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The Cake-eating problem: Non-linear sharing rules*

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Abstract

Consider the most simple problem in microeconomics, a maximization problem with an additive separable utility function over bundles of two goods which provide equal satisfaction to an agent. Although simple, this framework allows for a very wide range of applications, from the Arrow-Debreu contingent claims case to the risk-sharing problem, including standard portfolio choice, intertemporal individual consumption, demand for insurance and tax evasion. We show that any Engel curve can be generated through such a simple program and find the necessary and sufficient restrictions on the demand system to be the outcome of such a maximisation process. Moreover, we identify three classes of utility function that generate non-linear sharing rules. The gap between the two expenditure shares increases in absolute, average or marginal terms with the total amount of wealth, depending on whether DARA, DRRA and convex risk tolerance are considered. The extension of the different results to the case of more than two goods is provided.

Keywords*: cake-eating problem; sharing rules; concavity; convex risk tolerance.* **JEL Codes:** *D11; D81; D90; G12.*

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1 Introduction

Consider an investor who allocates an exogenous wealth over two assets carrying different risk or a consumer who chooses a consumption plan over two periods. Alternatively, look at a couple who share wealth among the two members with unequal weights and whose utility function is identical. In spite of the differences in the setting, these three simple decision-making problems all have the structure of a cake-sharing problem with the same features: a decision maker, two ways of allocating the exogenous wealth, the amounts of the two goods expressed in monetary units like the wealth in two states of the world, the same increasing and concave function representing the cardinal utility provided by the two attributes. Our set up encompasses such models as Arrow-Debreu contingent claim, the standard portfolio model, the intertemporal individual consumption choice problem, the demand for insurance, and the tax evasion decision. The same model may also be adapted to study the sharing of resources generated by the risk-sharing or the cake-sharing problem between two agents, the latter being highly commended from the prescriptive as well as the descriptive point of view.

The correlation in which we are interested is that between the allocation and the amount of wealth. A sharing function maps wealth into the quantity consumed or invested in one good. Under the assumption of identical increasing and strictly concave utility functions, the two sharing functions cannot intersect. If the decision maker prefers to consume more of an attribute for a given level of wealth, this holds for any level of wealth. For convenience, the attribute corresponding to the lower consumption will hereafter be referred to as the *less demanded* attribute. The model considered here is particularly simple in that the group utility function is supposed to be additive separable and the utility function attached to each person or to each good is the same. Can such a simple model recover any couple of non-intersecting sharing rules? When prices are fixed, the answer is positive. For any feasible sharing rule, a utility function can be found that generates this allocation rule as the solution to the optimization problem. Thus, the model's parsimony is not a "straitjacket" on its ability to explain empirical observed behavior. We prove that the unique restriction imposed by the assumption of identical utility functions is the presence of the same "less demanded attribute" for any income level.

However, the result does not hold when the more general issue is to derive any demand function depending on income and prices from an additive separable model with the same utility for the two goods (up to a multiplicative constant). Necessary and sufficient integrability conditions are found and discussed. The conditions are neat and provide a useful classroom exercise in demand theory. The result is somewhat unexpected. It means, for instance, that the usual intertemporal consumption model with discounted utility does not generate some empirically observable and well-shaped consumption patterns, even if we can choose the utility function from among the whole class of increasing and concave functions.

Additional results are obtained on the shape of sharing functions. Linear sharing functions occur in many contexts, as when preferences are homothetic in the context of consumption decision or when utility functions have constant absolute or relative risk aversion. Nevertheless, in general, the structure of the problem as such does not impose linear solutions. Under the assumption of identical utility functions, we show that three forms of non-linear sharing curves (with the linear case as a limit) emerge if and only if the utility function belongs to one of several well-known classes. In the wealth-sharing problem, the marginal propensity to consume the less demanded attribute is decreasing with wealth whenever the utility function has increasing and convex risk tolerance (CT). Suppose now that the utility function only satisfies decreasing relative risk aversion ($DRRA$). Then the propensity to consume the less demanded attribute decreases as wealth increases. Finally, when the distance between the two sharing curves increases with wealth, the utility function conveys decreasing absolute risk aversion ($DARA$).[1] Analogous results are already known for some peculiar frameworks. In an insurance context, Mossin (1968) showed that the amount of coverage is decreasing with wealth when the utility function satisfies $DARA$. A similar result was found by Arrow (1971) for portfolio choice. Gollier (2001b) and (2007) establishes the equivalence between the CT class and the concavity of the sharing function in asset pricing models and provides useful connections among many models, but it seems fair to say that the generality has not yet been fully anticipated nor the numerous applications.

[1] To be specific, in what follows "increasing" and "decreasing" respectively mean "non-decreasing" and "non-increasing". Absolute and relative risk aversion coefficients are equal to $-\frac{v''(x)}{v'(x)}$ and $-\frac{v''(x)}{v'(x)}x$, respectively. "Risk tolerance" is defined by $-\frac{v'(x)}{v''(x)}$ (see Wilson (1968) or Gollier (2001)).

The extension to more than two goods provides more good news than bad news. It is specially interesting for the intertemporal allocation interpretation of the problem. The integrability conditions may be generalized without difficulty and this time they also apply to the fixed prices case. This means that the assumption of the same utility function for all attributes is restrictive with at least three goods whereas it is not the case with two goods. We also find out that the three forms of non-linear sharing curves still emerge when the relationship is between the current consumption and the total wealth to be allocated at the current period.

The outline of the paper is as follows. The next section introduces the basic model, explores the constraint imposed by the condition that the utility functions must be the same and provides a characterization result of the non-linear sharing rules. We also point out the relevance of *linHARA* (linear plus Hyperbolic Risk Aversion) utility functions for the analysis. Section 3 provides various interpretations of the result. Section 4 develops the extension to more than two attributes. Section 5 provides some concluding comments. All proofs are relegated in the appendix.

2 The set up

Let $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Bernoulli utility function defined on the non-negative amounts of attributes 1 and 2. We assume that v is differentiable as many times as required, increasing, strictly concave and respects the "Inada conditions" $\lim_{x\rightarrow 0} v'(x) = \infty$ and $\lim_{x\rightarrow\infty} v'(x) = 0$. Given a weight $a \in [0, \frac{1}{2}]$ and a price vector $\mathbf{p} \in \mathbb{R}^2_+$, we set the constrained maximization problem:

$$\max_{x_1,x_2} av(x_1) + (1-a)v(x_2) \quad (\mathbf{P})$$

$$\text{s.t.} \quad p_1x_1 + p_2x_2 = y.$$

From the first order conditions of $(\mathbf{P})$, it follows that

$$\frac{v'(x_1^*)}{v'(x_2^*)} = \frac{p_1(1-a)}{p_2 a}. \quad (1)$$

Without loss of generality, we assume

$$\pi \equiv \frac{p_1(1-a)}{p_2 a} > 1. \quad (2)$$

Given the assumptions on the utility function, condition (1) guarantees that the two goods are normal. From (1) and (2), due to the strict concavity of v, we also get that the Engel curve for good 1 lies below that for good 2, *i-e.* $x_1^*(y,\pi) < x_2^*(y,\pi)$ for any $y \geq 0$ and any $\pi > 1$. Further, the graph of the outlay function $p_1 x_1^*(y;\pi)$ always lies below the egalitarian line in terms of expenditure. Since an outlay equal to $\frac{p_1}{p_1+p_2}y$ in good 1 ensures that the two attributes are consumed in equal amounts for any $y \geq 0$, we also get:

$$x_1^*(y,\pi) \leq \frac{p_1}{p_1+p_2}y \text{ for any } y \geq 0 \text{ and any } \pi > 1. \tag{3}$$

Differentiating with respect to y the budget constraint we also observe that the marginal rise of the outlay in each attribute cannot exceed the marginal rise of y. It follows by the Lagrange theorem that, for any $\pi > 1$ the demand $x_i^*(y,\pi)$ with $i = 1, 2$ satisfies:

$$(x_i^*(c,\pi) - x_i^*(b,\pi))/(c-b) \leq \frac{1}{p_i} \text{ for all } 0 \leq b < c. \tag{4}$$

Albeit elementary, these properties will be very useful in the following.

