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Running title: Monte Carlo derivative pricing in a class of marked DSPP

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Abstract

To model intraday stock price movements we propose a class of marked doubly stochastic Poisson processes, whose intensity process can be interpreted in terms of the effect of information release on market activity. Assuming a partial information setting in which market agents are restricted to observe only the price process, a filtering algorithm is applied to compute, by Monte Carlo approximation, contingent claim prices, when the dynamics of the price process is given under a martingale measure. In particular, conditions for the existence of the minimal martingale measure Q are derived, and properties of the model under Q are studied.

Key Words

Minimal martingale measure, News arrival, Marked point process, Nonlinear filtering, Reversible jump Markov chain Monte Carlo, Ultra-high frequency data

1 Introduction

Traditionally, in the literature, almost all financial models for asset prices have focused on processes with continuous sample paths, sometime allowing for the presence of jumps. In recent years, however, with the increasing availability of intraday information, in particular ultra-high-frequency (UHF), on financial asset price quotes, a part of the literature has moved its attention to pure jump models based on marked point processes (MPP) (see, for monographs on point processes in time, Cox and Isham (1980), Brémaud (1981) and Last and Brandt (1995)), in which price changes are assumed to take place only at discrete (generally irregularly spaced) instants of time. In this article we propose a framework for UHF stock price movements and for contingent claim pricing based on a particular class of MPP, namely on doubly stochastic Poisson processes (DSPP) with marks.

In the class of DSPP with marks, an early model for asset prices has been proposed by Rogers and Zane (1998). Subsequently, Frey and Runggaldier (2001) considered a 'shadow' logprice process given by a stochastic volatility model depending on a state variable which is assumed to be a diffusion which, in turn, drives the intensity process of the DSPP and when a jump occurs the logprice equals the shadow process. Within this model, the filtering of the underlying intensity is tackled following a so called 'reference probability' approach. A different model in which the intensity process is still driven by a diffusion, but the jumps are independently and identically distributed (i.i.d.), is considered instead in Frey (2000). In this last article, assuming that market agents cannot observe the intensity process, the hedging problem is tackled in the special case in which the price process is a martingale. Under the assumption of partial information, optimal (minimum quadratic risk) hedging strategies are computed using the filtering method described in Frey and Runggaldier (2001). A different interesting model in which the stochastic intensity is given by a non-Gaussian Ornstein-Uhlenbeck process has been proposed in Rydberg and Shephard (2000). There, the Authors approach the filtering problem (after time discretization in intervals of equal length) using a particle filter on the counting observations, based on a sampling importance resampling algorithm. Other works dealing with MPP but more concerned with option pricing and hedging can be found in Kirch and Runggaldier (2004), Prigent (2000) and Prigent *et al.* (2004). In Kirch and Runggaldier (2004), assuming a model in which the asset price follows a geometric Poisson process with unknown constant intensity, the optimal hedging strategy is constructed using stochastic control techniques. On the other hand, in Prigent (2000), in a very general context, the equivalent martingale measures are characterized by their Radon-Nykodim derivatives with respect to the natural probability, whereas in Prigent *et al.* (2004) the problem of option pricing is considered in the case in which the (dynamic) portfolios are adjusted only after fixed relative changes in the stock prices.

Following a different modelling strategy, in our model the intensity process δ of the DSPP with marks is characterized as a function of time and of another underlying MPP. To describe the logprice process, we associate to the marked DSPP a continuous time process with piecewise constant trajectories whose value, at any given time instant, is equal to the sum of the marks (logprice changes) associated to all past time events. This means that we are here referring mainly to the situation of a market maker updating her/his posted price at irregularly spaced time instants, where the price is constant between two successive adjustments (we do not consider here any bid-ask spread and we may think at the bid-ask mean value). This framework is rich enough to model many of the features of UHF data. For example, the dependence of δ upon time allows to incorporate deterministic seasonalities in the model without calling for ad-hoc methods, and it will be seen that by appropriately choosing the intensity process we can effectively capture the behavior of less liquid assets. At the same time the class proposed is mathematically tractable since a trajectory of the price process in any bounded time interval is characterized by a finite (although random) number of changes.

An interesting feature of the framework proposed is that it can be interpreted to account for the link between the information release and the changes in price volatility and trading activity, whose existence has been many times suggested in the economic literature (see, among others, Engle and Ng (1993) and Kalev *et al.* (2004)). In our model, this link is embodied by the intensity process δ governing the speed of price changes. In particular, if δ is a shot noise process, its sudden increases can be interpreted as perturbations in market activity caused by pieces of news reaching the market, being the size of each increase due to the importance and unexpectedness of the news, and its consequent exponential decays can be interpreted as progressive normalizations due to the absorption of the effect of the news by the market.

As far as the problem of pricing a contingent claim is concerned, a basic result of mathematical finance states that for a stochastic process S, representing the discounted stock price, the existence of an equivalent martingale measure, that is, of a measure equivalent to the 'natural' probability P, such that S is a local martingale, is essentially equivalent to the absence of arbitrage opportunities (see, for example, Harrison and Kreps (1979), Delbaen and Schachermayer (1994)). If the price of the risky asset follows a marked point process, the market model is in general incomplete and it can be shown that there exist more then one of such equivalent measures. Thus, the problem of pricing a contingent claim, under the no arbitrage assumption, is reduced to taking expected values under the 'right' measure among all existing equivalent martingale measures. One possibility is to choose the so called *minimal martingale measure* Q introduced by Föllmer and Schweizer (1991) which arises very often in the financial literature (see Prigent *et al.* (2004) for a discussion and for further references). In our probabilistic setting, for the case of partial information, in which market agents are allowed to observe only the history of the stock price (that is, all past times and sizes of price changes, but not the history of the intensity process), we propose to use as a pricing measure the restriction to the filtration representing the available information of the measure Q derived in the case of complete information. Indeed, this restriction is still a martingale measure and it can be seen as the best projection of Q over the coarser filtration.

With this choice, to effectively implement the pricing of a contingent claim, that is, to take expected values in the case of partial information, we can use a Monte Carlo procedure based on the reversible jump Markov chain Monte Carlo (RJMCMC) algorithm (first introduced in a Bayesian inferential context by Green(1995)), which allows the Monte Carlo evaluation of the conditional distribution of the intensity of the marked DSPP representing the stock prices, given a past realization of the times and sizes of price changes. This conditional evaluation can be reconduced to a nonlinear filtering problem similar to that considered in Centanni and Minozzo (2006a,b) since, under some conditions, the probabilistic structure of our model is the same under the natural measure and the probability measure Q.

The paper is organized as follows. In Section 2 we introduce our modelling framework, derive some fundamental properties and detail a basic class of models, which is particularly tractable and detains pleasant properties. In Section 3, conditions for the existence of the minimal martingale measure are given, and, under it, properties of the price and intensity processes of the discounted stock price are investigated. Then, considering a stylized market in which the only two assets available for trading are the stock and the bank account, in Section 4 we consider the computational problem of the pricing of a contingent claim in the case of partial information and propose a Monte Carlo procedure which involves the use of an RJMCMC algorithm. In Section 5 we illustrate the pricing algorithm by means of a simulation study in the case of the basic class of models. Finally, Section 6 concludes the paper.

2 The modelling framework

Given a probability space $(\Omega, \mathcal{G}, \mathrm{P})$ and a complete right continuous filtration $\{\mathcal{G}_t\}_{t\geq 0}$, let us consider an adapted marked point process $\Phi = (T_i, Z_i)_{i\in\mathbb{N}}$, where T_i are positive random variables satisfying $T_i < T_{i+1}$ and Z_i are $\mathbb{R}$-valued random variables; let us denote with N the counting process defined by $N_t = \#\{i : T_i \leq t\}$. Let also μ denote the counting measure associated to the point process Φ which, for all $A \in \mathcal{B}(\mathbb{R})$, is equal to $\mu(\omega, (0,t] \times A) = \sum_{i=1}^{N_t} \mathbf{1}_{\{Z_i \in A\}}$ (with the convention that the sum over an empty set is equal to 0). We assume in particular that Φ is a doubly stochastic marked Poisson process with respect to $\{\mathcal{G}_t\}_{t\geq 0}$ (see Last and Brandt (1995), Chapter 6).

