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A Counting Approach for Measuring Multidimensional Deprivation

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Abstract. This paper is concerned with the problem of ranking and quantifying the extent of deprivation exhibited by multidimensional distributions, where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. To this end we first aggregate deprivation for each individual into a "deprivation count" indicating the number of dimensions for which the individual suffers from deprivation. Then we compare distributions of deprivation counts through summary measures of deprivation, by drawing on the rank-dependent social evaluation framework (Sen 1974, Yaari 1987). This approach proves to allow decomposition of the summary measures into extent of and dispersion in the distribution of multiple deprivations. To provide a normative justification of the proposed deprivation measures, an intervention principle affecting the association between the different deprivation indicators is adopted. Moreover, we introduce a family of measures of concentration in the distribution of deprivation experienced by the population. Concentration is defined to occur if dispersion in the observed distribution of deprivation is higher than the dispersion attained when the single deprivation indicators are treated as independent random variables, under the constraint of unchanged marginal distributions.

KEY WORDS: *Multidimensional deprivation, counting approach, partial orderings, rank-dependent measures of deprivation, principles of association rearrangements*.
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1. Introduction

Since the seminal papers of Kolm (1968) and Atkinson (1970), a flourishing literature has been trying to extend the normative approach of inequality measurement to the multidimensional case. We address the reader to Weymark (2004) and Trannoy (2006) for overviews on multidimensional inequality indices and partial orderings, respectively. In this paper we focus on multidimensional poverty measurement, in the specific case where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. This practice is conventionally adopted by statistical agencies accounting for "material deprivation", defined as having or not basic goods or performing or not basic social activities. For example, information may be collected on how many people have income below a poverty threshold, suffer from poor health, lack social network etc. (see e.g. Guio et al., 2009, or Alkire and Santos, 2010). The number of dimensions for which each individual suffers from deprivation may be summarised in a "deprivation count". Atkinson (2003) labelled "counting approach" to multidimensional deprivation the analysis of the distribution of the "deprivation count" across the population. Bossert et al. (2006) use the counting approach to analyse social exclusion in a dynamic context. Alkire and Foster (2010), Bossert et al. (2009) and Lasso de La Vega and Urrutia (2010) provide different axiomatic foundations of deprivation measures based on the counting approach.

Being deprived on a single dimension could result from the combination of a threshold and a continuous or discrete variable (e.g. income, or number of healthy days for year). In what follows we suppose to be always able to determine if an individual is or is not deprived on each dimension, but also that this is all the available information. This simplification allows us to delve into the question underlying the "identification" of the poor. Should we define poor only as those people suffering from deprivation on all dimensions or those that suffer from at least one dimension? These two opposite views correspond to the so-called "intersection" and "union" approaches in multidimensional poverty assessment. A related issue associated with multidimensional poverty analysis concerns the sequence in which the individual observations are aggregated (see Weymark 2004). Let us consider n individuals and r dimensions. Aggregating first individuals' deprivation on each dimension, the resulting indicators can be subsequently aggregated over the r dimensions generating an overall deprivation measure. The Human Development Index (HDI) is a prominent example of this approach[1]. This paper relies on an alternative approach: First, by aggregating across the single dimensions for each individual a "deprivation count" is identified, indicating the number of dimensions for which the individual suffers from deprivation. Second, an overall measure of deprivation summarizes the distribution of deprivation counts across individuals. As apposed to the HDI this approach captures the association between the single deprivation indicators.

Atkinson (2003) investigated the relationship between expected utility type of summary measures of deprivation and the correlation between different attributes.[2] In the spirit of Bourguignon and Chakravarty (2003), Atkinson stressed the relevance of the sign of the cross derivatives of the individual "utility" function with respect to its arguments, leaving several doubts on the expected utility approach as the most attractive method for analysing counting data. However, by drawing on the rank-dependent normative theory of inequality

[1] See Anand and Sen (1993).
[2] See also Duclos et al. (2006).

measurement (Yaari 1987, Aaberge 2001) we introduce alternative ranking criteria for distributions of deprivation counts that do not suffer from similar drawbacks, simply because the conditions on the derivatives of the utility function arising from the expected utility model can be replaced by simple conditions on a weight function used to distort probabilities in the rank-dependent framework. Starting from a broad family of rank-dependent deprivation measures, we introduce two new criteria: the second-degree upward and downward count distribution dominance, to refine the trivial ranking imposed by Pareto dominance (or first order stochastic dominance) over the set of deprivation count distributions. We show that second-degree upward and downward dominance criteria generalize the union and intersection approaches in measuring deprivation. We also identify two alternative principles of association (correlation) rearrangements that are consistent with these criteria. Using the rearrangement principles allows us to disentangle the impact of the association between deprivations from different dimensions on the overall evaluation of deprivation. The rank-dependent approach proves to offer a description of several facets of multidimensional deprivation: its prevalence, extent and concentration.

The paper is organized as follows: In Section 2 we use an axiomatic characterization of the rank-dependent criteria to introduce suitable partial orders on deprivation count distributions. We also illustrate the importance of the shape of the "distortion function" to generalize the "union" and intersection" views. In Section 3 we identify the association principles linked to second-degree upward and downward dominance criteria. We also disentangle the contribution to the deprivation measure from the average deprivation share and from the dispersion of the total number of deprivations across the population. In Section 4 we assess the concentration of deprivation comparing the actual distribution of deprivation counts with an ideal distribution. The benchmark is obtained considering the single deprivation indicators as independent random variables and using the observed marginal distributions to generate a counterfactual distribution. In Section 5 we conclude the paper summarising the main results and discussing the main questions still open. Proofs are gathered in Appendix 1, while in Appendix 2 we provide hints on how to allow for different weights.