2.1 Is a one-utility model too restrictive?

An essential feature of the cake-eating problem (**P**) is the postulation of the same utility function for both attributes. As we will see, a number of applications can support this assumption. In the case of individual decision-making, this assumption is standard for intertemporal decisions or, more generally, when goods are expressed in monetary terms. For group decisions the assumption of identical preferences among agents is more questionable, but here we are interested in a different question: Might positing identical utility functions impose some further restrictions in addition to (3) on the classes of solutions of (**P**)? In what follows, to simplify the notation we will omit π when it is fixed, using the term *sharing function* to designate the quantity of the *less demanded attribute* consumed (invested) with respect to total wealth y, for a given π. We then denote $x_1^*(y;\pi) \equiv f(y)$. Starting from a reduced form with no specification of any particular structural model, we assume that a "less demanded attribute" exists and is unambiguously identified through the sharing function f. We will say that $f(y)$, which links the quantity of the *less consumed attribute* to the size of the cake y, is "regular" when: it starts at 0, is continuous, strictly increasing and it cannot rise more than y. More precisely,

Assumption 1 A "regular" sharing function f is such that $f(0) = 0$, is continuous, strictly increasing and satisfies conditions (3) and (4).

Our next results give mixed answers. Proposition 1 proves that the assumption of identical utility functions does not restrict the space of solutions when prices are fixed (as in the group decision-making framework). Setting $p_1 = p_2 = 1$, we get:

Proposition 1 *For any f satisfying Assumption 1 and $a \in (0, 1/2)$, there exists a differentiable and strictly increasing and concave utility function v such that, from the program (**P**) we get $x_1^*(y; a)) = f(y)$ for all $y \in \mathbb{R}_+$.*

Once a parameter $a \in (0, 1/2)$ is chosen, we can generate all sharing functions through Program (**P**). In this sense, the model with a unique utility function is parsimonious. The fact that a common utility function is used in Program (**P**) does not restrict the class of sharing rules under consideration, except that $x_1^*(y, \cdot) < x_2^*(y, \cdot)$ for any y. In the risk-sharing framework, Mazzocco and Saini (2007) use the fact that if two consumers (households) 1 and 2 have identical risk preferences but 2 is more wealthy, then $x_1(y) < x_2(y)$ for every y. Proposition 1 establishes that this property is, in fact, the only restriction empirically testable imposed by the assumption of identical risk preferences.[2]

It would be important to extend the previous result by introducing prices and requiring the complete recoverability of any "demand" function depending on wealth and prices.[3] When changes in prices are also allowed, the demand function is $x_1^*(y, \mathbf{p}, a)$ which can be written more compactly as $x_1^*(y, \pi)$, by posing $a = 1/2$, $p_1 = \pi$ and $p_2 = 1$. Then condition (1) becomes:

$$v'(f(y, \pi)) = \pi v'(y - \pi f(y, \pi)). \quad (5)$$

The following proposition establishes the necessary ans sufficient condition that a sharing function $f(y, \pi)$ must satisfy to be a solution of (**P**). It will be useful to introduce $h(x, \pi) = g(x, \pi) - \pi x$, where $g(x, \pi)$ is the inverse function of $f(y, \pi)$ wrt y using the fact that the two

[2] The above proposition was already proved in Eugenio Peluso's thesis (2004). The result was presenting in the PET conference in 2005 in Marseille (Peluso and Trannoy 2005). A similar result appears in Mazzoco and Saani (2012).

[3] Kernel prices in the contingent claims model, or interest rate in the intertemporal consumption setting.

goods are normal. The function h gives the demand of the second good as a function of the demand of the first good and of π. Let us also designate $h_t(t,\cdot) \equiv \frac{\partial h}{\partial t}(t,\cdot)$.

Proposition 2 *A function $f(y,\pi)$, strictly increasing with y and decreasing with π is a solution of (5) for all $y \in \mathbb{R}_+$ and for all $\pi > 1$, iff there exist a positive function $A(x)$ such that:*

$$\frac{h_x(x,\pi)}{h_\pi(x,\pi)} = A(x)\pi. \tag{6}$$

Then A represents the absolute risk aversion Arrow-Pratt coefficient of the generating utility function and $v'(x) = \exp \int_0^x - A(s)ds$.

Moreover, if h is a monotonic transformation of a separable function, it is a solution of (5) iff there exist F increasing such that :

$$h(x,\pi) = F(F^{-1}(x) - \ln(\pi)) + k \tag{7}$$

with $(F^{-1})' = \frac{1}{F'(F^{-1})} < 0$ and then

$$v(x) = \int_0^x e^{F^{-1}(s)} ds + C.$$

Proposition 2 allows us to test the consistency between demand functions adopted in economic applications and the model with a unique utility function. There are well-shaped demand functions that cannot be generated by the program (**P**) because they do not satisfy the integrability condition (6). This is the case, for instance, of the power function $x_1^*(y,p) = \frac{1}{2p}y^\gamma$, for $\gamma < 1$. However, condition (6) is satisfied in the linear case, when $\alpha = 1$ and $h(x,p) = (2G(p) - 1)x$, with $G(p)$ increasing and such that $G(1) = 1$. Linear sharing rules arise in individual and group decision making when utility functions are CARA or CRRA (see Borch 1968 or Eliashberg and Winkler (1981)). In some configurations, linear solutions are imposed by the structure of the problem (see Pratt and Zeckhauser (1989) or Pratt (2000) for group utility frontiers in the risk-sharing case). In general, decision models allow for non-linear solutions. In the following, we exhibit several non-linear demand functions implicitly defined by $h(x,\pi)$ that satisfy (6) and can be consequently generated by the program (**P**). We also provide the corresponding utility functions, which belong to some classes that will play an important role in the next paragraph.

1. If $h(x, \pi) = (1 + x)^{\pi} - 1 = \exp\exp[\ln(\ln(1 + x)) + \ln\pi] - 1$ (separable case). We get $\dfrac{h_x}{h_\pi} = \dfrac{\pi}{(1 + x)\ln(1 + x)}$. Then h is the solution of (**P**) with the log-integral utility function $v(x) = \int_0^x \frac{1}{\ln(1+s)} ds$.

2. If $h(x, \pi) = \ln(1+e^x-\pi)-\ln\pi$ (non-separable), we get $\dfrac{h_x}{h_\pi} = \dfrac{e^x}{1 + e^x}\pi$, and the integration gives the linex utility function $v(x) = x - e^{-x}$.

3. If $h(x, \pi) = \dfrac{x\pi}{x - x\pi + 1}$ (non-separable), we get $\dfrac{h_x}{h_\pi} = \dfrac{\pi}{x + x^2}$, which gives the lin-log utility function $v(x) = x + \ln x$.

All these solutions implicitly define concave sharing functions and the corresponding utility functions in the program (**P**) show convex risk tolerance. In the next section we clarify how the shape of the sharing function generated by the program (**P**) does depend on that of the utility function.

2.2 Characterizing non-linear sharing functions

We study three nested classes of non-linear sharing functions: the *moving away, progressive* and *concave*. They rely on simple properties that capture a growing divergence between the demand for the two attributes when wealth increases, and contain the linear class as a particular case.

First, a sharing function is of the *moving away* class $\mathcal{M}$, if the quantity of the less demanded good moves away from the equal split consumption $\frac{y}{p_1+p_2}$ as wealth increases. Then f belongs to this class when $\frac{y}{p_1+p_2} - f(y)$ is increasing with y. (See Panel (a) Figure 1). It also means that the gap between the demands of the two attributes widens as wealth increases. Equivalently, the moving away phenomenon can be described in terms of expenditure: as wealth increases, the expenditure on the less demanded attribute moves away from the amount corresponding to an equal split.

Second, a sharing function belongs to the *progressive*[4] class $\mathcal{P}$ if the average propensity to consume $\frac{f(y)}{y}$ is decreasing with y (see Panel (b), Figure 1). In that case, the ratio between the amounts invested in the two attributes rises with wealth. Equivalently, the difference between the proportions of wealth spent on the two goods increases.

