X - Y^{(T-1)}$. Given that after each stage the difference $X - Y^{(t-1)}$ is reduced by at least one unit, T equals at most $X - Y$, which is finite. As a result the final distribution $\bar{y}^{(T)}$ is obtained as $\bar{y} + \delta^*$ where the vector δ^* is such that $\delta_i^* = \sum_{t=1}^{T} \delta_i^{(t)}$ for all i associated with at least one positive $\delta_i^{(t)}$ and 0 for all the other elements of δ^*.

Part (B):

We consider now the case $X = Y$, where the distribution $\bar{y}$ can also be considered as obtained through the set of transformations in part A. We now apply a sequence of T_{NMS} transformations to $\bar{y}$ in order to obtain $\bar{x}$. Recall that by construction $\bar{x}$ is obtained from $\bar{y}$ through a T_{NMS} transformation if $\bar{x}_i - \bar{y}_i = 1$ and $\bar{x}_j - \bar{y}_j = -1$ where $j > i$ and $\bar{x}_h - \bar{y}_h = 0$ for all $h \neq i, j$. Furthermore, notice that $\sum_{i=1}^{k} \bar{x}_i \geq \sum_{i=1}^{k} \bar{y}_i$ for all $k = 1, 2, ..., n$, i.e. $\sum_{i=1}^{k} \delta_i \geq 0$ for all $k = 1, 2, ..n$ and $\sum_{i=1}^{n} \delta_i = 0$ implies that $\sum_{i=k}^{n} \delta_i \leq 0$ for all $k = 1, 2, ..n - 1$, i.e. the last element δ_i that is different from 0 is negative.

The algorithm is constructed letting $\bar{y}_i^{(t)} := \bar{y}_i^{(t-1)}$ for all $i \neq i^-, i^+$ and $\bar{y}_h^{(t)} := \bar{y}_h^{(t-1)} + \delta_h^{(t)}$ for $h = i^-$, and $h = i^+$, where $\bar{y}^{(0)} := \bar{y}$. We then denote $\delta_{i,t} := \bar{x}_i - \bar{y}_i^{(t-1)}$ and $\Delta_{i,t} := \bar{y}_{i-1}^{(t-1)} - \bar{y}_i^{(t-1)}$ for $i = 2, 3, ., n$. We let

$$i^- = \max \{ i : \delta_{i,t} < 0, \ \Delta_{i+1,t} > 0 \text{ if } i < n \},$$

and

$$i^+ = \max \{ i : \delta_{i,t} > 0, \ \Delta_{i,t} > 0 \text{ if } i > 1 \}$$

and therefore by construction $i^+ > i^-$. Then we write

$$-\delta_{i^-}^{(t)} = \delta_{i^+}^{(t)} = \min \{ \delta_{i^+,t}; \Delta_{i^+,t}; -\delta_{i^-,t} \}$$

which complete the algorithm for deriving the set of T_{NMS} transformations. Note that $-\delta_{i^-}^{(t)} = \delta_{i^+}^{(t)} > 0$ where $i^+ > i^-$, they are therefore obtained by $\delta_{i^+}^{(t)}$ T_{NMS} transformations. Furthermore these transformations are rank preserving given that by construction $\delta_{i^+}^{(t)} \leq \Delta_{i^+,t}$ and $\delta_{i^-,t} \leq \delta_{i^-} < 0$. Note that $\sum_{i=1}^{n} |\delta_i| / 2$ is the (finite) number of T_{NMS} transformations implementing $\bar{x}$ from $\bar{y}$.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$: By definition.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$:

We prove that the class of social exclusion measures in (1) is a special case of those in (4).

Let $E^n(x, v) := \sum_{i=1}^{n} v_i^n \cdot \bar{x}_i$. The index $E^n(x, v)$ satisfies NOM given that $E^n(01^n, v) = 0$, and ANY since it is defined in terms of ranked vectors $\bar{x}$.