2. Ranking distributions of deprivation counts

We consider a situation where individuals might suffer from r different dimensions of deprivation. Let X_i be equal to 1 if an individual suffers from deprivation in the dimension i and 0 otherwise. Moreover, let $X = \sum_{i=1}^{r} X_i$ with cumulative distribution function F and mean μ, and let F^{-1} be the left inverse of F. Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call X the deprivation count. Moreover, let $q_k = \Pr(X = k)$. Thus,

(2.1) $$F(k) = \sum_{j=0}^{k} q_j, \; k = 0,1,2...,r$$

and

(2.2) $$\mu = \sum_{k=1}^{r} k q_k \, .$$

Next, let $\boldsymbol{F}$ denotes the family of deprivation count distributions. A social planner's ranking over $\boldsymbol{F}$ may be represented by a preference relation $\succeq$, which will be assumed to satisfy the following axioms:

Axiom 1 (Order). *$\succeq$ is a transitive and complete ordering on $\boldsymbol{F}$.*

Axiom 2 (Continuity). *For each $F \in \boldsymbol{F}$ the sets $\{F^* \in \boldsymbol{F} : F \succeq F^*\}$ and $\{F^* \in \boldsymbol{F} : F^* \succeq F\}$ are closed (w.r.t. L_1-norm).*

Axiom 3 (Dominance). *Let $F_1, F_2 \in \boldsymbol{F}$. If $F_1(k) \geq F_2(k)$ for all $k = 0,1,2,...,r$ then $F_1 \succeq F_2$.*

Axiom 4 (Dual Independence). *Let F_1, F_2 and F_3 be members of $\boldsymbol{F}$ and let $\alpha \in [0,1]$. Then $F_1 \succeq F_2$ implies $\left(\alpha F_1^{-1} + (1-\alpha) F_3^{-1}\right)^{-1} \succeq \left(\alpha F_2^{-1} + (1-\alpha) F_3^{-1}\right)^{-1}$,*

Yaari (1987, 1988) introduced Axiom 4 as an alternative to the independence axioms of the expected utility theory. This axiom requires that the ordering of distributions is invariant with respect to certain changes in the distributions being compared. If F_1 is weakly preferred to F_2, then Axiom 4 states that any mixture on F_1^{-1} is weakly preferred to the corresponding mixture on F_2^{-1}. The intuition is that identical mixing interventions on the inverse distribution functions being compared do not affect the ranking of distributions. To illustrate this averaging operation, let us consider the problem of evaluating the average deprivation within couples obtained by matching men and women with the same rank in the male and female deprivation count distributions (in other terms, the most deprived man is matched with the most deprived woman, the second deprived man with the second deprived woman, and so on). Dual independence means that, given any initial distribution F_3 of deprivation over the female population, if within the male population, distribution F_1 is deemed to contain less deprivation than distribution F_2, this judgement is preserved after the matching with the women. Axiom 4 requires this property regardless of the initial patterns of deprivation and of the weights associated to male and female deprivation counts computing the average deprivation at the household level.

THEOREM 2.1. *A preference relation $\succeq$ on $\boldsymbol{F}$ satisfies Axioms 1-4 if and only if there exists a continuous and non-decreasing real function $\Gamma(\cdot)$ defined on the unit interval, such that for all $F_1, F_2 \in \boldsymbol{F}$*

$$F_1 \succeq F_2 \quad \Leftrightarrow \quad \sum_{k=0}^{r-1} \Gamma\left(\sum_{j=0}^{k} q_{1j}\right) \geq \sum_{k=0}^{r-1} \Gamma\left(\sum_{j=0}^{k} q_{2j}\right)$$

Moreover, Γ is unique up to a positive affine transformation.

For a proof of Theorem 2.1 we refer to Yaari (1987). Note, however, that Axiom 3 differs from the dominance axiom of Yaari (1987) and explains why Γ is non-decreasing.[3]

Theorem 2.1 shows that a social planner who supports Axioms 1 – 4 will rank count distributions of deprivation according to the criterion D_Γ defined by

$$D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma\left(\sum_{j=0}^{k} q_j\right), \tag{2.3}$$

where Γ, with $\Gamma(0)=0$ and $\Gamma(1)=1$, is a non-decreasing function that represents the preferences of the social planner. The social planner considers the distribution F that minimizes $D_\Gamma(F)$ to be the most favorable among those being compared. Since F denotes the distribution of the deprivation count, $D_\Gamma(F)$ can be considered as a summary measure of deprivation exhibited by the distribution F.

Atkinson et al. (2002) and Atkinson (2003) call attention to the distinction between the union and intersection approaches for measuring deprivation. A social planner who supports the union approach is particularly concerned with the number of people who suffer from at least one dimension of deprivation, whereas a social planner in favour of the intersection approach will focus attention on people deprived on all dimensions. By choosing the following specification for Γ

$$\Gamma(t) = \begin{cases} 0 & if\ 0 \le t < q_0 \\ q_0 & if\ t = q_0 \\ 1 & if\ q_0 < t \le 1, \end{cases} \tag{2.4}$$

it results: $D_\Gamma(F) = 1 - q_0$, which means that the union measure is a member of the D_Γ-family of deprivation measures. Alternatively, choosing the preference function

$$\Gamma(t) = \begin{cases} 0 & if\ 0 \le t < 1 - q_r \\ 1 - q_r & if\ t = 1 - q_r \\ 1 & if\ 1 - q_r < t \le 1, \end{cases} \tag{2.5}$$

yields $D_\Gamma(F) = r - 1 + q_r$, which means that the intersection measure also belongs to the D_Γ-family of deprivation measures.

Since deprivation count distributions might intersect each other, it will be useful to identify what restrictions a weaker dominance criterion than first-degree dominance (Axiom 3) places on the preference function Γ. Let us first introduce the "second-degree downward dominance" criterion:[4]

[3] Since the ordering relation defined on the set of inverse distribution functions is equivalent to the ordering relation defined on ***F***, the proof of Theorem 2.1 might alternatively be derived from the proof of the expected utility theory for choice under uncertainty.

[4] Note that second-degree downward dominance is analogous to the notion of second-degree downward Lorenz dominance introduced by Aaberge (2009).

DEFINITION 2.1A. *A deprivation count distribution* F_1 *is said to second-degree downward dominate a deprivation count distribution* F_2 *if*

$$\int_u^1 F_1^{-1}(t)dt \le \int_u^1 F_2^{-1}(t)dt \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

A social planner who implements second-degree downward count distribution dominance is especially concerned about those people who suffer from deprivation over many dimensions. However, an alternative ranking criterion that focuses attention on those who suffer deprivation from few dimensions can be obtained by aggregating the deprivation count distribution from below.