Definition 2.1 *A marked point process Φ adapted to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ is a doubly stochastic marked Poisson process if there exists a $\mathcal{G}_0$-measurable random measure υ on $\mathbb{R}^+ \times \mathbb{R}$ such that*

$$\mathrm{P}\left(\mu\left((s,t] \times A\right) = k \middle|\ \mathcal{G}_s\right) = \frac{\left(\upsilon\left((s,t] \times A\right)\right)^k}{k!} e^{-\upsilon((s,t]\times A)},$$

almost surely (a.s.), for every $A \in \mathcal{B}(\mathbb{R})$.

It is implicit in the definition that υ is a $\{\mathcal{G}_t\}$-compensator of Φ, that is, a $\{\mathcal{G}_t\}$-predictable random measure such that

$$\mathrm{E}\left(\int_0^t \int_{\mathbb{R}} f(s,z)\ \mu(\mathrm{d}s, \mathrm{d}z)\right) = \mathrm{E}\left(\int_0^t \int_{\mathbb{R}} f(s,z)\ \upsilon(\mathrm{d}s, \mathrm{d}z)\right), \tag{2.1}$$

for all predictable $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. Also, the process Φ has a finite number of points in any bounded interval and no fixed point of jump, and the compensator υ admits the disintegration

$$\upsilon(\mathrm{d}t, \mathrm{d}z) = \overline{\upsilon}(\mathrm{d}t) K(t, \mathrm{d}z), \tag{2.2}$$

where $\overline{\upsilon}(\cdot) = \upsilon(\cdot \times \mathbb{R})$ and K is a $\mathcal{G}_0 \otimes \mathcal{B}(\mathbb{R}^+)$-measurable stochastic kernel (see Last and Brandt (1995), Appendix A2) from $(\Omega \times \mathbb{R}^+)$ to $\mathbb{R}$.

Let us assume that $D_t = \overline{v}((0,t])$ has the form $D_t = \int_0^t \delta_s \mathrm{d}s$, and so that under the above assumptions the counting process N is a doubly stochastic Poisson process with intensity δ. Then, given the whole history of δ, the number of points in any time interval $(s,t]$ is a Poisson random variable (independent of $\mathcal{G}_s$) with mean $D_t - D_s$. Moreover,

$$\mathrm{P}(T_{N_s+1} > t \mid \mathcal{G}_s) = \mathrm{P}(N_t - N_s = 0 \mid \mathcal{G}_s) = \exp\left(- \int_s^t \delta_u \mathrm{d}u \right),$$

and $\mathrm{P}(Z_{N_s+1} \in B \mid \mathcal{G}_s, T_{N_s+1}) = \int_B K(T_{N_s+1}, \mathrm{d}z)$, for all $B \in \mathcal{B}(\mathbb{R})$.

Here, we are interested in using the marked point process Φ to model the logreturn changes of a given financial asset, for example, of a stock. That is, the price of the stock will be described by a process $S = (S_t)_{t\in\mathbb{R}^+}$ having the form $S_t = S_0 e^{Y_t}$, where $Y_t = \sum_{i=0}^{N_t} Z_i$ ($Z_0 = 0$), represents the logreturn process and the random variables Z_i and the process N are defined by the marked doubly stochastic Poisson process Φ. In our probability framework, the random variables T_i and Z_i represent the time and the size of the ith logreturn change whereas N_t represents the number of changes occurred up to time t. Applying Ito's formula to e^{Y_t} and observing that Y is a finite variation process, which implies that its continuous martingale part is zero, we can write

$$e^{Y_t} = 1 + \int_0^t e^{Y_{s-}} \, \mathrm{d}Y_s + \sum_{0<s\le t} \left(e^{Y_s} - e^{Y_{s-}} - e^{Y_{s-}} \Delta Y_s \right),$$

and so

$$\mathrm{d}S_t = S_{t-} \, \mathrm{d}Y_t + S_{t-} \left(e^{\Delta Y_t} - 1 - \Delta Y_t \right) = S_{t-} \, \mathrm{d}\widetilde{Y}_t, \tag{2.3}$$

where

$$\begin{aligned} \widetilde{Y}_t &= Y_t + \sum_{0<s\le t} \left(e^{\Delta Y_s} - 1 - \Delta Y_s \right) = \sum_{i=0}^{N_t} Z_i + \sum_{i=0}^{N_t} (e^{Z_i} - 1 - Z_i) \\ &= \sum_{i=0}^{N_t} (e^{Z_i} - 1) = \int_0^t \int_{\mathbb{R}} (e^z - 1) \, \mu(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{2.4}$$

In other words, we have $S_t = S_0 \mathcal{E}(\widetilde{Y})_t$ where, as usual, we denote with $\mathcal{E}(X)$ the stochastic exponential of a process X, that is, the unique solution to the stochastic differential equation

$\mathrm{d}L_t = L_{t-}\ \mathrm{d}X_t$, which is given by

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}\langle X^c\rangle_t\right) \cdot \prod_{0<s\leq t}(1+\Delta X_s)e^{-\Delta X_s} \tag{2.5}$$

(see, for example, Protter (1992) and Bühlmann *et al.* (1996)).

Under mild integrability assumptions, the process S admits a semimartingale representation.

Proposition 2.1 *If S is locally integrable, then it admits the decomposition $S = S_0 + M + B$, where M is a local martingale given by*

$$M_t = \int_0^t \int_{\mathbb{R}} S_{s-}(e^z - 1)\ (\mu - \upsilon)(\mathrm{d}s, \mathrm{d}z),$$

and the locally finite variation process B is given by

$$B_t = \int_0^t \int_{\mathbb{R}} S_{s-}(e^z - 1)\ \upsilon(\mathrm{d}s, \mathrm{d}z).$$

Moreover, the quadratic variation process is given by

$$[S, S]_t = S_0^2 + \int_0^t \int_{\mathbb{R}} S_{s-}^2(e^z - 1)^2\ \mu(\mathrm{d}s, \mathrm{d}z).$$

If S is also locally square integrable, then the angle process exists and is given by

$$\langle S, S\rangle_t = \int_0^t \int_{\mathbb{R}} S_{s-}^2(e^z - 1)^2\ \upsilon(\mathrm{d}s, \mathrm{d}z). \tag{2.6}$$

Proof If S is locally integrable then Equations (2.3) and (2.4) imply that

$$\mathrm{E}\bigg(\int_0^{t\wedge T_n} \int_{\mathbb{R}} |S_{s-}(e^z - 1)|\ \upsilon(\mathrm{d}s, \mathrm{d}z)\bigg)\ <\ +\infty,$$

in virtue of (2.1); then B_t is well defined and M_t is a local martingale (see Last and Brandt (1995), p. 126).