Finally, a sharing function is concave (see below Panel (c), Figure 1) when the difference between the marginal propensities to consume the two attributes is increasing in wealth. It may be observed further that the difference between the marginal expenditure on the two attributes must also be rising in wealth. Denoting by $\mathcal{C}$ the set of concave sharing functions, it is easy to show that $\mathcal{C} \subset \mathcal{P} \subset \mathcal{M}$.

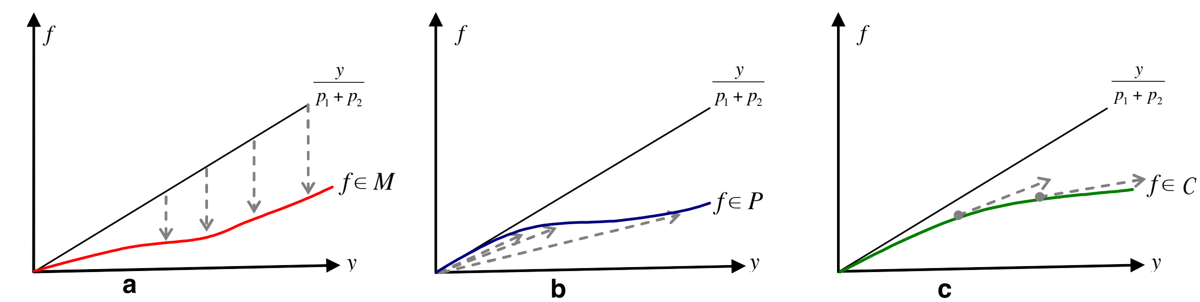

Figure 1. Three types of non-linear sharing functions

Notice that the ranking of the two attributes by demand does not translate directly into the same ranking by outlay. In fact, the higher price of the less demanded attribute may convey more expenditure in this attribute for lower levels of income. When the sharing function satisfies one of the above properties, this ranking can be reversed as income increases. It is also easy to see the existence of at most a single crossing among the outlay functions of the two attributes obtains if and only if the sharing function is concave.

The shape of the sharing function generated by the program (**P**) depends on that of the utility function. This link will be clarified by using some classes of utility functions that are well-known in the risk literature: $DARA$, decreasing absolute risk aversion; $DRRA$, decreasing

[4] The term originates in the public finance literature, where a progressive tax function means that the ratio of disposable income to gross income is decreasing with income.

relative risk aversion and CT, convex risk tolerance. When the three properties hold for the whole domain, we can state the following result.

Proposition 3 *i)* $v \in \text{DARA} \iff x_1^*(y;\cdot) \in \mathcal{M}$ for all $\pi \geq 1$

ii) $v \in \text{DRRA} \iff x_1^*(y;\cdot) \in \mathcal{P}$ for all $\pi \geq 1$.

iii) $v \in \text{CT} \iff x_1^*(y;\cdot) \in \mathcal{C}$ for all $\pi \geq 1$.

The three statements of Proposition 3 may also be expressed in terms of expenditure rather than consumption. The difference between the expenditure on the less invested attribute and that corresponding to an equal split in consumption decreases iff the utility function is $DARA$. A $DRRA$ utility function is necessary and sufficient for the share spent on the less demanded attribute to be decreasing in wealth as well; and a CT utility function is required for the marginal share of the less demanded attribute to decrease with wealth. It is easy to verify that convex risk tolerance insures that the graph of the outlay function of the less consumed good crosses at most once and from above that of the more consumed good.

The CT class is perhaps less well known than $DARA$ and DRRA, even though it includes some popular utility functions such as the $HARA$ class as a limit case and those expressed as a sum of linear and exponential functions (see Bell (1989)). An example of convex tolerance utility is the log-integral function: $v(x) = \int_0^x \frac{\alpha}{\log(1+t)} dt$ (with $\alpha > 0$). A much more flexible family of concave sharing functions is defined by introducing in program $(\mathbf{P})$ utility functions belonging to the *linHARA* class, that is the utility functions obtained by adding a linear term to $HARA$ utility functions. We get three cases:

1) linex $v(x) = \alpha x - e^{-\beta x}$

2) linpower $v(x) = \frac{k}{1-a} x^{1-a} + bx$, with parameters $a > 1$, b and $k > 0$.

3) linlog $v(x) = \alpha x + \log x$.

The relevance of Proposition 3 to individual and collective decision-making is now discussed.

3 Interpretations

Arrow-Debreu contingency claims

Given two states of the world 1 and 2, with probability a and $1-a$, let x_1 and x_2 be the quantities of the Arrow-Debreu securities demanded in these states and p_1 and p_2 their respective prices. Let y be the initial wealth of the investor, v a state-independent utility function. $x_1^*(y, \mathbf{p}; a)$ gives the demand for the contingent claim with the highest "kernel price" $\frac{p_1}{a}$ namely the price per probability unit. The condition (2) indicates that any risk-adverse decision maker will invest less in the more expensive than in the less expensive asset. In this context, the interpretation of Proposition 3 is the following. The discrepancy between the demand for the cheaper contingent claim and the more expansive one is increasing if and only if the utility function is *DARA*. The average propensity to consume this contingent claim is decreasing with wealth if and only it is *DRRA*. Finally, the marginal propensity to consume the more expensive contingent claim is decreasing with wealth if and only if it is *CT*.

The standard portfolio model

Consider an agent with initial wealth y that she can invest in a risk-free asset (asset 1) and a risky asset (asset 2). There are two states of the world with probability a and $1-a$, respectively. The excess return of the risky asset is negative in state 1(the gross return is equal to $1+r-r_1$, with $r_1 > 0$) and positive in state 2 and equal to r_2 (the gross return is equal to $1+r+r_2$). Let z_1 and z_2 be the investment in the two assets, x_1 the final wealth in state 1 and x_2 the final wealth in state 2. They are related by the following constraints: $z_1(1+r)+z_2(1+r-r_1) = x_1$ and $z_1(1+r)+z_2(1+r+r_2) = x_2$. This gives

$$z_2 = \frac{x_2 - x_1}{r_1 + r_2}; \text{and } z_1 = \frac{x_1(1+r+r_2) - x_2(1+r-r_1)}{(1+r)(r_1+r_2)} \tag{8}$$

which substituted into $z_1 + z_2 = y$ leads to

$$x_1 \frac{r_2}{(1+r)(r_1+r_2)} + x_2 \frac{r_1}{(1+r)(r_1+r_2)} = y. \tag{9}$$

The portfolio problem to solve is

$$\max_{x_1, x_2} av(x_1) + (1-a)v(x_2)$$

under constraint (9) as in program $(\mathbf{P})$, with $p_1 = \frac{r_2}{(1+r)(r_1+r_2)}$ and $p_2 = \frac{r_1}{(1+r)(r_1+r_2)}$. The initial condition $\frac{p_1(1-a)}{p_2 a} > 1$ means $r_2(1-a) > ar_1$, that is the expected return on the risky asset must be greater than on the risk-free asset.

Part (i) of Proposition 3 means that the investment in the risky asset (see 8) is increasing with wealth iff the utility function is *DARA*. Part (ii) states that the proportion of wealth in the good state must be increasing with income iff the utility function is *DRRA*. It translates into an increasing proportion of investment in the risky asset, as it is easy to check by plugging the expression of x_1 from (9) into (8) which results in

$$\frac{z_2}{y} = \frac{x_2}{yr_2} - \frac{1}{r_1 + r_2}. \tag{10}$$

Part (iii) means that wealth in the good state is a convex function of initial wealth, and so wealth in the bad state is a concave function of initial wealth iff the utility function has convex risk-tolerance. Using (8) again this means that convex risk tolerance is necessary and sufficient to ensure that the marginal propensity to consume the risky asset (the risk-free asset) is increasing (decreasing) in initial wealth.