DEFINITION 2.1B. *A deprivation count distribution* F_1 *is said to second-degree upward dominate a deprivation count distribution* F_2 *if:*

$$\int_0^u F_1^{-1}(t)dt \le \int_0^u F_2^{-1}(t)dt \text{ for all } u \in [0,1],$$

and strict inequality holds strictly for some $u \in \langle 0,1 \rangle$.

Note that second-degree downward as well as upward count distribution dominance preserves first-degree dominance (Axiom 3) since first-degree dominance implies second-degree downward and upward dominance. A social planner who supports the condition of second-degree downward count distribution dominance will consider a distribution F_1 where individual *i* suffers from *h* deprivations and individual *j* from *l* (*l*<*h*) deprivations to be preferable to a distribution F_2 where individual *i* suffers from *h+1* deprivations and individual *j* from *l-1* deprivations, provided that the remaining individuals of the population have identical status in F_1 and F_2. By contrast, a social planner who supports the condition of second-degree upward count distribution dominance will prefer F_2 to F_1. Accordingly, second-degree upward and downward count distribution dominance might be considered as generalizations of the *union* and the *intersection* approach, respectively.
Let Ω_1 be a family of preference functions related to D_Γ and defined by

$$\Omega_1 = \left\{ \Gamma : \Gamma'(t) > 0, \Gamma''(t) > 0 \text{ for } t \in \langle 0,1 \rangle, \text{ and } \Gamma'(0) = 0 \right\}.$$

Note that $\Gamma'(0) = 0$ can be considered as a normalization condition. The following result provides a characterization of second-degree downward distribution dominance.

THEOREM 2.2A. Let F_1 and F_2 be members of $\boldsymbol{F}$. Then the following statements are equivalent:

(i) F_1 *second-degree downward dominates* F_2

(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ *for all* $\Gamma \in \Omega_1$.

(*Proof in Appendix*).

To ensure equivalence between second-degree downward deprivation dominance and D_Γ-measures as ranking criteria, Theorem 2.2A shows that it is necessary to restrict the preference function Γ to be increasing and convex. If, by contrast, Γ is increasing and concave then Theorem 2.2B provides the analogy to Theorem 2.2A for upward dominance. Let Ω_2 be defined by

$$\Omega_2 = \left\{ \Gamma : \Gamma'(t) > 0, \Gamma''(t) < 0 \ for\ t \in \langle 0,1 \rangle, and\ \Gamma'(1) = 0 \right\}.$$

THEOREM 2.2B. Let F_1 and F_2 be members of $\boldsymbol{F}$. Then the following statements are equivalent:

(i) F_1 *second-degree upward dominates* F_2

(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ *for all* $\Gamma \in \Omega_2$.

(*Proof in Appendix*).

To ease the comparability of data from different years and countries and to quantify various aspects of the distribution of multidimensional deprivation, the measures of extent of deprivation presented above will be complemented in next section by a framework for measuring the prevalence, the dispersion and the concentration in distributions of multidimensional deprivation.

3. Summary measures of deprivation

The following example motivates the methods introduced in this section:

Example 1. Two alternative policies produce the following distributions of two-dimensional deprivation: F_1, where 50% of the population suffers from one dimension and the remaining 50% suffers from the other dimension, and F_2 where 50% of the population does not suffer from any deprivation and the remaining 50% suffers from both dimensions. Thus, the mean number of deprivation is 1 for both distributions, but the intersection measure ranks F_1 to be preferable to F_2 whereas the union measure ranks F_2 to be preferable to F_1. An interesting question is which restrictions on Γ guarantee that D_Γ ranks F_1 to be preferable to F_2 or vice versa.

As it will be demonstrated below, the ranking of F_1 and F_2 provided by D_Γ depends on whether Γ is convex or concave, which according to Theorems 2.2A and 2.2B depend on whether the social planner favors second-degree downward or upward count distribution dominance. This judgment can be equivalently expressed in terms of the mean and the

dispersion of the deprivation count distributions: the intuition of this result is now presented through the two-dimensional case, then the general r-dimensional case follows.

3.1. The two dimensional case

Let $r = 2$, i.e. $X = X_1 + X_2$, and let
$p_{ij} = Pr\left(\left(X_1 = i\right) \cap \left(X_2 = j\right)\right)$, $p_{i+} = \Pr\left(X_1 = i\right)$, $p_{+j} = \Pr\left(X_2 = j\right)$.
Thus, $q_k = \Pr\left(X = k\right)$ can be expressed by $p_{ij}, i, j = 1,2$ in the following way:

$$
\begin{aligned}
q_0 &= p_{00} \\
q_1 &= p_{10} + p_{01} \\
q_2 &= p_{11}.
\end{aligned}
\tag{3.1}
$$

The 2x2 case can be illustrated by the following table:

Table 3.1. The distribution of deprivation in two dimensions

		X_2		
		0	1	
X_1	0	p_{00}	p_{01}	p_{0+}
	1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

The distribution F of X is given by

$$
F(k) = \Pr(X \le k) = \sum_{j=0}^{k} q_j,\ k = 0,1,2, \tag{3.2}
$$

where $F(2) = 1$ and the mean $\mu = q_1 + 2q_2$.

In this case $D_\Gamma(F)$ defined by (2.3) is given by

$$
D_\Gamma(F) = 2 - \Gamma(1 - q_2) - \Gamma(q_0). \tag{3.3}
$$

Note that Γ can be interpreted as a preference function of a social planner that assigns lower weights for one than for two deprivation counts.

To supplement the information provided by $D_\Gamma(F)$ and μ, it will be useful to introduce the following measure of dispersion,

$$(3.4)\quad \Delta_\Gamma(F)=\begin{cases}\sum_{k=0}^{1}\left[\sum_{j=0}^{k}q_j-\Gamma(\sum_{j=0}^{k}q_j)\right]=q_0-\Gamma(q_0)+(1-q_2)-\Gamma(1-q_2) & \text{when } \Gamma \text{ is convex}\\ \sum_{k=0}^{1}\left[\sum_{j=0}^{k}\Gamma(\sum_{j=0}^{k}q_j)-\sum_{j=0}^{k}q_j\right]=\Gamma(q_0)-q_0+\Gamma(1-q_2)-(1-q_2) & \text{when } \Gamma \text{ is concave}\end{cases}$$