Since S is adapted, càdlàg (that is, having trajectories which are right continuous with left hand limits) and with trajectories of finite variation on bounded intervals, S is also a quadratic pure jump process (see Protter (1992), p. 63, Theorem 26), that is,

$$[S,S]_t = S_0^2 + \sum_{0<s\le t} (\Delta S_s)^2 = S_0^2 + \sum_{n=1}^{N_t} S_{T_n-}^2 (e^{Z_n}-1)^2 = S_0^2 + \int_0^t \int_{\mathbb{R}} S_{s-}^2 (e^z-1)^2 \; \mu(\mathrm{d}s,\mathrm{d}z),$$

where we used the convention that the sum over an empty set is equal to zero. Also, using Ito's formula we can write

$$S_t^2 = S_0^2 + \int_0^t 2S_{s-}\;\mathrm{d}S_s + \sum_{0<s\le t} (S_s^2 - S_{s-}^2 - 2S_{s-}\Delta S_s) = S_0^2 + \int_0^t \int_{\mathbb{R}} S_{s-}^2 (e^{2z}-1)\; \mu(\mathrm{d}s,\mathrm{d}z);$$

hence if S is locally square integrable, $[S,S]$ is locally integrable. Then $\langle S,S\rangle$ exists and (2.6) holds since we can write

$$\begin{aligned}[S,S]_t &= S_0^2 + \int_0^t \int_{\mathbb{R}} S_{s-}^2 (e^z-1)^2 \; \mu(\mathrm{d}s,\mathrm{d}z) \\ &= S_0^2 + \int_0^t \int_{\mathbb{R}} S_{s-}^2 (e^z-1)^2 \; (\mu-\upsilon)(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{\mathbb{R}} S_{s-}^2 (e^z-1)^2 \; \upsilon(\mathrm{d}s,\mathrm{d}z). \quad \text{QED}\end{aligned}$$

Remark 2.1 *S is a special semimartingale since it admits a (unique) decomposition such that the finite variation process B is also predictable.*

Remark 2.2 *Observing that B_t is a continuous process, we have that $[M,M]=[S,S]$ and also that $\langle M,M\rangle = \langle S,S\rangle$.*

As far as the form of the intensity process is concerned, we assume that the intensity δ is given by $\delta_t = h(t, \Phi_0'^t)$, where $\Phi' = (\tau_j, X_j)_{j\in\mathbb{N}\cup\{0\}}$, with $\tau_0 = 0$, is an MPP with a finite number of points in bounded intervals and $\Phi_0'^t$ is the restriction of Φ' to $[0,\ t]$. For example, δ can be a generalization of the classical shot noise process

$$\delta_t = a(t) + b\lambda_t, \tag{2.7}$$

where $a(\cdot)$ is an integrable $\mathbb{R}^+$-valued deterministic function, b is a nonnegative parameter, and the process λ is given by

$$\lambda_t = \sum_{j=0}^{N_t'} X_j \; e^{-k(t-\tau_j)}, \tag{2.8}$$

where $k > 0$, $X_j > 0$, for all $j \geq 0$, $N'_t = \#\{j > 0 : \tau_j \leq t\}$. Hence, the integral D_t assumes the form $D_t = A(t) + b\Lambda_t$, where $A(t) = \int_0^t a(s)\ \mathrm{d}s$ and where

$$\begin{aligned} \Lambda_t &= \int_0^t \lambda_s\ \mathrm{d}s = \int_0^t \left(\sum_{j=0}^{N'_s} X_j e^{-k(s-\tau_j)} \right) \mathrm{d}s \\ &= \frac{1}{k} \left[\sum_{j=0}^{N'_t} X_j \left(1 - e^{-k(t-\tau_j)} \right) \right]. \end{aligned} \tag{2.9}$$

Let us observe that λ is the solution of the stochastic differential equation $\mathrm{d}\lambda_t = -k\lambda_t\ \mathrm{d}t + \mathrm{d}J_t$, where $J_t = \sum_{j=0}^{N'_t} X_j$.

This formulation of the stochastic part of the intensity allows a natural interpretation for the stochastic changes of the intensity δ in terms of market perturbations caused by the arrival of relevant news. When the jth piece of news reaches the market, a sudden increase X_j in trading activity occurs, the size of which can be interpreted as the effect of the piece of news on the market. After each jump, a progressive normalization of the trade activity follows, which can be thought to be due to the absorption of the piece of news by market agents. The random variable τ_j represents the time of arrival of the jth piece of news. The parameter k expresses the speed of absorption of the effect of the pieces of news by the market, while $a(\cdot)$ represents the activity that the market would have had in absence of random perturbations caused by the arrival of relevant news. By adequately choosing the function $a(t)$, it is possible to take into account the seasonalities and the other features that often characterize intraday price data (see, for example, Guillaume *et al.* (1999)).

Among the many specifications allowed in the present modelling framework to account for news arrival, for the perturbing potential of the intensity jumps, as well as for the marks of the DSPP, we define the following simple subclass of models to which we will later refer to as the *basic class*.

Definition 2.2 *A marked DSPP is said to belong to the basic class if it has an intensity of the* *form (2.7) and satisfies:*

A1. N' *is a Poisson process with constant intensity* ν.

A2. X_j, $j > 0$, *are i.i.d. Exponential random variables with mean* $1/\gamma$ *(independent from* τ_j*).*

A3. The initial value X_0 *of the process* λ *has a Gamma distribution with parameters* ν/k *and* γ *(that is,* $\mathrm{E}(X_0) = \nu/(k\gamma)$*).*

A4. $a(t) \equiv 0$ *and* $b = 1$ *(so that* $\delta = \lambda$*).*

A5. Z_i, $i \in \mathbb{N}$, *are i.i.d. random variables (independent from the processes* N *and* δ*).*

This definition specifies a class of models which, apart from their simplicity, have also some nice properties. Indeed, under Assumptions $A1$ and $A2$, λ is an affine process (see Duffie *et al.* (2003)), while under Assumptions $A1$–$A3$, λ is a stationary Ornstein-Uhlenbeck process with univariate marginal Gamma distribution (see Barndorff-Nielsen and Shephard (2001), and Centanni and Minozzo (2006a) for further details).

3 The minimal martingale measure

Assuming that the price dynamic of the underlying financial asset can be described in the modelling framework presented in Section 2, in this section we discuss the existence of the minimal martingale measure Q, and derive, under it, some properties of the model. We recall the following definitions (see Schweizer (1995)).

Definition 3.1 *A process* S *is said to satisfy the structure condition (SC) if its finite variation part* B *is absolutely continuous with respect to* $\langle M, M\rangle$, *that is,* $S_t = M_t + \int_0^t c_s \,\mathrm{d}\langle M, M\rangle_s$, *where the predictable process* c *satisfies* $\widetilde{K}_t := \int_0^t c_s^2 \,\mathrm{d}\langle M, M\rangle_s < +\infty$, *a.s. (with respect to* P*), for each* $t > 0$.

In this case the stochastic integral $\int c \, \mathrm{d}M$ is well defined and its angle process is equal to $\widetilde{K}$.

Definition 3.2 *A process q is called a martingale density for S if q is a local* P*-martingale with $q_0 = 1$ a.s.* P*, such that the product qS is also a local* P*-martingale. If q is in addition strictly positive, q is called a strict martingale density for S.*

Remark 3.1 *If q is a martingale density for S, it defines a (possibly signed) measure* Q *that is locally absolutely continuous with respect to* P *($\mathrm{Q} \overset{loc}{\ll} \mathrm{P}$) by*

$$q_t = \frac{\mathrm{dQ} \mid_{\mathcal{G}_t}}{\mathrm{dP} \mid_{\mathcal{G}_t}}$$

(see Schweizer (1995) and Jacod and Shiryaev (1980)). Since every strictly positive local martingale is a supermartingale, if q is a strict martingale density for S with $\mathrm{E}(q_t) = 1$, $\forall t > 0$, then Q *is locally equivalent to* P*.*

In the following lemma we show that, under some integrability conditions, the price process S, as defined in Section 2, satisfies condition (SC).

Lemma 3.1 *Let $\Phi = (T_i, Z_i)_{i \in \mathbb{N}}$ be a DSPP with marks with respect to the filtration $\{\mathcal{G}_t\}_{t \geq 0}$ as given in Section 2 and let S be defined by $S_t = S_0 \exp(\sum_{i=0}^{N_t} Z_i)$. If S is locally square integrable, then it satisfies (SC) and the process c is given by*

$$c_t = \frac{\int_{\mathbb{R}} (e^z - 1) \; K(t, \mathrm{d}z)}{S_{t-} \int_{\mathbb{R}} (e^z - 1)^2 \; K(t, \mathrm{d}z)},$$

where K is the stochastic kernel deriving from the disintegration (2.2).