Note that $E^n(x, v)$ satisfies MON if $E^n(x, v) \leq E^n(x', v)$ for all $x, x' \in D^n$ s.t. x is obtained from x' reducing the level of exclusion of at least one individual in at least one characteristic. It follows that $E^n(x', v) - E^n(x, v) = \sum_{i=1}^{n} v_i^n \cdot (\bar{x}_i' - \bar{x}_i) \geq 0$. Suppose that $\bar{x}_i - \bar{x}_i' = 0$ for all $i \neq j$ and $\bar{x}_j' - \bar{x}_j = 1$. Then $E^n(x', v) - E^n(x, v) \geq 0$ implies that $v_j^n \geq 0$. Given that we can choose any $j = 1, 2, ..., n$, it follows that a necessary condition for satisfying MON is $v_j^n \geq 0$ for all j. This condition is also sufficient.

Suppose that x is obtained from x' through a T_{NMS} transformation involving individuals i and $i + 1$ for $i \leq n - 1$. The condition $E^n(x', v) - E^n(x, v) \geq 0$ implies that $v_i^n - v_{i+1}^n \geq 0$, that is, $v_i^n \geq v_{i+1}^n$ for all $i = 1, 2, ..., n-1$ is necessary for satisfying NMS. It is also sufficient. ■

Let u and v be two non-decreasingly ordered income distributions over a given population size n. Shorrocks (1983) showed that u generalized Lorenz dominates v (that is, $\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i$, $1 \leq k \leq n$) if and only if $F(u) \geq F(v)$ for all S-concave social welfare functions F, where S-concavity of F demands that a rank preserving transfer of income from a person to anyone with a lower income does not decrease welfare. It may be noted that all S-concave functions satisfy anonymity. Marshall and Olkin's (1979, p.108) Proposition A.2 shows that generalized Lorenz domination of u over v is equivalent to the condition that u can be obtained from v by applications of a finite number of T transformations of the form $T(z_1, z_{2,...}, \alpha z_i, z_{i+1,...}, z_n)$, where $\alpha > 1$. Therefore, equivalence between conditions (2), (3) and (4) of Theorem 1 can be regarded as social exclusion analogue to the Marshall and Olkin (1979) and Shorrocks (1983) demonstrations. Moreover, it generalizes Muirhead 1903 result (see Marshall and Olkin, 1979) linking majorization and T transformations for distributions whose components are non negative integers with fixed equal sums.

We now extend the previous results in order to compare distributions with different population sizes.

Let $\mathbb{Q}_+$ denote the set of non-negative rational numbers.

We construct the Social Exclusion curve $SE_x(p)$ of distribution $x \in D^n$ for $p \in [0, 1]$ in the following way:

$$SE_x(\frac{k}{n}) = \frac{1}{n} \sum\nolimits_{i=1}^{k} \bar{x}_i \text{ for } k = 1, 2, ..., n, \quad SE_x(0) = 0$$

and $SE_x(p) = SE_x(\frac{k}{n}) + \left(p - \frac{k}{n}\right) \left[SE_x(\frac{k+1}{n}) - SE_x(\frac{k}{n})\right]$ for all $p \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$ for $k = 0, 1, 2, ..., n-1$.

That is, the social exclusion curve is a plot of the cumulative sum of personal exclusion levels, expressed as a fraction of the population size, against the corresponding population proportions, where the persons are arranged non-increasingly with respect to their exclusions. Theorem 2 shows that y is not more excluded than x socially by all exclusion indices satisfying NOM, MON, ANY, NMS and POP if and only if the social exclusion curve of x dominates that of y, that is, the social exclusion curve of x is nowhere below that of y (condition 2). This is equivalent to the condition that the weighted sum of individual exclusions in x is at least as high as that in y, where the rank dependent weights are increasing and concave.