It can easily be observed from (3.4) that $\Delta_\Gamma(F)=0$ if and only if q_0, q_1 or q_2 is equal to 1, which means that every individual suffers from 0, 1 or 2 deprivations. Since $q_0+(1-q_2)=2-q_1-2q_2=2-\mu$, by inserting (3.4) in (3.3) it results that the deprivation measure D_Γ admits the following decomposition:

$$(3.5)\qquad D_\Gamma(F)=\begin{cases}\mu+\Delta_\Gamma(F) & \text{when } \Gamma \text{ is convex}\\ \mu-\Delta_\Gamma(F) & \text{when } \Gamma \text{ is concave.}\end{cases}$$

Thus, by using (3.5) we may identify the contribution to D_Γ from the average number of deprivations (μ) as well as from the dispersion of deprivations across the population. Moreover, expression (3.5) demonstrates that a social planner who is concerned about reducing the mean number of deprivations as well as the dispersion of deprivations across the population will use the criterion D_Γ with a convex Γ; i.e. in this case the social planner pays particular attention to people who suffer from many deprivations. By contrast, when the social planner uses criterion D_Γ with a concave Γ he/she is more concerned about the number of people who are deprived on at least one dimension (the union approach) than about those deprived on all dimensions (the intersection approach). In this case D_Γ can be expressed as the difference between the mean number of deprivations in the population and the dispersion of deprivations across the population, which means that employment of D_Γ with a concave Γ corresponds to making a trade-off between reducing the total number of deprivations (μ) and their relative dispersion across the population ($1-\Delta_\Gamma/\mu$). Thus, D_Γ decreases if Δ_Γ increases. By employing the criterion $D_\Gamma(F)$ defined by (3.5) for Example 1, it follows that F_1 is preferred if the social planner relies on a convex Γ. By contrast, F_2 is considered to be preferable if a concave Γ represent the preferences of the social planner.

By inserting for $\Gamma(t)=2t-t^2$ or $\Gamma(t)=t^2$ in (3.3) and (3.4) we get the following expressions for the Gini measure of deprivation and the Gini measure of dispersion[5] corresponds to Gini's mean difference $\int F(x)(1-F(x))dx$),

$$(3.6)\qquad D_\Gamma(F)=\begin{cases}\mu+q_1(1-q_1)+2q_2(1-q_2)-2q_1q_2 & \text{when } \Gamma(t)=t^2\\ \mu-q_1(1-q_1)-2q_2(1-q_2)+2q_1q_2 & \text{when } \Gamma(t)=2t-t^2\end{cases}$$

and

$$(3.7)\qquad \Delta_G(F)=q_0(1-q_0)+q_2(1-q_2)=q_1(1-q_1)+2q_2(1-q_2)-2q_1q_2.$$

Note that Δ_G takes its maximum value when $q_0 = q_2 = \frac{1}{2}$.

3.2. The r dimensional case

This sub-section considers the r dimensional case formed by the multinomial distribution of r deprivation indicators $X_1, X_2, ..., X_r$. In this case $\sum_{k=0}^{r} q_k = 1$ and the mean μ is given by (2.2). Similarly as in the 2x2 case we get that $D_\Gamma(F)$ admits the decomposition

$$D_\Gamma(F) = \begin{cases} \mu + \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is convex} \\ \mu - \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is concave,} \end{cases} \tag{3.8}$$

where the dispersion measure $\Delta_\Gamma(F)$ is defined by

$$\Delta_\Gamma(F) = \begin{cases} \sum_{k=0}^{r-1} \left[\sum_{j=0}^{k} q_j - \Gamma(\sum_{j=0}^{k} q_j) \right] & \textit{when } \Gamma \textit{ is convex} \\ \sum_{k=0}^{r-1} \left[\Gamma(\sum_{j=0}^{k} q_j) - \sum_{j=0}^{k} q_j \right] & \textit{when } \Gamma \textit{ is concave,} \end{cases} \tag{3.9}$$

Note that $D_\Gamma(F) \geq r - \sum_{k=0}^{r-1} \sum_{j=0}^{k} q_j = \mu$ and $\mu \leq D_\Gamma(F) \leq r$ when Γ *is convex*, and $0 \leq D_\Gamma(F) \leq \mu$ when Γ *is concave*. When Γ *is convex* the minimum value of $D_\Gamma(F)$ is attained when $\Delta_\Gamma(F) = 0$; i.e. when each individual of the population suffers from the same number of deprivations. By contrast, the maximum value of $D_\Gamma(F)$ is attained when $\Delta_\Gamma(F) = 1/2$; i.e. when 50 per cent of the population does not suffer from any deprivation and the remaining 50 per cent suffer from every dimension of deprivation. By contrast, for concave Γ the minimum and maximum values of $D_\Gamma(F)$ are attained when $\Delta_\Gamma(F)$ is equal to $1/2$ and 0.

As for the two-dimensional case we get by inserting for $\Gamma(t) = t^2$ and $\Gamma(t) = 2t - t^2$ in (3.8) and (3.9) the following convenient expressions for the Gini measures of deprivation and dispersion,

$$D_\Gamma(F) = \begin{cases} \mu + \Delta_G(F) & \textit{when } \Gamma(t) = t^2 \\ \mu - \Delta_G(F) & \textit{when } \Gamma(t) = 2t - t^2. \end{cases} \tag{3.10}$$

where

(3.11) $$\Delta_G(F) = \sum_{k=0}^{r-1} kq_k(1-q_k) - 2\sum_{j=0}^{r-1}\sum_{k=j+1}^{r-1} jkq_j q_k \ .$$

The decomposition (3.8) suggests that $D_\Gamma(F)$ obeys the principle of mean preserving spread when Γ is convex; i.e. $D_\Gamma(F)$ increases when the number of deprivations at the middle of the count distribution is shifted towards the tails, under the condition of fixed total number of deprivations. However, when Γ is concave, the summary measure $D_\Gamma(F)$ decreases as a consequence of a mean preserving spread. This is due to the fact that such an operation will increase the number of people who don't suffer from any deprivation and/or suffer from a few dimensions of deprivation. The next sub-section will clarify the relationship between a mean preserving spread, second-degree upward and downward count distribution dominance and association rearrangements.