Proof Since S is locally square integrable, in virtue of Proposition 2.1, S is a semimartingale with decomposition $S = S_0 + M + B$, the process $\langle M, M \rangle$ exists and we can write

$$c_t := \frac{\mathrm{d}B_t}{\mathrm{d}\langle M, M \rangle_t} = \frac{\int_{\mathbb{R}} (e^z - 1) \; K(t, \mathrm{d}z)}{S_{t-} \int_{\mathbb{R}} (e^z - 1)^2 \; K(t, \mathrm{d}z)}.$$

Moreover, being K a stochastic kernel,

$$\begin{aligned}\int_0^t c_s^2 \, \mathrm{d}\langle M, M\rangle_s &= \int_0^t \frac{\left[\int_{\mathbb{R}}(e^z-1)\ K(s,\mathrm{d}z)\right]^2}{S_{s-}^2 \left[\int_{\mathbb{R}}(e^z-1)^2\ K(s,\mathrm{d}z)\right]^2}\ S_{s-}^2 \delta_s \left[\int_{\mathbb{R}}(e^z-1)^2\ K(s,\mathrm{d}z)\right] \mathrm{d}s \\ &\leq \int_0^t \delta_s \ \mathrm{d}s \ < \ +\infty. \quad \text{QED}\end{aligned}$$

Let us turn now to the study of the properties of our model under the minimal martingale measure Q. To this end, we need first to compute the Radon-Nikodym derivative of Q with respect to P. If the process S satisfies the (SC) condition, we can define the process

$$q_t = \mathcal{E}\left(-\int c \ \mathrm{d}M\right)_t, \tag{3.1}$$

where the expression $\mathcal{E}(\cdot)$ is defined in (2.5). This is a martingale density for S which defines the minimal martingale measure; as explained in Remark 3.1 it is, in general, a signed measure. A necessary and sufficient condition that guarantees that the process q is a strict martingale density for S is given in the following theorem due to Schweizer (1995).

Theorem 3.1 *Suppose that S is a locally square integrable special semimartingale satisfying condition (SC). Then $q_t = \mathcal{E}\left(-\int c \ \mathrm{d}M\right)$ is a strict martingale density for S if and only if*

$$1 - c\ \Delta M > 0, \quad a.s. \ \ \text{P}. \tag{3.2}$$

Proof See Schweizer (1995), Proposition 2. QED

Lemma 3.1 and Theorem 3.1 play a crucial role in our modelling framework (see also Prigent (2000)).

Proposition 3.1 *Under the hypotheses of Lemma 3.1, let us define the process U by*

$$U(t,z) = \frac{\int_{\mathbb{R}}(e^z-1)\ K(t,\mathrm{d}z)}{\int_{\mathbb{R}}(e^z-1)^2\ K(t,\mathrm{d}z)} \cdot (e^z-1)\,.$$

Then Condition (3.2) can be expressed as

$$U(T_i, Z_i) < 1 \quad \forall i \in \mathbb{N}, \quad a.s. \ \ \text{P}. \tag{3.3}$$

Moreover, when this is satisfied, the compensator $\widetilde{\upsilon}$ of Φ, under Q, *assumes the form*

$$\widetilde{\upsilon} = (1-U)\upsilon. \tag{3.4}$$

Proof Observing that

$$\Delta M_t = \sum_i \mathbf{1}_{\{t=T_i\}} S_{t-}(e^{Z_i}-1),$$

and recalling that

$$c_t = \frac{\int_{\mathbb{R}}(e^z-1)\ K(t,\mathrm{d}z)}{S_{t-}\int_{\mathbb{R}}(e^z-1)^2\ K(t,\mathrm{d}z)},$$

we can write

$$1 - c_t\ \Delta M_t = 1 - \sum_i \mathbf{1}_{\{t=T_i\}} \frac{\int_{\mathbb{R}}(e^z-1)\ K(T_i,\mathrm{d}z)}{\int_{\mathbb{R}}(e^z-1)^2\ K(T_i,\mathrm{d}z)}(e^{Z_i}-1).$$

Then (3.2) holds if and only if $U(T_i, Z_i) < 1$. Moreover, since

$$\mathcal{E}\left(-\int_0^t c_s\ \mathrm{d}M_s\right) = \mathcal{E}\left(-\int_0^t\int_{\mathbb{R}} \frac{\int(e^z-1)\ K(t,\mathrm{d}z)}{\int(e^z-1)^2\ K(t,\mathrm{d}z)}\cdot(e^z-1)\ \ (\mu-\upsilon)(\mathrm{d}t,\mathrm{d}z)\right),$$

by the very definition of stochastic exponential and applying Theorem 10.2.2 in Last and Brandt (1995), relation (3.4) can be derived. QED

The following proposition states that the process δ has the same distribution under both P and Q. Let us note that, since, under P, δ is defined trajectory-wise by $\delta_t = h(t, \Phi_0'^t)$, we can focus our attention on the distributional properties of the MPP Φ', that is, on its compensator.

Proposition 3.2 *Let us consider the counting random measure m, together with its* P*-compensator n, associated to the point process $\Phi' = (\tau_j, X_j)_{j\in\mathbb{N}\cup\{0\}}$. Under the hypotheses of Lemma 3.1, let us assume that Condition (3.3) is satisfied. Then n is also the* Q*-compensator of m.*

Proof For each $B \in \mathcal{B}(\mathbb{R}^+)$ let the process L^B be defined by $L^B_t = m(\cdot,(0,t],B) - n(\cdot,(0,t],B)$. Then L^B is a local P-martingale for each $B \in \mathcal{B}(\mathbb{R}^+)$. Applying Girsanov's theorem for general semimartingales (see Protter (1992), p. 109), the process $\widetilde{L}^B$ given by

$$\widetilde{L}_t^B = L_t^B - \int \frac{1}{q_t} \, \mathrm{d}[q, L^B]_t$$

must be a Q-martingale. So, since L^B and q are pure jump local martingales without common jumps, we have $\mathrm{d}[q, L^B] = 0$ and n is also the compensator of Φ' under Q. QED

As far as the process Φ of the logerturn changes is concerned, let us now assume that the random variables Z_i are i.i.d (independent also from T_i and δ) with finite first and second exponential moments. In this case it can be shown that $\mathrm{E}(q_t) = 1$, for all $t \geq 0$, and that, under Q, the process Φ is still a DSPP with marks and Z_i are still independent.

Corollary 3.1 *Under the hypotheses of Proposition 3.2, let us assume that Z_i are i.i.d with common distribution $G(\mathrm{d}z)$ such that $\mathrm{E}(\exp(Z_i)) = \alpha < \infty$ and $\mathrm{E}(\exp(2Z_i)) = \varsigma < \infty$. Then the process U is given by*

$$U(\omega, t, z) = U(z) = \frac{\alpha - 1}{\varsigma^2 - 2\alpha + 1} \, (e^z - 1), \tag{3.5}$$

whereas the density process q is given by

$$q_t = \exp\left((1 - R) \int_0^t \delta_u \, \mathrm{d}u\right) \cdot \prod_{i=0}^{N_t} (1 - U(Z_i)),$$

where $R = 1 - (\alpha - 1)^2 / (\varsigma^2 - 2\alpha + 1)$, and $\mathrm{E}(q_t) = 1$, $\forall \; t \geq 0$.