Tax evasion

The similarity of the tax evasion problem with the portfolio problem has long been noted (See Cowell 1990). The taxpayer is confronted with a classic economic problem of choice under risk. Consider a taxpayer who has a fixed gross income y subject to a proportional income tax at rate t. The taxpayer can conceal part of his income, e, while declaring the rest d. There are two states of the world, getting caught (state 1) and not (state 2). The probability of being caught is a and is assumed to be independent of any action by the taxpayer. When caught, the income concealed is subject to surcharge at a rate s. In state 1, the taxpayer pays a tax ty $+se$, whereas in state 2 he pays a tax of td. Declared income and concealed income are thus equivalent respectively to a safe asset with negative return and a risky asset. The return to the safe asset is equal to $1-t$. The excess return to a dollar of evaded with respect to declared income is negative in state 1 and is equal to s (the gross return is $1 - t - s$) and positive in state 2 and is equal to t (the gross return is 1). Let x_1 and x_2 be the final wealth in the two states. They are defined by the following constraints $x_1 = d(1 - t) + e(1 - t - s)$ and $x_2 = d(1 - t) + e$ which give

$$e = \frac{x_2 - x_1}{s + t} \text{ and } d = \frac{x_1 - x_2(1 - t - s)}{(1 - t)(s + t)}$$

which if substituted into $d + e = y$ results in

$$x_1 \frac{t}{(1-t)(s+t)} + x_2 \frac{s}{(1-t)(s+t)} = y. \tag{11}$$

The tax evasion reduces to

$$\max_{x_1, x_2} av(x_1) + (1-a)v(x_2)$$

under the same constraint (11) as in program (**P**), with $p_1 = \frac{t}{(1-t)(s+t)}$ and $p_2 = \frac{s}{(1-t)(s+t)}$. The initial condition $\frac{p_1(1-a)}{p_2 a} > 1$ means $t(1-a) > as$, that is, the net expected return of the concealed income must be positive.

Part (i) of Proposition 3 tells us that evaded income is increasing with wealth iff the utility function is *DARA*. Part (ii) states that an increasing proportion of income is concealed iff the utility function is *DRRA*. Part (iii) tells us that convex risk tolerance is necessary and sufficient to ensure that the marginal propensity to evade is increasing with wealth.

Insurance

Consider an agent with initial wealth Y who faces the risk of a loss of $-X$ (with $X > 0$) in state 1 with probability a. This loss can be covered by an insurance contract where the policyholder can choose the optimal absolute coverage $0 \leq C \leq X$. The premium βC is proportional to the coverage, with $\beta < 1$. The final wealth available in the "bad" state 1 is

$$x_1 = Y - X + (1-\beta)C, \tag{12}$$

and

$$x_2 = Y - \beta C \tag{13}$$

in the "good" state 2. Observe that the uninsured loss $X - C$, denoted z_1, is simply

$$z_1 = x_2 - x_1$$

while $z_2 = Y - X + C$ is the wealth covered by insurance or risk-free. From (13) we get $z_1 = X - \frac{Y - x_2}{\beta}$, and by using (12) we can write $z_2 = \frac{\beta(X-Y) + x_1}{(1-\beta)}$. By substituting into $z_1 + z_2 = Y$, we get :

$$\beta x_1 + (1-\beta)x_2 = Y - \beta X = y \tag{14}$$

The decision problem faced by the policyholder then becomes: $\max\limits_{x_1,x_2} av(x_1) + (1-a)v(x_2)$, under constraint (14). The initial condition $\frac{p_1(1-a)}{p_2 a} > 1$ translates into $\frac{\beta}{a} > \frac{(1-\beta)}{(1-a)}$, that is $\beta > a$, which means a positive loading factor charged by the insurance company over and above the fair insurance premium. In this context, part (i) of Proposition 3 means that the uninsured wealth is increasing with wealth iff the utility function is *DARA*. Part (ii) states that the proportion of insured wealth decreases with income iff the utility function is *DRRA*. Part (iii) means that the insured wealth is a concave function of wealth iff the utility function is *CT*. Observe that due to (14), these results cover the case of a change in expected loss.

The intertemporal consumption setting

Proposition 3 above clarifies the connection between consumption and wealth, in a simple intertemporal setting. An agent lives two periods 1 and 2 and wishes to smooth consumption. His exogenous wealth is y, consumption in the two periods is x_1 and x_2. His saving in the first period is $y - x_1$. The agent has an intertemporal separable utility function where the subjective discount utility factor is $\beta < 1$, which implies $a = \frac{1}{1+\beta}$ and $1 - a = \frac{\beta}{1+\beta}$. There is a risk-free asset bringing an interest r. With $p_1 = 1$ and $p_2 = \frac{1}{1+r}$ the market discount factor, the budget constraint is written as in program **P.** The initial condition $\pi > 1$, which ensures lower consumption in the first period (when the agent is younger) than in the second period, is obtained when the subjective discount factor β is greater than the market discount factor. Hence the marginal opportunity cost of saving is lower than the intertemporal marginal rate of substitution, meaning that the agent will consume less than half of his wealth in the first period. Statement (i) of Proposition 3 can thus be interpreted as follows: The positive difference between the future and current period consumption is increasing iff the decision maker has a *DARA* utility function. In other words, this class of utility functions is the largest one that ensures that saving is globally increasing with wealth. Statement (ii) says that the Keynesian concept of average propensity to consume is decreasing for decision makers who have decreasing relative risk aversion. Statement (iii) means that the marginal propensity to consume (in the first period) is decreasing iff the risk tolerance of the decision maker is convex.

Intra-household allocation

This model posits two spouses with the same cardinal utility function. This assumption may reflect a normative point of view, i.e. that one euro of expenditure procures the same marginal utility to any adult person with similar needs. This paves the way to a normative interpretation of the results in this specific model. The spouses have to decide the allocation of the household budget y between them. There are no externalities or public consumption. The private expenditures of the two individuals are x_1 and x_2 and prices are equal to 1. The balance of power among them is captured by the weight a and it is assumed that individual 1 is the "weaker" individual, that is $a < 1/2$. $x_1^*(y,a)$ gives the private expenditure of the weaker individual as a function of the household budget and the weight a. The proposition illuminates the importance of the properties of the cardinal utility function for describing how the consumption of the weaker party relates to the household budget. His marginal part is always less than 1/2 in other words, the difference between the two is increasing with household income iff the utility function satisfies $DARA$. The share of the weaker party decreases iff the utility function belongs to the $DRRA$ type. The marginal portion devoted to the weaker party is decreasing[5] iff the utility function exhibits convex risk tolerance.

Risk-sharing

The second intra-household model is placed in the same framework, but now the income of the household is risky. The spouses earn a random individual incomes z, which are contingent on the realization of θ belonging to a set of states of the world Θ and are not perfectly correlated. So household income becomes a random variable $\gamma : \Theta \to \mathbb{R}$. The two individuals agree to represent risk by a cumulative distribution function $F : \Theta \to [0,1]$. Let $v(x)$ be the identical vNM utility of the two spouses. Hence, the household solves the following program*:*

$$\max_{x_1,x_2} a \int_\Theta v(x_1(\theta))dF(\theta) + (1-a)\int_\Theta v(x_2(\theta))dF(\theta), \; \textit{with } a \in (0, \frac{1}{2}] \tag{15}$$

$$\text{s.t.} \quad z_1(\theta) + z_2(\theta) = y(\theta) = x_1(\theta) + x_2(\theta), \; \forall\theta \in \Theta; \quad x_1 \geq 0; \quad x_2 \geq 0.$$

By the Pareto-efficiency condition obtained by Borch (1960, 1962), the consumption in each state of the world should depend only on the total wealth in that state. That is, the

[5] The concavity of the intra-household sharing function is shown to be crucial in welfare analysis involving stochastic dominance criteria (See Peluso and Trannoy (2007)).

function to be maximized can be written

$$a\int_{\Theta} v(x_1(y))dF(\gamma^{-1}(y)) + (1-a)\int_{\Theta} v(x_2(y))dF(\gamma^{-1}(y) \tag{16}$$

under

$$y = x_1(y) + x_2(y),\ \forall y \in \gamma(\Theta).$$

Since wealth is not transferable from one state to another, solving the above program requires solving Program P for any feasible household income y. Then for a given household income, the problem reduces to the simple intra-household allocation model described above. Then, for instance Proposition 3 *(iii)* tells us that a concave risk-sharing function for the weaker individual arises if and only if the utility function is CT.