3.2. Principles of association rearrangements

To provide a normative justification of upward and downward count distribution dominance as well as for employing the deprivation measures D_Γ for concave and convex Γ, a correlation intervention principle will be introduced as in Epstein and Tanny (1980), Boland and Proschan (1988) and Tsui (1999, 2002). However, the previous literature does not distinguish between positive and negative association (or correlation). A distinction will be made between whether an association rearrangement comes from a distribution characterized by positive or negative association between two or several deprivation indicators, in he spirit of the statistical literature on measurement of association in multidimensional contingency tables (formed by two or several 0-1 variables). Various authors (see e.g. Yule, 1910 and Mosteller, 1968) have emphasized the importance of separating the information of a 2x2 table provided by the association between the social indicators X_1 and X_2 from the information provided by the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}). For 2x2 tables (see Table 3.1) this objective corresponds to introducing measures of association that are invariant under the transformation

(3.12) $$p_{ij} \to a_i b_j p_{ij}$$

for any set of positive numbers $\{a_i\}$ and $\{b_j\}$ such that $\sum_{i=0}^{1}\sum_{j=0}^{1} a_i b_j p_{ij} = 1$.

The cross-product α introduced by Yule (1900) and defined by

(3.13) $$\alpha = \frac{p_{00}p_{11}}{p_{01}p_{10}},$$

is a measure of association that satisfies the invariance condition (3.12), whereas the correlation coefficient does not. Thus, the association measure α and the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}) together provide complete information of Table 3.1. Note that $\alpha \in [0,\infty)$, $\alpha = 1$ if the indicators X_1 and X_2 are independent, $\alpha = 0$ when $p_{00} = p_{11} = 0$ and $\alpha \to \infty$ when $p_{01} = p_{10} = 0$. In the former case there is perfect negative association between the two indicators, whereas it is perfect positive association in the latter case. Accordingly, it is required to make a distinction between positive association increasing rearrangements, positive association decreasing rearrangements, negative association increasing rearrangements and negative association decreasing rearrangements[6].

DEFINITION 3.1A. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha > 1$. *The following marginal-free change* $(p_{00} + \delta, p_{01} - \delta, p_{10} - \delta, p_{11} + \delta)$ *is said to provide marginal distributions preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.1B. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha < 1$. *The following marginal-free change* $(p_{00} + \delta, p_{01} - \delta, p_{10} - \delta, p_{11} + \delta)$ *is said to provide a marginal distributions preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$
$(\delta > 0)$.

An illustartion is provided in the tables below, where the right (left) panel of Table 3.2 is obtained from the left (right) panel by a positive association increasing (decreasing) rearrangement, whereas the right (left) panel of Table 3.3 can be obtained from the left (right) panel by a negative association increasing (decreasing) rearrangement.

Table 3.2. Rearrangements that modify positive association

	0	1				0	1	
0	.30	.20	.50		0	.31	.19	.50
1	.20	.30	.50		1	.19	.31	.50
	.50	.50	1			0.50	0.50	1

Table 3.3. Rearrangements that modify negative association

	0	1			0	1	
0	.20	.30	.50	0	.19	.31	.50
1	.30	.20	.50	1	.31	.19	.50
	.50	.50			.50	.50	1

The association increasing/decreasing rearrangement principles defined by Definitions 3.1A and 3.1B will be proved to support second-degree downward/upward dominance under the condition of unchanged marginal distributions; i.e. the number of people suffering from each of the deprivation indicators are kept fixed. However, since real world interventions normally

[6] For similar definitions of association increasing rearrangements based on the correlation coefficient we refer to Atkinson and Bourguignon (1982), Dardanoni (1995) , Tsui (1999, 2002), Bourguignon and Chakravarty (2003), Duclos et al. (2006) and Kakwani and Silber (2008). See also Tchen (1980) who deal with positive association (or concordance) between bivariate probability measures.

concern trade-offs that allows reduction in one deprivation indicator at the cost of a rise in another deprivation indicator, we find it attractive to introduce association increasing/decreasing rearrangement principles that rely on the condition of fixed number of total deprivations rather than on the condition of keeping the number suffering from each of the indicators fixed. The more general mean preserving versions of Definitions 3.1A and 3.1B are defined as follows.

DEFINITION 3.2A. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha > 1$. *The following change* $(p_{00} + \delta, p_{01}, p_{10} - 2\delta, p_{11} + \delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.2B. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha < 1$. *The following marginal-free change* $(p_{00} + \delta, p_{01}, p_{10} - 2\delta, p_{11} + \delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ $(\delta > 0)$.

It follows straightforward from Definitions 3.2A and 3.2B that the mean preserving association principles make a mean preserving rearrangement that reduces the number of people suffering from indicator X_1 at the cost of increasing the number of people suffering from indicator X_2 when $\delta > 0$ and vice versa when $\delta < 0$. As illustrated by Table 3.4 the right (left) panel can be obtained from the left (right) panel by a mean preserving positive increasing (decreasing) rearrangement, since the association is negative and the mean is kept fixed equal to 1 under the rearrangement where $\delta = .01$.

Table 3.4. Illustration of mean preserving decreasing negative association rearrangements

	0	1			0	1	
0	.20	.30	.50	0	.21	.30	.51
1	.30	.20	.50	1	.28	.21	.49
	.50	.50	1		.49	.51	1

Since the condition of fixed marginal distributions also implies that the means are kept fixed, it follows that Definitions 3.1A and 3.1B can be considered as a special case of Definitions 3.2A and 3.2B, respectively. Thus, we will focus attention on Definitions 3.2A and 3.2B below.

Definitions 3.2A and 3.2B can readily be extended to higher dimensions. However, for a large number of dimensions the standard subscript notation becomes cumbersome. Thus, we find it convenient to introduce the following simplified subscript notation p_{ijk}, where i and j represents two arbitrary chosen deprivation dimensions and m represents the remaining s-2 dimensions and α_{ijm} is defined by

(3.14) $$\alpha_{ijm} = \frac{p_{iim} p_{jjm}}{p_{ijm} p_{jim}},$$

where *m* is a s-2 dimensional vector of any combination of zeroes and ones. In this case association is defined by *s(s-1)/2* cross-products.

In order to deal with s-dimensional counting data we introduce the following generalization of Definitions 3.2A and 3.2B,

DEFINITION 3.3A. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} > 1$. *The following change* $(p_{iim} + \delta, p_{ijm}, p_{jim} - 2\delta, p_{jjm} + \delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.3B. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} < 1$. *The following change* $(p_{iim} + \delta, p_{ijm}, p_{jim} - 2\delta, p_{jjm} + \delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ $(\delta > 0)$.