Proof The independence assumption implies that $\int (e^z - 1) \; K(t, \mathrm{d}z) = \int (e^z - 1) \; \mathrm{d}G(z) = \alpha - 1$, for each $t > 0$, and that $\int (e^z - 1)^2 \; K(t, \mathrm{d}z) = (\varsigma^2 - 2\alpha + 1)$, and so that (3.5) holds true. Moreover, observing that $\iint U \; \mathrm{d}(\mu - \upsilon)$ is a finite variation process, and so that its continuous martingale part is equal to zero (see, for example, Protter (1992)), from (2.5) we have

$$
\begin{aligned}
q_t &= \mathcal{E}\left(-\int_0^t c_s\ \mathrm{d}M_s\right) = \mathcal{E}\left(-\int_0^t\!\!\int_{\mathbb{R}} U(z)\ (\mu - \upsilon)(\mathrm{d}s,\mathrm{d}z)\right) \\
&= \exp\left(-\int_0^t\!\!\int_{\mathbb{R}} U(z)\ (\mu-\upsilon)(\mathrm{d}s,\mathrm{d}z)\right) \prod_{i:T_i\le t}(1-U(Z_i))e^{U(Z_i)} \\
&= \exp\left(-\int_0^t\!\!\int_{\mathbb{R}} U(z)\ (\mu-\upsilon)(\mathrm{d}s,\mathrm{d}z)\right)\cdot\exp\left(\int_0^t\!\!\int_{\mathbb{R}} U(z)\ \mu(\mathrm{d}s,\mathrm{d}z)\right)\prod_{i:T_i\le t}(1-U(Z_i)) \\
&= \exp\left(\int_0^t\!\!\int_{\mathbb{R}} U(z)\ \upsilon(\mathrm{d}s,\mathrm{d}z)\right) \prod_{i:T_i\le t}(1-U(Z_i)) = \exp\left((1-R)\int_0^t \delta_u\ \mathrm{d}u\right)\prod_{i=0}^{N_t}(1-U(Z_i));
\end{aligned}
$$

moreover

$$
\begin{aligned}
\mathrm{E}(q_t|\mathcal{G}_t) &= \mathrm{E}\Big[\exp\left((1-R)\int_0^t\delta_u\ \mathrm{d}u\right)\cdot\prod_{i=0}^{N_t}(1-U(Z_i))\Big|\mathcal{G}_t\Big] \\
&= \exp\left((1-R)\int_0^t \delta_u\ \mathrm{d}u\right)\cdot\exp\left(-\int_0^t\delta_u\ \mathrm{d}u\right) \\
&\quad \cdot\left[1+\sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^k}\prod_{i=1}^{k}(1-U(z_i))\frac{\left(\int_0^t\delta_u\ \mathrm{d}u\right)^k}{k!}G(\mathrm{d}z_1)\dots G(\mathrm{d}z_k)\right)\right] \\
&= \exp\left(-\int_0^t R\delta_u\ \mathrm{d}u\right)\cdot\sum_{k=0}^{\infty}\frac{\left(\int_0^t R\delta_u\ \mathrm{d}u\right)^k}{k!} = 1. \quad \text{QED}
\end{aligned}
$$

Corollary 3.2 *Under the hypotheses of Corollary 3.1, Φ is still a marked DSPP whose intensity process is given by $\widetilde{\delta}_t = R\ \delta_t$, and Z_i are i.i.d. random variables with common distribution*

$$
\widetilde{G}(\mathrm{d}z) = \frac{1-U(z)}{R}\ G(\mathrm{d}z).
$$

Proof By Proposition 3.1, under Q, the compensator of Φ is given by $\mathrm{d}\widetilde{\upsilon} = \delta_t\ \mathrm{d}t(1-U(z))G(\mathrm{d}z)$, which admits the disintegration $\widetilde{\upsilon}(\mathrm{d}t,\mathrm{d}z) = \widetilde{\delta}_t\ \mathrm{d}t\cdot\widetilde{G}(\mathrm{d}z)$, where $\widetilde{\delta}_t = R\ \delta_t$ and $\widetilde{G}(\mathrm{d}z) = R^{-1}(1-$

$U(z))G(\mathrm{d}z)$; hence Z_i are i.i.d. and independent from T_i, also under Q. Moreover

$$\begin{aligned}
\mathrm{Q}(N_t - N_s = k|\ \mathcal{G}_s) &= \mathrm{E}^{\mathrm{P}}\left(\mathbf{1}_{\{N_t-N_s=k\}}\ \frac{q_t}{q_s}\middle|\ \mathcal{G}_s\right)\\
&= \mathrm{E}^{\mathrm{P}}\left(\mathbf{1}_{\{N_t-N_s=k\}}\exp\left((1-R)\int_s^t \delta_u\ \mathrm{d}u\right)\prod_{i=0}^{k}(1-U(Z_i))\right)\\
&= \exp\left((1-R)\int_s^t \delta_u\ \mathrm{d}u\right)\int_{\mathbb{R}^k}\prod_{i=1}^{k}(1-U(z_i))\frac{\left(\int_s^t \delta_u\ \mathrm{d}u\right)^k}{k!}\exp\left(-\int_s^t \delta_u\ \mathrm{d}u\right)G(\mathrm{d}z_1)\ldots G(\mathrm{d}z_k)\\
&= \frac{\left(\int_s^t R\delta_u\ \mathrm{d}u\right)^k}{k!}\exp\left(-\int_s^t R\delta_u\ \mathrm{d}u\right),
\end{aligned}$$

where E^{P} denotes the expected value under P, with the convention that the product on an empty set is equal to 1. QED

4 Pricing through the RJMCMC algorithm

Let us consider a stylized market in which a risky asset (the stock) and a riskless bank account are available for trading. For the bank account, we will assume, without loss of generality, that it has constant value equal to one. In this market, we will also consider an European contingent claim with fixed maturity $T \in \mathbb{R}^+$ and payoff of the form $H = H(S_T)$. Assuming that the price of the stock is described by a process S satisfying the hypotheses of Corollary 3.1 under the natural probability measure P, in a complete information setting we could use as a pricing measure the minimal martingale measure Q analyzed in Section 3. However, in a financial context it is much more reasonable to assume that market agents are restricted to observe only the history of the stock price S, that is, all past times and sizes of price changes, and not the history of the intensity process δ. Due to this partial information constrain, we are actually restricted to the filtration $\{\mathcal{F}_t^S\}_{t\geq 0}$, where $\mathcal{F}_t^S = \sigma(S_u, 0 \leq u \leq t)$, generated by the price process. Though Q is not minimal with respect to $\{\mathcal{F}_t^S\}_{t\geq 0}$, here we suggest to perform derivative pricing resorting to the pricing measure defined by the Radon-Nykodym derivative $\mathrm{E}\left(\frac{\mathrm{dQ}}{\mathrm{dP}}\middle|\mathcal{F}_t^S\right)$, which provides the best projection of the minimal martingale measure Q on the filtration $\{\mathcal{F}_t^S\}_{t\geq 0}$ generated by the observations. Indeed, in virtue of what observed at the end of Section 3 and since we assume that the bank account, which is the numeraire, is a constant (and therefore adapted to the partial information filtration), this pricing measure is a martingale measure for S with respect to the new filtration $\{\mathcal{F}_t^S\}_{t\geq 0}$. Thus an arbitrage-free value for the contingent claim at a given time instant $t < T$ is given by $\mathrm{E}^Q(H(S_T)|\mathcal{F}_t^S)$. In the following, to solve our pricing problem, we will need to filter the intensity under Q of the DSPP Φ, that is $\widetilde{\delta} = R\delta$ (or its stochastic part δ), that is, to evaluate the conditional distribution of δ, given a past realization of Φ. This is a nonlinear filtering problem which can be solved by stochastic simulation using an RJMCMC algorithm run on the space of the trajectories of the intensity process (see also Centanni and Minozzo (2006a)).

Given a measurable space $(X, \mathcal{X})$, let $\pi(\mathrm{d}x)$ denote a target distribution of interest (which will be in our case the conditional distribution, under Q, of δ from 0 to t, given the observed price history $\mathcal{F}_t^S$). Markov chain Monte Carlo techniques are based on the construction, through the Metropolis-Hastings algorithm, of a Markov chain with an aperiodic and irreducible transition kernel $P(x, \mathrm{d}x)$ satisfying the detailed balance condition

$$\int_A \int_B \pi(\mathrm{d}x) P(x, \mathrm{d}x') = \int_B \int_A \pi(\mathrm{d}x') P(x', \mathrm{d}x), \tag{4.1}$$

for all $A, B \in \mathcal{X}$, having $\pi(\mathrm{d}x)$ as its limiting distribution. The simulation of this chain will provide, after a sufficiently long initial run (*burn in*), an approximate sample (generally dependent) from $\pi(\mathrm{d}x)$.