4 The case with several attributes/periods

In this section we extend the model to a finite number $n > 2$ of agents (states or periods). Extending the previous assumptions on utility, the constrained maximization problem becomes:

$$\max_{\mathbf{x}} \sum_{i=1}^{n} a_i v(x_i) \tag{P Bis}$$

$$\text{s.t. } \mathbf{p}'\mathbf{x} = y \text{ and } \sum_{i=1}^{n} a_i = 1.$$

From the first order conditions of (**P**), it follows that

$$\tfrac{v'(x_i^*)}{v'(x_j^*)} = \tfrac{p_i a_j}{p_j a_i} = \pi_{ij},\ \forall i \neq j. \tag{17}$$

Proposition 1 is no more true with more than two goods. The assumption of identical utility functions restricts the set of admissible Engel curves. To save on notations, we will illustrate how to recover the utility function associated to a set of well-defined Engel curves when there are three goods. Setting the prices to be equal, the weights may be different and are represented by the vector π as before. The relationship between x_1^* and x_2^* is denoted by ϕ, $x_1^* = \phi(x_2^*)$, while $h_1(x_1^*) = g_1(x_1^*;\pi) - x_1^* - \phi^{-1}(x_1^*)$ is the demand of good 3 as a function of the demand of good 1 and $h_2(x_2^*) = g_2(x_2^*;\pi) - \phi(x_2^*) - x_2^*$ represents the demand of good 3 as a function of the demand of good 2. Obviously $h_1(\phi(x_2^*)) = h_2(x_2^*)$.

Denoting h_{11} the derivative of h_1 with respect to x_1 and h_{22} the derivative of h_2 with respect to x_2, we can now state the following extension of Proposition 1.

Proposition 4 (1bis) *For any vector f of strictly increasing and differentiable sharing functions satisfying $f_i(0) = 0$ for all i, there exists a differentiable and strictly increasing and concave utility function v and a vector π such that the program (P Bis) admits the solutions $x_i^*(y;\pi)) = f_i(y)$ for all $y \in \mathbb{R}_+$ and $i = 1, 2, 3$ if only if the following functional equation can be solved*

$$A(\phi(x)) = A(x)\frac{h_{11}(\phi(x))}{h_{22}(x)},$$

and the solution satisfies the equality

$$A(x_2^*) = A(h_2)h_{22}. \tag{18}$$

Then A represents the absolute risk aversion coefficient of the generating utility function and $v'(x) = \exp\int_0^x - A(s)ds$.

The above proposition provides a way to recover the utility function and the weights that generate the Engel curves. Solving the above functional equation for any possible $\phi(.)$, $h_{11}(.), h_{22}(.)$ does not help too much since these functions are not independent.

The following examples illustrate the powerfulness of the result. Example 1 exhibits a vector of sharing functions for which the functional equation cannot be solved. In example 2 this step is overcomed but (18) is not verified. Finally, the utility function and the weights are recovered in example 3.

Let f be defined by $x_1^* = \frac{y}{4}; x_2^* = \frac{\sqrt{y}}{2}; x_3^* = \frac{3y-2\sqrt{y}}{4}$. By using the fact that $x_1^* = x_2^{*2}$ we easily get: $h_1 = 3x_1 - \sqrt{x_1}$ and $h_2 = 3x_2^2 - x_2$ and therefore $h_{11} = 3 - \frac{1}{2}x_1^{-\frac{1}{2}}$ and $h_{22} = 6x_2 - 1$. The functional equation to solve is: $\frac{A(x_2^2)}{A(x_2^*)} = \frac{3-\frac{1}{2x_2}}{6x_2-1} = \frac{1}{2x_2}$. Then, for $x_2 = 1$, we get: $A(x_2^2) = A(x_2)$ and the above equation reduces to $1 = 1/2$ which is impossible. Then, the function A, if it exists, is not defined for $x = 1$. We conclude that there exists values of y for which there does not exist a utility function generating the above Engel curves through program P Bis.

Let f be defined by $x_1^* = \frac{\sqrt{y}}{2\alpha}; x_2^* = \frac{\sqrt{y}}{2}; x_3^* = y - \frac{(1+\alpha)\sqrt{y}}{2\alpha}$, with $\alpha > 1$. From $x_2^* = \alpha x_1^*$ we easily get: $h_1(x_1) = 4\alpha^2 x_1^2 - (1+\alpha)x_1$ and $h_2(x_2) = 4x_2^2 - \frac{\alpha+1}{\alpha}x_2$ and therefore $h_{11} = 8\alpha^2 x_1 -$

$(1+\alpha)$ and $h_{22} = 8x_2 - \frac{\alpha+1}{\alpha}$. The functional equation to solve is: $\frac{A(x)}{A(\alpha x)} = \frac{8\alpha^2 x_1 - (1+\alpha)}{8x_2 - \frac{\alpha+1}{\alpha}}$. Fixing $x = 1$, we get $\frac{A(1)}{A(\alpha)} = \frac{8\alpha^2-(1+\alpha)}{8\alpha - \frac{\alpha+1}{\alpha}} = \alpha$. Then $A(x) = \frac{k}{x}$ and we must restrict the possible solutions within the crra functions. However condition (18) is not verified, as $\frac{1}{x_1} = \frac{8\alpha^2 x_1 - (1+\alpha)}{4\alpha^2 x_1^2 - (1+\alpha)x_1}$ is false and the problem does not admit solutions.

Let $(1+x_1)^4 + (1+x_1)^2 + x_1 - 2 = y$ implicitly define f_1. We also fix $x_2 = (1+x_1)^2 - 1$ and $x_3 = (1+x_1)^4 - 1$. This implies $x_3 = (1+x_2)^2 - 1$. The functional equation is $\frac{A(x)}{A((1+x)^2-1)} = \frac{4(1+x)^3}{2(1+x)^2} = 2(1+x)$. It is solved for $A(x) = \frac{1}{(1+x)\ln(1+x)}$,which gives a log-integral utility. (18) is checked since $\frac{1}{(1+x_2)\ln(1+x_2)} = \frac{2(1+x_2)}{(1+x_2)^2 \ln[(1+x_2)^2]}$. Finally, from the f.o.c, by exploiting $x_2 = (1+x_1)^2 - 1$ and $x_3 = (1+x_1)^4 - 1$, we easily derive $\frac{a_2}{a_1} = 2$ and $\frac{a_3}{a_1} = 4$, that is, $a_1 = 1/7$, $a_2 = 2/7$ and $a_3 = 4/7$ which generates the assigned sharing rules under the log-integral utility.

To provide extensions of Proposition 2, we need further notation. We set $a_i = 1\ \forall i$ and choose the good n as numéraire, so that $p_n = 1$. The price vector is then denoted as $\mathbf{p} = (p_1, ..., 1)$, the marshallian demand of a generic good i as $x_i^*(y, \mathbf{p})$ and the vector of marshallian demands as $\mathbf{x} = (x_1^*, .., x_n^*)$. It will also be useful to designate by $\mathbf{p}_{\neq n}$ and $\mathbf{x}_{\neq n}$ the truncated price and demand vectors without their last element, that is 1 and the numeraire good quantity x_n^*, respectively. The notation of Section 2 is then completed by defining for any $i = 1, ..., n-1$ the function $g_i(x_i, \mathbf{p})$ as the inverse function of $x_i^*(y, \mathbf{p})$ wrt y and $h_i(x_i, \mathbf{p}, \mathbf{x}_{\neq \mathbf{n},\mathbf{i}}(x_i)) = g_i(x_i, \mathbf{p}) - \mathbf{x}'_{\neq \mathbf{n},\mathbf{i}}(x_i) \cdot \mathbf{p}_{\neq n}$. This function expresses the demand of the most preferred good, good n, in terms of good i and the prices of the other goods, the quantity of other goods than i and n also being equilibrium functions of x_i.