As is demonstrated by Theorems 3.1A below a social planner who is in favour of second-degree downward dominance will consider a mean preserving positive association increasing rearrangement as well as a mean preserving negative association decreasing rearrangement as a rise in overall deprivation. By contrast, a planner who favours upward second-degree dominance will consider such rearrangement as a reduction in the overall deprivation. Moreover, it is proved that the principles of mean preserving association increasing/decreasing rearrangement are equivalent to the mean preserving spread/contraction defined by

DEFINITION 3.4. Let F_1 and F_2 be members of the family ***F*** of count distributions based on s deprivation indicators and where F_1 and F_2 are assumed to have equal means. Then F_2 is said to differ from F_1 by mean preserving spread (contraction) if $\Delta_\Gamma(F_2) > \Delta_\Gamma(F_1)$ for all convex Γ ($\Delta_\Gamma(F_2) < \Delta_\Gamma(F_1)$ for all concave Γ).

Note that Definition 3.4 is equivalent to a sequence of the mean preserving spread introduced by Rothschild and Stiglitz (1970). This is easily seen by combining statements (ii) and (iii) of Theorem 3.1A and equation (A5) of the Appendix.

THEOREM 3.1A. Let F_1 and F_2 be members of the family ***F*** of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent:

(i) F_1 *second-degree downward dominates* F_2

(ii) F_2 *can be obtained from* F_1 *by a sequence of mean preserving positive association increasing rearrangements when* $\alpha > 1$ *and a sequence of mean preserving negative association decreasing rearrangements when* $\alpha < 1$

(iii) F_2 *can be obtained from* F_1 *by a mean preserving spread.*

(*Proof in Appendix*).

THEOREM 3.1B. Let F_1 and F_2 be members of the family $\boldsymbol{F}$ of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent:

(i) F_1 *second-degree upward dominates* F_2

(ii) F_2 *can be obtained from* F_1 *by a sequence of mean preserving positive association decreasing rearrangements when* $\alpha > 1$ *and a sequence of mean preserving negative association increasing rearrangements when* $\alpha < 1$.

(iii) F_2 *can be obtained from* F_1 *by a mean preserving contraction*

(*Proof in Appendix*).

The links between the extent, the dispersion and the concentration of deprivation will be illustrated in the next section.

4. Measures of concentration

Development of measures of concentration in the multivariate distribution of deprivations calls for a definition of the state of no concentration. The state of "no concentration" is defined to occur if the observed extent of deprivation is equal to the extent attained when the single deprivation indicators are treated as independent random variables, under the constraint of fixed marginal distributions equal to the observed ones. Thus, concentration is defined to occur if the observed extent of deprivation is higher than the extent of deprivation attained under the state of "no concentration".

4.1. The two dimensional case

This section relies on the notation introduced in Section 3.2. If X_1 and X_2 are stochastically independent, then $p_{ij} = p_{i+}p_{+j}$. In this case D_Γ obeys the following expression:

(4.1) $$D_\Gamma(F_{independ}) = 2 - \Gamma(p_{0+}p_{+0}) - \Gamma(1 - p_{1+}p_{+1}).$$

Thus, concentration can be measured as the deviation between $D_\Gamma(F)$ and $D_\Gamma(F_{independ})$, where F is the observed deprivation distribution. A normalized $[0,1]$ version is obtained by dividing the difference between $D_\Gamma(F)$ and $D_\Gamma(F_{independ})$ by the difference between $D_\Gamma(F)$ and $D_\Gamma(F_{\max})$, where $F_{\max}$ is the hypothetical extreme distribution that produces the highest deprivation (largest value of D_Γ), given that the marginal distributions (p_{0+}, p_{1+} and p_{+0}, p_{+1}) are kept fixed. To this end, assume that the probability of suffering from deprivation

1 ($X_1 = 1$) is larger than the probability of suffering from deprivation 2 ($X_2 = 1$), i.e. $p_{1+} > p_{+1}$. Under the constraint of fixed marginal distributions, D_Γ will attain its largest value when $q_2 = p_{+1}$, $q_1 = p_{1+} - p_{+1}$ and $q_0 = 1 - p_{1+}$. Thus, $D_\Gamma(F_{\max})$ is given by

(4.2) $$D_\Gamma(F_{\max}) = 2 - \Gamma(1 - p_{1+}) - \Gamma(1 - p_{+1}),$$

and the normalized measure of concentration of deprivation is defined by

(4.3) $$C_\Gamma(F) = \frac{D_\Gamma(F) - D_\Gamma(F_{indep})}{D_\Gamma(F_{\max}) - D_\Gamma(F_{indep})} = \frac{\Gamma(p_{00}) - \Gamma(p_{0+}p_{+0}) + \Gamma(1 - p_{11}) - \Gamma(1 - p_{1+}p_{+1})}{\Gamma(p_{0+}) - \Gamma(p_{0+}p_{+0}) + \Gamma(1 - p_{+1}) - \Gamma(1 - p_{1+}p_{+1})}.$$

Since the mean μ is identical for F, F_{indep} and $F_{\max}$, it follows from (3.10) that the concentration measure $C_\Gamma(F)$ also can be expressed in terms of the dispersion measure Δ_Γ,

(4.4) $$C_\Gamma(F) = \frac{\Delta_\Gamma(F) - \Delta_\Gamma(F_{indep})}{\Delta_\Gamma(F_{\max}) - \Delta_\Gamma(F_{indep})}.$$

For the Gini case, where $\Gamma(t) = t^2$ we get

(4.5)

$$C_G(F) = \frac{p_{00}(1 - p_{00}) - p_{0+}p_{+0}(1 - p_{0+}p_{+0}) + p_{11}(1 - p_{11}) - p_{1+}p_{+1}(1 - p_{1+}p_{+1}) + (p_{00} - p_{0+}p_{+0}) - (p_{11} - p_{1+}p_{+1})}{p_{+1}(1 - p_{+1}) - p_{1+}p_{+1}(1 - p_{1+}p_{+1}) + (p_{0+} - p_{0+}p_{+0}) - (p_{+1} - p_{+1}p_{1+})}$$