In detail, in our case, the sample space X is the subspace of the Skorokhod space of càdlàg functions in a given time interval $[0, t]$. In particular, this is the space of all the trajectories of δ from 0 to t, which we denote with $\boldsymbol{\delta}_0^t$, which are of the form $\boldsymbol{\delta}_0^t(s) = h(s, \phi_0'^s)$, where $\phi_0'^t = (\tau_j, x_j)_{j\in\{0,\ldots,n\}}$, $n \in \mathbb{N}$, $x_j \in \mathbb{R}$, $j = 0, \ldots, n$, $\tau_j \in \mathbb{R}^+$, $j = 1, \ldots, n$, and $\tau_0 = 0 <$

$\tau_1 < \tau_2 < \cdots < \tau_n \leq t$. Before proceeding, we identify each trajectory $x \in X$ with the vector $\boldsymbol{\beta}(n) = (\tau_1, \ldots, \tau_n, x_0, \ldots, x_n) \in \mathbb{R}^{2n+1}$, and the price history with the vector of the times and sizes of logreturn jumps $\boldsymbol{\Phi} = (T_1, \ldots, T_N, Z_1, \ldots, Z_N) \in \mathbb{R}^{2N}$. In this way we can equivalently consider, instead of X and of $\pi(\mathrm{d}x)$, the space $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$, where $\mathcal{C}_n = \{n\} \times \mathbb{R}_+^{2n+1}$, so that each element $x \in \mathcal{C}$ is of the form $x = (n, \boldsymbol{\beta}(n))$, $n = 0, 1, 2, \ldots$, and the target distribution $\pi(\mathrm{d}x; \boldsymbol{\Phi})$ on the space $\mathcal{C}$ depending on the vector $\boldsymbol{\Phi}$ representing the observations which is assumed to be known.

With this identification, to obtain a random sample from the conditional distribution of δ, given a past realization of the MPP Φ, we can implement a particular version of the Metropolis-Hastings algorithm called RJMCMC, which allows for efficient transitions between spaces of different dimension, that is, between trajectories of δ having a different number of jumps. To this end, we define a set of types of transition moves taking the chain from the current state x to the state $\mathrm{d}x'$, which is proposed with an essentially arbitrary probability $q_m(x, \mathrm{d}x')$ depending on the type m of move. As usual with Metropolis-Hastings algorithms, the proposed new state is not automatically accepted; it is instead accepted with probability $\alpha_m(x, x')$, which is constructed in such a way that (4.1) is satisfied. Under the assumption that $\pi(\mathrm{d}x; \boldsymbol{\Phi}) q_m(x, \mathrm{d}x')$ has a finite density $f_m(x, x')$ with respect to a symmetric measure ξ_m on $\mathcal{C} \times \mathcal{C}$, in Green (1995) it is shown that such an acceptance probability is given by

$$\alpha_m(x, x') = \min\left\{1, \frac{f_m(x', x)}{f_m(x, x')}\right\}.$$

Let us now suppose that the distribution π can be characterized through proper densities $\pi(\boldsymbol{\beta}(n) | n; \boldsymbol{\Phi})$ on the subspaces $\mathbb{R}^{2n+1}$, $n = 0, 1, \ldots$, that the current state of the chain is $x = (n, \boldsymbol{\beta}(n))$, and that we have defined one or more different move types allowing transitions between states of the chain belonging to different-dimensional spaces. Then the RJMCMC update proceeds with the following steps:

1. With probability $p(m|n)$ choose to perform a move of type (m).

2. Generate a random vector $\mathbf{u}$ from a specified proposal density $q(\mathbf{u}|m, n, \boldsymbol{\beta}(n))$.

3. Set $(x', \mathbf{u}') = g_{m,n,n'}(x, \mathbf{u})$, where $x' = (n', \boldsymbol{\beta}'(n'))$ and $g_{m,n,n'}$ is an invertible function with $(2n+1) + \dim(\mathbf{u}) = (2n'+1) + \dim(\mathbf{u}')$.

4. Accept x' as the new state of the chain with probability $\min\{1, A\}$, where the acceptance probability ratio A is given by

$$\frac{p(\boldsymbol{\Phi}|n', \boldsymbol{\beta}'(n'))}{p(\boldsymbol{\Phi}|n, \boldsymbol{\beta}(n))} \frac{p(n', \boldsymbol{\beta}'(n'))}{p(n, \boldsymbol{\beta}(n))} \frac{p(m|n)q(\mathbf{u}|m, n, \boldsymbol{\beta}(n))}{p(m'|n')q(\mathbf{u}'|m', n', \boldsymbol{\beta}'(n'))} \left| \frac{\partial g_{m,n,n'}(\boldsymbol{\beta}(n), \mathbf{u})}{\partial(\boldsymbol{\beta}(n), \mathbf{u})} \right|, \tag{4.2}$$

and where $p(\boldsymbol{\Phi}|n, \boldsymbol{\beta}(n))$ is the conditional distribution under Q of the data $\boldsymbol{\Phi}$ given the intensity x, and $p(n, \boldsymbol{\beta}(n))$ is the distribution under Q of x.

Observe that, borrowing from the Bayesian terminology, the acceptance ratio is expressed as the product of four terms: likelihood ratio, prior ratio, proposal ratio and Jacobian.

Thus, denoting with $Q_T(\mathrm{d}s, \mathrm{d}x; \boldsymbol{\Phi})$ the joint distribution under Q of S_T and $\boldsymbol{\delta}_0^t$, given the past observations $\boldsymbol{\Phi}$, and with $Q_T(\mathrm{d}s; x, \boldsymbol{\Phi})\ \pi_T(\mathrm{d}x; \boldsymbol{\Phi})$ its factorization, to evaluate the conditional expectation $\mathrm{E}^Q(H(S_T)|\mathcal{F}_t^S)$, where $t < T$, we can proceed as follows:

(i) draw a sample $\boldsymbol{\delta}_0^{t(1)}, \ldots, \boldsymbol{\delta}_0^{t(M)}$ of size M from $\pi(\mathrm{d}x; \boldsymbol{\Phi})$ using the RJMCMC filtering algorithm just described;

(ii) simulate a continuation of each of these trajectories $\boldsymbol{\delta}_0^{t(i)}$, $i = 1, \ldots, M$, from t to T, according to the law (under Q) of the intensity process, obtaining a sample $\boldsymbol{\delta}_0^{T(i)}$, $i = 1, \ldots, M$ from $\pi_T(\mathrm{d}x; \boldsymbol{\Phi})$;

(iii) for each $\boldsymbol{\delta}_0^{T(i)}$, simulate a continuation of the price S from t to T, extending the observed trajectory $\boldsymbol{\Phi}$, and take the final value $s_T^{(i)}$ assumed by the price in T, as a realization from $Q_T(\mathrm{d}s; \boldsymbol{\Phi})$;

(iv) compute the approximation

$$\mathrm{E}^Q(H(S_T)|\mathcal{F}_t^S) \approx \frac{1}{M}\sum_{i=1}^{M} H\left(s_T^{(i)}\right).$$

5 Simulation experiments

To illustrate the filtering and pricing algorithms described in the previous section, we consider here a simulation experiment assuming for the price process two models belonging to the basic class. Let us recall that, in virtue of Assumption $A4$, the shot noise type intensity δ now coincides with its stochastic part λ given by (2.8). Fixing a value for the parameters ν, k and γ, and choosing a distribution for the random variables Z_i satisfying condition (3.3), we simulated a trajectory S_0^T of the price process from time 0 to T. In particular, we expressed time in minutes and considered $T = 2400$, corresponding to one week of market activity, that is, to 5 days, 8 hours a day. Figure 1 (middle) shows a simulated trajectory of λ, whereas Figure 1 (top) shows a simulated trajectory of the price process S for $\nu = 1/60$, $k = 0.0035$ and $\gamma = 2.50$, with a starting value of $S_0 = 100$. For the distribution of the logreturn jumps Z_i, we chose a binomial scheme, allowing only 'up' and 'down' movements, with $\mathrm{P}(Z_i = 0.0035) = 0.55$ and $\mathrm{P}(Z_i = -0.0041) = 0.45$.

[**FIGURE 1 ABOUT HERE**]

Instead, Figure 2 (middle) shows a simulated trajectory of λ, whereas Figure 2 (top) shows a simulated trajectory of the price process S for $\nu = 1/150$, $k = 0.0030$ and $\gamma = 45.89$, with a starting value of $S_0 = 100$. Here, for the distribution of Z_i, we chose a binomial scheme with $\mathrm{P}(Z_i = 0.0275) = 0.60$ and $\mathrm{P}(Z_i = -0.0410) = 0.40$.