Proposition 5 (2bis) *A demand vector $\mathbf{x} = (x_1^*, .., x_n^*)$ with generic element $x_i^*(y, \mathbf{p})$, strictly increasing with y and decreasing with p_i is a solution of (P Bis) for all $y \in \mathbb{R}_+$ and for all positive $\mathbf{p}$ iff there exists a real function A such that, for any $i = 1, ..., n-1$, the following condition holds:*

$$\frac{\frac{\partial h_i(x_i, \mathbf{p})}{\partial x_i}}{\frac{\partial h_i(x_i, \mathbf{p})}{\partial p_i})} = A(x_i) p_i \tag{19}$$

Then A is the absolute risk aversion Arrow-Pratt coefficient of the generating utility function and $v'(x) = \exp \int_0^x A(s) ds$.

We then characterize non-linear sharing functions when we assume that the kernel prices $\frac{p_i}{a_i}$

are strictly decreasing wrt i. It follows that $\pi_{ij} > 1$ if $i < j$.[6] This case covers the intertemporal consumption model and models with unitary prices (risk-sharing,intra-household). It is easy to find that, solving the program (P Bis), the derivative with respect to y of the demand of each good i is:

$$\frac{\partial x_i^*}{\partial y} = \frac{T(x_i^*)}{\sum_{j=1}^{n} p_j T(x_j^*)}. \tag{20}$$

This rule was found by Wilson (1966) in the special case of risk sharing (where $p_j = 1$ for all j) and by Gollier (2001) in the asset-pricing model (when $\sum p_j = 1$). Notice first that only prices explicitly appear in condition (20). However, we know from (17) that the ranking among any x_i^* and x_j^* is determined by π_{ij}. Having in mind these preliminary facts, we now study how Moving Away, Progressive and Concave sharing functions may emerge as comparative static properties. The following Proposition analyzes the demand function of the less demanded good with respect to the exogenous wealth.

Lemma 1 *i)* $v \in \text{DARA} \Longrightarrow x_1^*(y; \cdot) \in \mathcal{M}$

ii) $v \in \text{DRRA} \Longrightarrow x_1^*(y; \cdot) \in \mathcal{P}$

iii) $v \in \text{CT} \Longrightarrow x_1^*(y; \cdot) \in \mathcal{C}$.

For the concave class, Gollier pointed out that a sufficient condition to get concave sharing functions is convex risk tolerance. However, with a finite number of agents (or periods), this cannot be true for all agents (or periods) as it can be trivially shown by differentiating the FOC wrt y. In the next proposition we shed light on the case of intertemporal consumption or more generally on sequential decisions by clarifying the influence of risk attitude on individual decisions taken at each period.

Proposition 6 (3bis) *Let (P Bis) represent an intertemporal consumption choice, with $n = T$ periods and initial wealth y_1. Let us consider the associated dynamic programming problem where the consumer chooses at each time t the optimal consumption pattern $c_t, c_{t+1}, ..., c_T$ of the remaining $T - t$ periods as a function of the current wealth y_t. Then the conditions of*

[6] This assumption implies a simple rearrangement of the kernel prices and is natural in the intertemporal consumption application.

the Lemma 1 apply for each period $t = 1...T-1$ to the sharing function linking the current consumption c_t to the current wealth y_t.

The risk attitudes framed in Lemma 1 determine the consumption shape at time t as a function of the current wealth y_t and therefore insure a moving away, progressive or concave pattern with respect to that wealth. We cannot say more about the relation between the current consumption and initial wealth since for instance the current wealth is a convex function of initial wealth in the case of convex tolerance. At this stage our concavity result is different from the result obtained by Carroll and Kimball (1996). It is not implied by their result since their theorem is about utility functions belonging to the HARA class with risk aversion and prudence. On the otherway, they consider that prices and incomes are random.

5 Concluding remarks

One aim of decision theory is to find regularities that explain the behavior of the decision-maker independently of the context. This article finds one common feature in a cake-sharing problem when the utility maximizing decision-maker is:

1) an *individual* who allocates an exogenous quantity of wealth among attributes providing utility through the same cardinal function.

2) a *group* with a exogenous wealth to share between agents with different weights, whose utility is identical.

We have examined the impact of a change in wealth on the optimal allocation among the attributes. This relation is encapsulated by the sharing function, which can be viewed as a reduced form of the decision process. The model is shown to be very parsimonious in the context of collective decision-making theory when there are only two goods. In the framework of individual decision-making, the assumption of identical utility functions prevents some sharing functions from being reproduced. We derive a general necessary and sufficient condition for integrability. Very neat non-linear sharing rules emerge whereby the divergence between the two demands increases either in absolute, average or marginal terms as the size of the cake increases. We also provide the first description of the nice properties of the "*linHARA*" utility functions. This flexible class of utility functions with convex risk tolerance provides useful functional forms in all the contexts described above generating a rich family of concave well-shaped demand functions.

A natural question that arises is whether the characterization result still obtains when some heterogeneity of preference is allowed. An easy extension is given in Peluso and Trannoy (2005), when the weights and the utility functions depend on the same parameter in the context of risk-sharing. This is a matter for further research.

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Appendix

Proof of Proposition 1

Without loss of generality, we assume unitary prices and we pose $\pi \equiv \frac{1-a}{a} > 1$. We show that for any continuous and increasing $f(y) < \frac{1}{2}y$ and for any $\pi > 1$, there exists at least one function $u'(f(y);\pi\ )$ that satisfies the the f.o.c. of the program (P), that is

$$u'(f(y);\pi) = \pi u'(y - f(y);\pi), \text{ for all } y \in \mathbb{R}_{++}. \tag{21}$$

We divide the proof into three steps.

Step 1. Let g be the inverse function of f w.r.t. y. g is increasing and such that $g(x) \geq 2t$, $\forall t \in R_{++}$. We denote $h(x) \equiv g(x) - x$.[7] A solution u' of (21) must satisfy;

$$\tfrac{u'(x;\pi)}{\pi} = u'(h(x);\pi), \text{ for all } x \in \mathbb{R}_{++}. \tag{22}$$

Since π is fixed we pose $u'(x;\pi\ ) \equiv v'(x)$ and $j \equiv h^{-1}$, writing (22) as

$$\tfrac{v'(x)}{\pi} = v'(h(x)),$$

after abuse of notation we get

$$\tfrac{v'(j(x))}{\pi} = v'(x). \tag{23}$$

Since j is a one-to-one mapping of $\mathbb{R}_+$, it is meaningful to consider its iterative compositions. Since $\frac{v'(x)}{v'(j(x))} = \frac{1}{\pi}$ holds for any $x > 0$, it is also true $\frac{v'(j(x))}{v'(j \circ j(x))} = \frac{1}{\pi}$, which gives, combined with (23), $\frac{v'(x)}{v'(j \circ j(x))} = \frac{1}{\pi^2}$. More generally, we get:

$$\tfrac{v'(x)}{v'(j_{n-1}(x))} = \tfrac{1}{\pi^n} \forall\ n \in \mathbb{N}. \tag{24}$$

where $j_0(x) \equiv j(x)$ and $j_{n-1}(x) \equiv j \circ .... \circ j$, for $n-1$ times. Using (23) and (24), we now construct a solution $u'(x;\pi\ )$ of (22).

Step 2. Starting from an arbitrary point $(x_a, v'(x_a))$, with x_a and $v'(x_a) > 0$, a second point $(x_b, v'(x_b))$ is uniquely determined posing $x_b \equiv h(x_a)$. Then, from (23) it follows $v'(x_b) = v'(x_a)/\pi$ (see Figure 3.a). We now join the points $(x_a, v'(x_a))$ and $(x_b, v'(x_b))$ by a decreasing segment (for convenience, we choose a segment belonging to an arbitrary strictly decreasing and continuous function w defined on $[x_a, x_b]$, such that $w(x_a) = v'(x_a)$ and $w(x_b) = v'(x_b)$). Using these ingredients, we

[7] Observe that Assumption 1 guarantees that h is increasing.

now show that, for any $\bar{x} > 0$ it is possible to recover $v'(\bar{x})$.