4.2. The r dimensional case

Let $q_k = \Pr(X = k)$; i.e. $q_0 = p_{00\ldots0}$, $q_1 = p_{100\ldots0} + p_{010\ldots0} + \cdots + p_{000\ldots01}, \ldots, q_r = p_{1111\ldots1}$. Under the condition of independence we have that

(4.6) $$p_{ijk\ldots t} = p_{i++\ldots+}p_{+j+\ldots+}p_{++k\ldots+} \cdots p_{+++\ldots t}.$$

Thus, $D_\Gamma(F_{independ})$ is formed by inserting for (4.6) in (2.3), where $q_o = p_{o++\ldots}p_{+0+\ldots} \cdots p_{+++\ldots0}$, $q_1 = p_{1++\ldots}p_{+00\ldots}p_{++0\ldots} \cdots p_{++\ldots0} \cdots p_{0++\ldots}p_{+0+\ldots} \cdots p_{++\ldots1}$, etc.

Next, let p_j denote the probability of suffering from deprivation *j*, and assume that $p_1 > p_2 > \cdots > p_r$, i.e $p_1 = p_{1++\ldots} > p_2 = p_{+1+\ldots}$, etc. Under the constraint of unchanged marginal distributions D_Γ will attain its maximum value when

$q_r = p_r, q_{r-1} = p_{r-1} - p_r, q_{r-2} = p_{r-2} - p_{r-1}, etc., i.e. q_j = p_j - p_{j+1}$ for $j = 1, 2, ..., r-1, q_r = p_r$, $q_0 = 1 - p_1$ and $\sum_{j=0}^{k} q_k = 1 - p_{k+1}$, $k = 0, 1, ..., r-1$. Thus, in this case D_Γ is given by

$$D_\Gamma(F_{\max}) = r - \sum_{k=0}^{r-1} \Gamma(1 - p_k), \tag{4.7}$$

and the measure of concentration C_Γ is given by

$$C_\Gamma(F) = \frac{\sum_{k=0}^{r-1} \Gamma(\sum_{j=0}^{k} q_j) - \sum_{k=0}^{r-1} \Gamma(\sum_{j=0}^{k} q_{j,indep})}{\sum_{k=0}^{r-1} \Gamma(1 - p_k) - \sum_{k=0}^{r-1} \Gamma(\sum_{j=0}^{k} q_{j,indep})}. \tag{4.8}$$

5. Summary and discussion

The conventional approach in official statistics as well as in most empirical studies of multidimensional deprivation is focusing on the distribution of the number of dimensions in which people suffer from deprivation. This paper is concerned with the problem of ranking and quantifying the extent of deprivation exhibited by multidimensional distributions of deprivation where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. To this end summary measures of deprivation are proposed, by drawing on the rank-dependent social evaluation framework that originates from Sen (1974) and Yaari (1988). This approach proves to allow decomposition of the summary measures into extent of and dispersion in the distribution of multiple deprivations. To provide a justification of the proposed deprivation measures two intervention principles affecting respectively the association (correlation) between the different deprivation indicators and the spread of the deprivation counts are adopted. Moreover, a family of measures of concentration in the distribution of deprivations experienced by the population is introduced, where concentration is defined to occur if dispersion in the observed distribution of deprivations is higher than the dispersion attained when the single deprivation indicators are treated as independent random variables, under the constraint of unchanged marginal distributions.

Notice that that the deprivation indicators are assumed to be perfect substitutes by construction, since the counting approach attaches an equal weight to each of the single indicators. An interesting question is whether or not the framework in this paper can be extended to allow for different weighting profiles across the multidimensional distribution of deprivations. Appendix 2 offers a first positive answer, showing the computational difficulties implied by this extension.

Appendix 1- Proofs

LEMMA 1. *Let H be the family of bounded, continuous and non-negative functions on [0,1] which are positive on* $\langle 0,1\rangle$ *and let g be an arbitrary bounded and continuous function on* $[0,1]$. *Then*

$$\int g(t)h(t)dt > 0 \quad \textit{for all } h \in H$$

implies

$$g(t) \geq 0 \quad \textit{for all } t \in [0,1]$$

and the inequality holds strictly for at least one $t \in \langle 0,1\rangle$.

Proof of Theorems 2.2A and 2.2B. Using integration by parts, we get:

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 (1-\Gamma(t))d(F_2^{-1}(t) - F_1^{-1}(t)) = -\Gamma'(0)\int_0^1 (F_2^{-1}(t) - F_1^{-1}(t))dt + \int_0^1 \Gamma''(u)\int_u^1 (F_2^{-1}(t) - F_1^{-1}(t))dtdu.$$

Thus, if (i) holds then $D_\Gamma(F_1) < D_\Gamma(F_2) \quad \textit{for all } \Gamma \in \Omega_1$.

To prove the converse statement we restrict to preference functions $\Gamma \in \Omega_1$. Hence,

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 \Gamma''(u)\int_u^1 (F_2^{-1}(t) - F_1^{-1}(t))dtdu,$$

and the result is obtained by applying Lemma 1.

The proof of Theorem 2.2B is analogous to the proof of Theorem 2.2A, and is based on the expression

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 (1-\Gamma(t))d(F_2^{-1}(t) - F_1^{-1}(t)) = -\Gamma'(1)\int_0^1 (F_2^{-1}(t) - F_1^{-1}(t))dt - \int_0^1 \Gamma''(u)\int_0^u (F_2^{-1}(t) - F_1^{-1}(t))dtdu,$$

which is obtained by using integration by parts. Thus, by using arguments like those in the proof of Theorem 2.2A the results of Theorem 2.2B are obtained.

Proof of Theorems 3.1A and 3.1B.

As demonstrated by Hardy, Littlewood and Polya (1934) an equivalent condition of Definition 2.1A is given by

(A1) $$\int_y^\infty F(x)dx \geq \int_y^\infty \tilde{F}(x)dx \quad for\ all\ y \in [0,\infty),$$

where $F(k) = F_1(k) = \sum_{j=0}^{k} q_j \quad and \quad \tilde{F}(k) = F_2(k) = \sum_{j=0}^{k} \tilde{q}_j$.