Theorem 2 *Let $x \in D^{n_x}, y \in D^{n_y}$. The following statements are equivalent:*

(1) $\sum_{i=1}^{n_x} \left[V(\frac{i}{n_x}) - V(\frac{i-1}{n_x})\right] \cdot \bar{x}_i \geq \sum_{i=1}^{n_y} \left[V(\frac{i}{n_y}) - V(\frac{i-1}{n_y})\right] \cdot \bar{y}_i$ for all $V : \mathbb{Q}_+ \to \mathbb{R}$ increasing and concave.

(2) $SE_x(p) \geq SE_y(p)$ for all $p \in [0, 1]$.

(3) There exist x' and y' s.t. x' and y' can be obtained, respectively from x and y, through T_{PR} transformations and x' can be obtained from y' through a finite sequence of T_A,T_M and T_{NMS} transformations.

(4) $E(x) \geq E(y)$ for all indices $E(.)$ satisfying NOM, MON, ANY, NMS and POP.

Proof of Theorem 2. Note that with $v_i^n := \left[V(\frac{i}{n}) - V(\frac{i-1}{n})\right]$, $E^n(x,v)$ satisfies POP, i.e. $E^{rn}(x^r, v) = E^n(x,v)$. Suppose that $x' = x^r$ and $y' = y^s$ s.t. $rn_x = sn_y = n$. Therefore condition (1) is equivalent to $\sum_{i=1}^n \left[V(\frac{i}{n}) - V(\frac{i-1}{n})\right] \cdot \delta_i \geq 0$ for all $V : \mathbb{Q}_+ \to \mathbb{R}$ increasing and concave where $\delta_i := \bar{x}'_i - \bar{y}'_i$. Note that if we let $V(\frac{i}{n}) := \sum_{j=1}^i v_j^n$ we get precisely the same conditions specified in part (1) of Theorem 1.

Furthermore note that $SE(p)$ satisfies POP, therefore $SE_{x'}(p) = SE_x(p) \geq SE_y(p) = SE_{y'}(p)$.

Similarly for all indices $E(x)$ satisfying POP we get $E(x') = E(x) \geq E(y) = E(y')$. Given that the population size of x' and y' is the same, all the implications in Theorem 1 apply giving the desired results. ■

From Theorem 2 it emerges that under the conditions stated in part (1) of the theorem, we can regard $\sum_{i=1}^{n_x} \left[V(\frac{i}{n_x}) - V(\frac{i-1}{n_x})\right] \cdot \bar{x}_i$ as a social exclusion index. If we assume that $V(\frac{i}{n}) = (\frac{i}{n})^\beta$, where $0 < \beta < 1$, then the resulting index becomes the social exclusion counterpart to the Donaldson and Weymark (1980) generalized Gini inequality index.

It may now be worthwhile to compare our approach with that of Bossert et al.(2007) who argued that persistence in the state of deprivation forms the basis of social exclusion. They started with characterization of individual deprivation measures, which ultimately have been transformed into social exclusion measures. While in our framework, minimal level of social exclusion is achieved when nobody is excluded from any functioning, according to the Bossert et al. (2007) formulation individual deprivation is minimized when everybody has the same number of capability failures. There are other differences as well- they assume a linear homogeneity condition, whereas we do not make any such assumption. Furthermore, they are concerned only with characterization of measures and we deal mainly with ordering of exclusion profiles. To conclude, our approach generalizes Muirhead integer majorization result in a different direction with respect to what has been done in Savaglio and Vannucci (2007). In their work the distribution of concern considers the height functions (of non negative integer value) of each individual opportunity set with respect to some opportunity sets thresholds. The majorization result holds for a fixed sum of heights for each opportunity profile, while transformations of the opportunity profiles, in order to lead to the dominance condition, need to be devised so to avoid that equalizing opportunity transfers do not destroy total aggregate height (see Savaglio and Vannucci 2007, p. 484). In our work this condition is always satisfied but our result holds also for social exclusion profiles exhibiting different aggregate deprivation scores.

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