[**FIGURE 2 ABOUT HERE**]

The first model, whose price process has a high intensity, might correspond to the behaviour of a liquid asset in which the stock price changes very often, with small logreturn jumps. Instead, the second model might correspond to an illiquid asset in which the price changes less often, but with bigger jumps.

To show the behaviour of the pricing algorithm, under the equivalent martingale measure Q, we now assume as the observed data the simulated trajectory of the price process S_0^T, and, obviously, also assume that the (simulated) trajectory of λ (from which S_0^T has been generated) is unknown and that the parameters of the model are known. To actually implement the RJMCMC filtering algorithm, under Q, we need to detail the transition moves and to specify the distributions in (4.2). From Assumption $A5$ of the basic class and Corollary 3.2 it follows that, since Z_i are independent from λ, given T_i, the target distribution $\pi(\mathrm{d}x; \mathbf{\Phi})$ actually depends only on the jump times $T_1, \ldots, T_N$. Thus, being N a DSPP both under P and Q, the likelihood ratio can be computed observing that, given a trajectory of λ in $[0, T]$, the conditional distribution under Q of the jump times can be written as

$$p(T_1, \ldots, T_N | n, \boldsymbol{\beta}(n)) = \left[\prod_{i=1}^{N} f(T_i | T_{i-1}, n, \boldsymbol{\beta}(n))\right] \mathrm{P}(T_{N+1} > T | T_N, n, \boldsymbol{\beta}(n)) = \exp\left(-R \int_0^T \lambda_s \mathrm{d}s\right) \prod_{i=1}^{N} R\lambda_{T_i}.$$

Moreover, the prior ratio can be computed observing that

$$p(n, \boldsymbol{\beta}(n)) = f(x_0) \left[\prod_{j=1}^{n} f_X(x_j) f_\tau(\tau_j | \tau_{j-1})\right] \mathrm{P}(\tau_{n+1} > T | \tau_n),$$

where $f(x_0) = \gamma^{\nu/k} x_0^{\nu/k-1} \exp(-\gamma x_0)/\Gamma(\nu/k)$, $f_X(x_j) = \gamma \exp(-\gamma x_j)$ and $f_\tau(\tau_j | \tau_{j-1}) = \nu \exp(-\nu(\tau_j - \tau_{j-1}))$, $j = 1, \ldots, n$ (with $\tau_0 = 0$), and $\mathrm{P}(\tau_{n+1} > T | \tau_n) = \exp(-\nu(T - \tau_n))$.

As for the transition moves, following the strategy of Centanni and Minozzo (2006a), let us define the following five types of transition moves which are naturally suggested by the structure of the basic class:

(s) Change of the starting value of the intensity λ.

(h) Change of the height of a randomly chosen intensity jump.

(p) Change of the position in time of a randomly chosen intensity jump.

(b) Inclusion in the intensity of a new jump at a randomly chosen time in $(0, T]$ ('birth' move).

(d) Suppression from the intensity of a randomly chosen jump ('death' move).

At each transition of the sampler, we propose one of the five move types with probability $p(m|n)$, where (m) stays for (s), (h), (p), (b) or (d), such that $\sum_m p(m|n) = 1$; obviously, if the number of intensity jumps n is equal to 0, the only move types available for a proposal are the change of the starting value (s) and the birth of a jump (b), and $p(m|n = 0) = 0$, for $m = h, p, d$.

With the current specification of the model, assuming that a move of type (s) has been selected, we draw a value at random from a Gamma distribution with mean $\nu/(k\gamma)$ and variance $\nu/(k\gamma^2)$, and accept it as the new initial value of the intensity by considering as prior ratio $\exp(-\gamma(x' - x))(x'/x)^{\nu/k-1}$ and as proposal ratio $\exp(-\gamma(x - x'))(x/x')^{\nu/k-1}$ in the acceptance probability ratio (4.2). For the move of type (h), we may choose a random number j in $\{1, \ldots, n\}$ and draw a value x'_j from an Exponential distribution with mean $1/\gamma$. The value x'_j is accepted as the new size of the jth intensity jump with acceptance probability ratio A taking as prior ratio $\exp(-\gamma(x'_j - x_j))$ and as proposal ratio $\exp(-\gamma(x_j - x'_j))$. Analogously, for the move of type (p), we may choose a random number j in $\{1, \ldots, n\}$ and a random time τ'_j uniformly in (τ_{j-1}, τ_{j+1}), where $\tau_0 = 0$ and $\tau_{n+1} = T$. The new position in time τ'_j for the jth jump is accepted with acceptance probability ratio A taking the prior ratio and the proposal ratio both equal to one. Note that for the moves (s), (h) and (p), which do not involve a change of dimension, the Jacobian in (4.2) is identically equal to one.

Now, for the move of type (b), which considers the inclusion of a new intensity jump, we can draw a position time τ^* uniformly in $(0, T)$ and a jump size x^* from an Exponential distribution with mean $1/\gamma$. The new intensity jump is accepted with acceptance probability ratio (4.2) with prior ratio $\nu\gamma\exp(-\gamma x^*)$ and proposal ratio

$$\frac{p(d|n+1)\cdot T}{(n+1)\cdot p(b|n)\gamma\exp(-\gamma x^*)}.$$

Lastly, for the move of type (d), regarding the suppression of one of the n intensity jumps, we can draw uniformly a number j from $\{1,\ldots,n\}$ and suppress the jth jump with acceptance probability ratio A, where the prior ratio is given by $(\nu\gamma\exp(-\gamma x_j))^{-1}$ and the proposal ratio by

$$\frac{p(b|n-1)\gamma\exp(-\gamma x_j)\cdot n}{T\cdot p(d|n)}.$$

For these latter moves (b) and (d), which involve a change of dimension of the vector $\boldsymbol{\beta}(n)$, the Jacobian in (4.2) is still equal to one.

Having detailed an RJMCMC sampler for our class of models, let us remember that the last M updates (after burn in) of a run of this sampler provide a set of trajectories $\{\boldsymbol{\lambda}_0^{T^{(i)}}, i = 1,\ldots,M\}$ which can be considered as a (dependent) sample from the conditional distribution, under Q, of λ, given the data S_0^T. This sample may be viewed as the support of a discrete distribution, assigning mass $1/M$ to each element, that approximates the conditional distribution of interest and can be used to approximate many quantities of interest. For instance, for any given time instant $s \in [0, T]$, an approximation of the conditional distribution, under Q, of λ_s given the observations (up to time T) can simply be obtained by considering the values of the sampled trajectories in s, that is, the values $\boldsymbol{\lambda}_0^{T^{(i)}}(s)$, $i = 1,\ldots,M$.

Let us now use the above filtering algorithm for the pricing of an European call expiring at T with strike price $K = 105$. For the two sets of values chosen for the parameters ν, k and γ, Condition (3.3) is satisfied since $U(Z_i) \leq 0.0214$ for the first set of parameters, and $U(Z_i) \leq 0.0160$ for the second set of parameters. Under the martingale measure Q, the distribution of the logreturn jumps is characterized by a probability of the 'up' event equal to 0.5385 for the first set of parameters, and to 0.5906 for the second set of parameters. For the actual pricing of the call, we fixed a set of 20-minutes spaced time instants $\mathbb{T} = \{0, 20, \ldots, 2400\}$ where to compute the price of the call. For any time instant $t \in \mathbb{T}$, considering as the observed data the simulated price trajectory S_0^t, we can obtain a sample from the conditional distribution, under Q, of S_T given the observations, running the pricing algotithm of Section 4 which exploits the RJMCMC filtering algorithm just detailed. Here, after a burn in of 5,000 updates, we run the chain for an additional 50,000 updates. Figure 1 (bottom) and Figure 2 (bottom) show (for the two sets of parameter values considered) the approximated (Monte Carlo) trajectory of the value process C_t of the call for each $t \in \mathbb{T}$, corresponding to the simulated trajectory of the stock price given in Figure 1 (top) and Figure 2 (top). It can be noticed that in both figures the trajectory of the value process of the call, as expected, properly captures the peculiarities of the trajectory of the stock price process in all the time interval $[0, T]$. For instance, in Figure 2 the option value falls close to zero near to maturity when the price of the stock goes under the stike price .