By inserting for F and $\tilde{F}$ in (A1) we get that F second-degree downward dominates $\tilde{F}$ if and only if

(A2) $$\sum_{j=i}^{r-1}\sum_{k=0}^{j} q_k \geq \sum_{j=i}^{r-1}\sum_{k=0}^{j} \tilde{q}_k \quad for\ i = 0,1,...,r-1.$$

Next, assume that (ii) is true; i.e.
$\tilde{p}_{iim} = p_{iim} + \delta, \tilde{p}_{ijm} = p_{ijm}, \tilde{p}_{jim} = p_{jim} - 2\delta\ and\ \tilde{p}_{jjm} = p_{jjm} + \delta$ which we assume corresponds to changes in the number of people suffering from t (p_{iim}), $t+1$ $(p_{ijm} + p_{jim})$ and $t+2$ (p_{jjm}) deprivations such that

(A3) $$\tilde{q}_t = q_t + \delta, \tilde{q}_{t+1} = q_{t+1} - 2\delta, \tilde{q}_{t+2} = q_{t+2} + \delta\ and\ \tilde{q}_k = q_k\ for\ all\ k \neq t, t+1, t+2,$$

which means that the mean of $\tilde{F}$ is equal to the mean of F.
Inserting for (A3) in $\tilde{F}$ yields

(A4) $$\tilde{F}(k) = \sum_{j=0}^{k} \tilde{q}_j = \begin{cases} \sum_{j=0}^{k} q_j & for\ k = 0,1,...,t-1 \\ \sum_{j=0}^{k} q_j + \delta & for\ k = t \\ \sum_{j=0}^{k} q_j - \delta & for\ k = t+1 \\ \sum_{j=0}^{k} q_j & for\ k = t+2, t+3, ..., r. \end{cases}$$

It follows by straightforward calculations that (A4) implies (A2) and thus that (ii) implies (i).

To prove the converse statement, assume that (i) is true, i.e. that (A2) is valid. Since F and $\tilde{F}$ are step functions it can be demonstrated that there exists a sequence of discrete distribution functions $F_0^*, F_1^*, ..., F_s^*$ such that $F = F_0^*$, $\tilde{F} = F_s^*$ and F_{i+1}^* differs from F_i^* by a mean preserving positive association increasing rearrangement, i.e. $F_{i+1}^* - F_i^*$ is given by

(A5) $$F_{i+1}^{*}(k)-F_{i}^{*}(k)=\begin{cases} 0 & for\ k=0,1,\ldots,t-1 \\ \delta & for\ k=t \\ -\delta & for\ k=t+1 \\ 0 & for\ k=t+2,t+3,\ldots,r. \end{cases}$$

Next, we use (A5) to construct F_1^* from F, F_2^* from F_1^* and finally $\tilde{F}$ from F_{s-1}^*. The required number of iterations (s) depends on the number of steps exhibited by the difference $\tilde{F}-F$.

The equivalence between (i) and (iii) follows directly from Theorem 2.2 A.

The proof of Theorem 3.1B is analogous to the proof of Theorem 3.1A. Thus, by using arguments like those in the proof of Theorem 3.1A the results of Theorem 3.1B are obtained.

Appendix 2- Accounting for different weights

Replacing 1 by the weights w_1 and w_2 as outcomes for the marginal indicator distributions in the two-dimensional case, the distribution of deprivation for two dimensions is given by the following table,

Table 5.1. The distribution of weighted deprivation in two dimensions

	$\tilde{X}_1$			
		0	w_2	
$\tilde{X}_2$	0 w_1	p_{00} p_{10}	p_{01} p_{11}	p_{0+} p_{1+}
		p_{+0}	p_{+1}	1

Next, by assuming that $w_1 \leq w_2$ the variable $\tilde{X}$ defined by $\tilde{X} = \tilde{X}_1 + \tilde{X}_2 = w_1 X_1 + w_2 X_2$ might be considered as a weighted counting variable. The distribution $\tilde{F}$ of $\tilde{X}$ is given by

(A6)
$$\tilde{F}(z) = \begin{cases} p_{00} & if\ z = 0 \\ p_{00} + p_{10} & if\ z = w_1 \\ p_{00} + p_{10} + p_{01} & if\ z = w_2 \\ 1 & if\ z = w_1 + w_2. \end{cases}$$

Theorem 2.1 shows that a social planner who supports Axioms 1 – 4 will rank count distributions of deprivation according to the criterion D_Γ defined by

(A7)
$$\tilde{D}_\Gamma(\tilde{F}) = \int (1 - \Gamma(\tilde{F}(z)))dz,$$

where Γ, with $\Gamma(0) = 0$ and $\Gamma(1) = 1$, is *a non-decreasing* function that represents the preferences of the social planner. Thus, the social planer considers the distribution $\tilde{F}$ that minimizes $\tilde{D}_\Gamma(\tilde{F})$ to be the most favorable among those being compared. Since F denotes the weighted count variable distribution of deprivation, $\tilde{D}_\Gamma(\tilde{F})$ can be considered as a measure of the extent of deprivation exhibited by the distribution $\tilde{F}$. Now, by inserting the mean $\tilde{\mu} = \int (1 - \tilde{F}(z))dz$ in (A7) we obtain the following decomposition

(A8)
$$\tilde{D}_\Gamma(\tilde{F}) = \begin{cases} \tilde{\mu} + \tilde{\Delta}_\Gamma(\tilde{F}) & when\ \Gamma\ is\ convex \\ \tilde{\mu} - \tilde{\Delta}_\Gamma(\tilde{F}) & when\ \Gamma\ is\ concave \end{cases}$$

where $\tilde{\Delta}_\Gamma(\tilde{F})$ is defined by

(A9) $$\tilde{\Delta}_{\Gamma}(\tilde{F}) = \begin{cases} \int \left(\tilde{F}(z) - \Gamma(\tilde{F}(z)) \right) dz & \textit{when } \Gamma \textit{ is convex} \\ \int \left(\Gamma(\tilde{F}(z)) - \tilde{F}(z) \right) dz & \textit{when } \Gamma \textit{ is concave.} \end{cases}$$

Expressions (A8) and (A9) demonstrate that Theorems 2.2A, 2.2 B, 3.1A and 3.1B are valid for weighted count distributions as well. However, by extending the dimensions from 2 to r it becomes cumbersome to provide a convenient formal description of the weighted count distribution $\tilde{F}$.

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