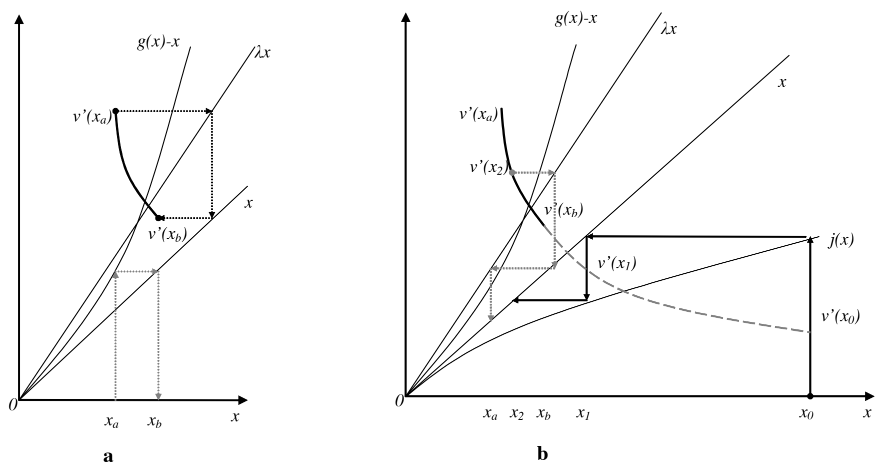

Figure 3

Step 3. Two cases are possible.

a) $\bar{x} > x_b$. j is the inverse function of h. It is increasing and respects $j(x) < x$ for all $x > 0$. The real sequence x_n, defined by $x_{n+1} = j(x_n)$ decreases and converges at 0. We set $x_0 = \bar{x}$, Since $x_a = j(x_b)$ and j is strictly increasing, then $j(\bar{x}) > j(x_b)$,that is $x_1 > x_a$. Even if $x_1 > x_b$, there exists a k such that $x_k \in [x_a, x_b]$. Then $v'(x_k) = w(x_k)$. By (24), it follows $v'(\bar{x}) = v'(x_k)/\pi^{\ k}$. An example with $k = 2$ is sketched in Figure 3.b.

b) $\bar{x} < x_a$. Let x_n be the real sequence defined by $x_{n+1} = h(x_n)$. It is increasing and diverges at ∞. By setting $x_0 = \bar{x} < x_a$, since $x_b = h(x_a)$, then there exists an integer k such that $x_k \in [x_a, x_b]$ and, by reasoning as in the case a, $v'(\bar{x}) = \pi^{\ k} v'(x_k)$. In general, choosing an arbitrary strictly decreasing and continuous function w, such that $w(x_a) = q$ and $w(h(x_a)) = \frac{q}{\pi}$, for some $q > 0$, the solution is found:

$$u'(x;\pi) = \begin{cases} \ldots\ldots & \ldots\ldots \\ \pi^{\ 2} w \circ h_1(x) & j_1(x_a) \leq x < j(x_a) \\ \pi\ w \circ h(x) & j(x_a) \leq x < x_a \\ w(x) & x_a \leq x < h(x_a) \\ \frac{w \circ j_1(x)}{\pi} & h(x_a) \leq x < h_1(x_a) \\ \ldots\ldots & \ldots\ldots \end{cases}$$

$u'(x;\pi)$ is right-continuous since w and h are continuous. Let $(h_1(x_a);\pi)$ be a possible point of non-continuity, with $u'(h_1(x_a);\pi) = \frac{w \circ j_1(h_1(x_a))}{\pi^2} = \frac{q}{\pi^2}$. By construction, $\lim\limits_{x \to h \circ h(x_a)^-} u'(h_1(x_a);\pi$

$) = \lim_{x \to h \circ h(x_a)^-} \frac{w \circ j(x)}{\pi} = \lim_{z \to x_a^-} \frac{w \circ j(h_1(z))}{\pi} = \lim_{z \to x_a^-} \frac{w(h(z))}{\pi} = \frac{q}{\pi^2}$. Since the previous argument applies to any other point where a discontinuity could appear, u' is continuous on R_{++}. It is also immediate to see that it is decreasing.Q.E.D

Proof of Proposition 2

Using $h(x, \pi)$, condition (5) becomes:

$$v'(x) = \pi v'(h(x, \pi)). \tag{25}$$

By differentiating both sides of (25) wrt x, we get

$$v''(x) = \pi v''(h(x, \pi)) h_x(x, \pi). \tag{26}$$

By dividing (26) by (25) it follows

$$A(x) = A(h(x, \pi)) h_x(x, \pi). \tag{27}$$

with

$$A(x) = \frac{v''(x)}{v'(x)} < 0.$$

From (27) we deduce

$$h_x(x, \pi) = \frac{A(x)}{A(h(x, \pi))}. \tag{28}$$

We differentiate now both sides of (25) wrt π

$$0 = v'(h(x, \pi)) + \pi h_\pi(x, \pi) v''(h(x, \pi)) \tag{29}$$

and we get:

$$h_\pi(x, \pi) = -\frac{1}{\pi A(h(x, \pi))}. \tag{30}$$

Since $A < 0$, it follows $h_x(x, \pi)$ and $h_\pi(x, \pi) > 0$. We get

$$\frac{h_x(x, \pi)}{h_\pi(x, \pi)} = -A(x)\pi \tag{31}$$

from which we deduce $v'(x) = \exp \int\limits_0^x A(s)ds$. Condition (31) is then both necessary and sufficient.

We focus now on solutions of the type $h(x, \pi) = F(g(x)+k(\pi))$ that is, F is additively separable.

Let $z = g(x) + k(\pi)$

Hence

$h_x(x, \pi) = g'(x)F'_z(g(x) + k(\pi))$

and

$h_\pi(x, \pi) = k'(\pi)F'_z(g(x) + k(\pi))$

implying that $\frac{h_x(x,\pi)}{h_\pi(x,\pi)} = \frac{g'(x)}{k'(\pi)}$.

Recalling that $\frac{h_x(x,\pi)}{h_\pi(x,\pi)} = -A(x)\pi$, we deduce that $g'(x) = A(x) < 0$; $k'(\pi) = -\frac{1}{\pi}$ are solutions of the problem.

2) we solve in $A(.)$ the functional equation

$F'(z) = \frac{1}{A(F(z))}$.

Using the fact that F is monotone and defining $z = F^{-1}(u)$

we deduce

$$A(u) = \frac{1}{F'(F^{-1}(u))} = (F^{-1}(u))'$$

with the restriction that $(F^{-1}(u))' < 0$.

Using the previous results we can conclude

$g(x) = \int\limits_0^x A(t)dt = F^{-1}(x)$

$k(\pi) = -\ln \pi$.

Since

$$A(x) = \frac{v''(x)}{v'(x)} = \left(\log v'(x)\right)'$$

We can retrieve v from A by performing the following integration

$$v(x) = \int\limits_0^x e^{F^{-1}(s)}ds + C.$$

Q.E.D

Proof of Proposition 3 The two first statements have been known since Mossin (1968) and Arrow (1970) in risk bearing and portfolio choice theory, respectively, and are easily proved in the general case. In particular i) is an immediate consequence of Wilson (1968), Theorem 5, p.128 and of the f.o.c. of (**P**). iii) Can be obtained as a corollary of Theorem 2, Chevallier and Muller (1994). For completeness, we provide a simple proof of ii) and iii).

ii) To simplify the notation we assume unitary prices. Differentiating the f.o.c. of (**P**) with respect to y and solving a standard comparative static problem, we get:

$$\frac{\partial x_1^*}{\partial y}(y,a) = \frac{-(1-a)v''(x_2^*)}{-av''(x_1^*)-(1-a)v''(x_2^*)}; \qquad \frac{\partial x_2^*}{\partial y}(y,a) = \frac{-a_1 v''(x_1^*)}{-av''(x_1^*)-(1-a)v''(x_2^*)}. \tag{32}$$

From (32), we deduce:

$$\frac{\partial x_1^*}{\partial y}(y,a) \leq \frac{x_1^*(y,a)}{y} \iff \frac{-(1-a)v''(x_2^*)}{-av''(x_1^*)-(1-a)v''(x_2^*)} \leq \frac{x_1^*(y,a)}{y}.$$