6 Conclusions

In this paper we considered a model for the price evolution of a financial asset where the intraday price movements occur only at irregularly spaced time instants. Considering a stylized market with only a stock and a riskless bank account available for trading, we tackled the problem of pricing a contingent claim in a partial information setting in which only the history of price changes is available, but not the intensity process underlying the times of price changes. Among all existing equivalent martingale measures, we chose to use as a pricing measure the restriction to the coarser filtration of the minimal martingale measure Q derived in the complete information case. In order to perform the actual calculation of the contingent claim prices, we proposed a Monte Carlo simulation approach based on a filtering procedure of the unobserved intensity, (derived with respect to $\{\mathcal{G}_t\}_{t\geq 0}$), exploiting the capability of the RJMCMC algorithm.

Let us note that in this paper we implicitely assumed that the available data have been generated by a stochastic process S which, under the "natural" measure P, is not, in general, a martingale, and we tackled the pricing problem deriving, under the conditions introduced in Section 3, a martingale measure under which the price process maintains a DSPP structure. Alternatively, regardless of how the data have been generated, for pricing purposes we might use directly the model introduced in Section 2 with the assumption that the (discounted) price process is a martingale, that is, with the constraint that B_t in Proposition 2.1 is equal to 0 (since we assumed the numeraire to be constant and equal to 1). Let us remark that following this pricing approach we do not need to impose any other condition on the model, apart from the above martingale assumption, and we can compute contingent claim prices by Monte Carlo simulations using a numerical procedure similar to that considered in Section 4.

As far as the parameters of the model, in this paper we assumed that they are known. Assuming that the data have been generated under the "natural" measure P, a likelihood based estimation procedure exploiting the RJMCMC algorithm has been developed in Centanni and Minozzo (2006a). On the other hand, in the case in which we are using the model in Section 2 itself as a pricing measure, and we do not know the model generating the data, we could resort to option data to estimate the parameters.

An important point which has not been taken into account in this paper is the evaluation of the error made in the pricing of the contingent claim using the proposed Monte Carlo procedure. We just remark that this error is affected by the length of the RJMCMC chain, as well as, in the case of real (non simulated) data, by the error in the estimation of the parameters of the model.

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Figure Legends

Figure 1. Simulated trajectory of the price process S (top), with starting value $S_0 = 100$, obtained by conditioning to a simulated trajectory of the intensity process λ (middle), assuming a model of the basic class with $\nu = 1/60$, $k = 0.0035$ and $\gamma = 2.50$, under the natural probability P, with $T = 2400$. The bottom graph shows the approximated (Monte Carlo) trajectory of the value process C_t, $t \in \mathbb{T} = \{0, 20, \ldots, 2400\}$, for an European call expiring at $T = 2400$ with strike price $K = 105$.

Figure 2. Simulated trajectory of the price process S (top), with starting value $S_0 = 100$, obtained by conditioning to a simulated trajectory of the intensity process λ (middle), assuming a model of the basic class with $\nu = 1/150$, $k = 0.0030$ and $\gamma = 45.89$, under the natural probability P, with $T = 2400$. The bottom graph shows the approximated (Monte Carlo) trajectory of the value process C_t, $t \in \mathbb{T} = \{0, 20, \ldots, 2400\}$, for an European call expiring at $T = 2400$ with strike price $K = 105$.

Figure 1

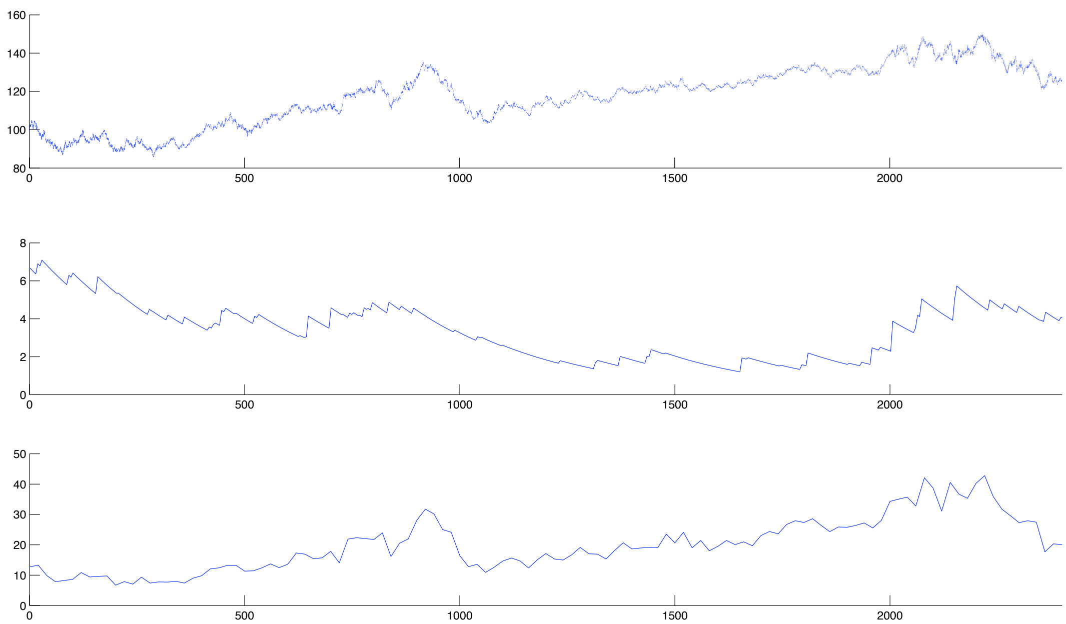
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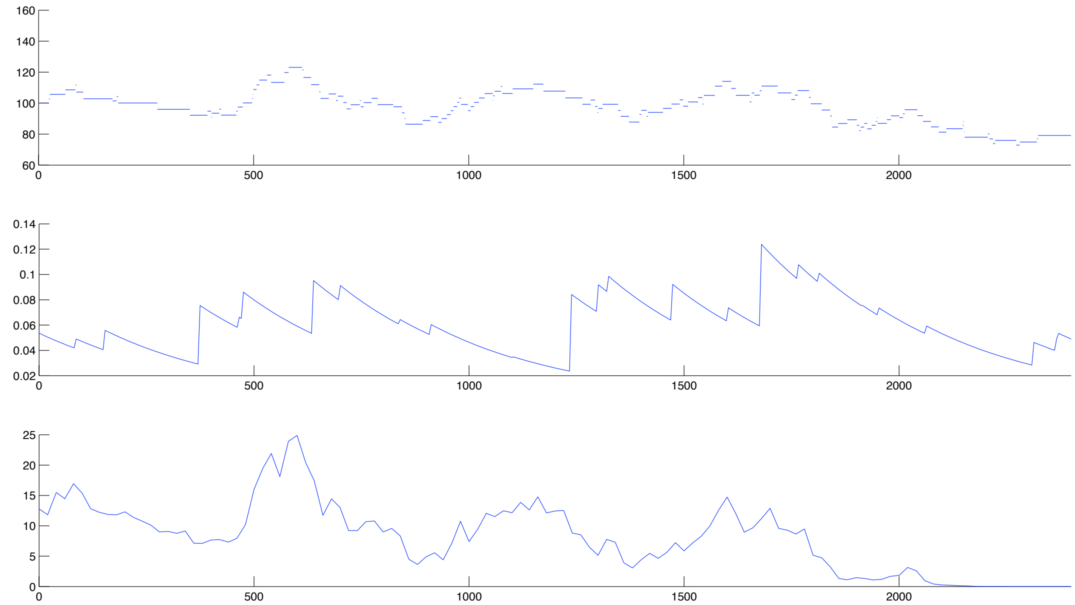
Figure 2



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