Since

$$\frac{-(1-a)v''(x_2^*)}{-av''(x_1^*)-(1-a)v''(x_2^*)} \leq \frac{x_1^*(y,a)}{y} \iff -(1-a)v''(x_2^*)(y - x_1^*(y,a)) \leq -av''(x_1^*)x_1^*(y,a)$$

using (1), we get

$$\frac{\partial x_1^*}{\partial y}(y,a) \leq \frac{x_1^*(y,a)}{y} \iff -\frac{v''(x_2^*)}{v'(x_2^*)}x_2^*(y,a)) \leq -\frac{v''(x_1^*)}{v'(x_1^*)}x_1^*(y,a). \tag{33}$$

Sufficiency of DRRA is then deduced. Necessity can be easily proved by contradiction: suppose that for some $\bar{x}_1$ and $\bar{x}_2$ (with $\bar{y} = \bar{x}_1 + \bar{x}_2$) we get $-\frac{v''(\bar{x}_2)}{v'(\bar{x}_2)}\bar{x}_2(y,a)) > -\frac{v''(\bar{x}_1)}{v'(\bar{x}_1)}\bar{x}_1(y,a)$. Given continuity and monotonicity of $x_1^*(y,a)$ with respect to a, there exists $\bar{a} \in (0,1)$ such that $\bar{x}_1 = x_1^*(\bar{y},\bar{a})$ is the solution of the problem (**P**). From (33) we get $\frac{\partial x_1^*}{\partial y}(\bar{y},\bar{a}) > \frac{\bar{x}_1(y,a)}{y}$.

iii) From (32), we get:

$$\frac{\partial^2 x_1^*}{\partial y^2}(y,a) = \frac{(1-a)v'''(x_2^*)\frac{\partial x_2^*}{\partial y}\left[av''(x_1^*)+(1-a)v''(x_2^*)\right]-(1-a)v''(x_2^*)\left[av'''(x_1^*)\frac{\partial x_1^*}{\partial y}+(1-a)v'''(x_2^*)\frac{\partial x_2^*}{\partial y}\right]}{\left[-av''(x_1^*)-(1-a)v''(x_2^*)\right]^2}.$$

Then

$$\frac{\partial^2 x_1^*}{\partial y^2}(y,a) \leq 0 \Leftrightarrow v'''(x_2^*)v''(x_1^*)\frac{\partial x_2^*}{\partial y} - v'''(x_1^*)v''(x_2^*)\frac{\partial x_1^*}{\partial y} \leq 0.$$

Replacing $\frac{\partial x_1^*}{\partial y}$ and $\frac{\partial x_2^*}{\partial y}$ and using (32), it follows:

$$\frac{\partial^2 x_1^*}{\partial y^2}(y,a) \leq 0 \Leftrightarrow (1-a)v'''(x_1^*)\left[v''(x_2^*)\right]^2 - av'''(x_2^*)\left[v''(x_1^*)\right]^2 \leq 0.$$

Using again (1), we conclude:

$$\frac{\partial^2 x_1^*}{\partial y^2}(y,a) \leq 0 \Leftrightarrow \frac{v'(x_1^*)v'''(x_1^*)}{\left[v''(x_1^*)\right]^2} \leq \frac{v'(x_2^*)v'''(x_2^*)}{\left[v''(x_2^*)\right]^2}.$$

We then deduce sufficiency. Necessity can be easily proved by contradiction as in ii). Q.E.D

Proof of Proposition 4

The two f.o.c are

$v'(x_1^*) = \pi_1 v'(h_1(x_1^*;\pi))$

$v'(x_2^*) = \pi_2 v'(h_2(x_2^*;\pi))$

By differentiating

$v''(x_1^*) = \pi_1 v''(h_1(x_1^*;\pi))h_{11}$

$v''(x_2^*) = \pi_2 v''(h_2(x_2^*;\pi))h_{22}$

Dividing we obtain

$$\begin{aligned} A(x_1^*) &= A(h_1)h_{11} \qquad (34) \\ A(x_2^*) &= A(h_2)h_{22} \end{aligned}$$

Since h_1 and h_2 represent the demand for the same good, good 3, we get

$$\frac{h_{11}(x_1^*)}{h_{22}(x_2^*)} = \frac{A(x_1^*)}{A(x_2^*)}$$

and by replacing $x_1^* = \phi(x_2^*)$ we obtain the functional equation to solve. If the solution $A(x)$ exists and satisfies either (18) or (34) (if it satisfies one of the two, the other is satisfied because $A(x)$ is the solution of the functional equation), the utility function can be fully recovered and $v'(x) = \exp \int_0^x -A(s)ds$. The weights are obtained by replacing $v'(x)$ and f in the foc (17) and using the fact they sum to 1. Indeed, notice that if a weight, let us say π_1, depends on x_1^*, the foc would be $v'(x_1^*) = \pi_1(x_1^*)v'(h_1(x_1^*;\pi))$ and (34) would be false. QED.

Proof of Proposition 5

The focs of (P Bis) amount to the list of $n-1$ equations $v'(x_i^*) = p_i v'(x_n^*)$, for $i = 1, ..., n-1$, plus the budget constraint. By replacing the latter in each of the $n-1$ former equations, we get $v'(x_i^*) = p_i v'(y - \mathbf{x}'_{\neq n}\mathbf{p}_{\neq n})$, for $i = 1, ..., n-1$. By introducing the inverse function g_i and simplifying the notation, we get at the equilibrium $v'(x_i) = p_i v'(g_i(x_i, p) - \mathbf{x}'_{\neq n}(x_i)\mathbf{p}_{\neq n})$ and finally a list of $n-1$ equations $v'(x_i) = p_i v'(h_i(x_i, p))$. By differentiating each of these equations with respect to x_i and p_i and repeating the same operations as in Proposition 2, we get (19).

Proof of Lemma 1

For completeness, we first derive the Wilson result in our general set up. Differentiating the FOC we get:

$$1 - p_1 \frac{\partial x_1^*}{\partial y}(y, \cdot) - ..... - p_n \frac{\partial x_n^*}{\partial y}(y, \cdot) = 0$$

$$a_1 v''(x_1^*(y, \cdot)) \frac{\partial x_1^*}{\partial y}(y, \cdot) - p_1 \frac{\partial \lambda(y, \cdot)}{\partial y} = 0$$

....

$$a_n v''(x_n^*(y, \cdot)) \frac{\partial x_n^*}{\partial y}(y, \cdot) - \frac{\partial \lambda}{\partial y}(y, \cdot) = 0$$

In matrix form:

$$\begin{bmatrix} 0 & p_1 & ... & p_n \\ p_1 & a_1 v''(x_1^*) & ... & 0 \\ ... & 0 & ... & 0 \\ ... & 0 & ... & 0 \\ p_n & 0 & ... & a_n v''(x_n^*) \end{bmatrix} \begin{matrix} \frac{\partial \lambda}{\partial y}(y, \cdot) \\ \frac{\partial x_1^*}{\partial y}(y, \cdot) \\ ... \\ \frac{\partial x_n^*}{\partial y}(y, \cdot) \end{matrix} = \begin{bmatrix} 1 \\ 0 \\ ... \\ ... \\ 0 \end{bmatrix},$$

Applying the Cramer rule, we find, for any $k = 1, ...n$:

$$\frac{\partial x_k^*}{\partial y}(y, \cdot) = \frac{\prod_{i \neq k} a_i v''(x_i^*) p_k}{\sum_{j=1}^{n} \prod_{i \neq j} a_i v''(x_i^*) p_j^2}.$$

This gives

$$\frac{\partial x_k^*}{\partial y}(y, \cdot) = \frac{1}{p_k + \frac{a_k v''(x_k^*) p_1^2}{a_1 v''(x_1^*) p_k} + \frac{a_k v''(x_k^*) p_2^2}{a_2 v''(x_1^*) p_k} + ... + \frac{a_k v''(x_k^*) p_n^2}{a_n v''(x_n^*) p_k}}$$

Denoting $T_j \equiv -\frac{v'(x_j^*)}{v''(x_j^*)}$, and using the FOC condition we get

$$\frac{\partial x_k^*}{\partial y}(y,\cdot) = \frac{1}{p_k + \frac{p_1 T_1}{T_k} + \frac{p_2 T_2}{T_k} + ... + \frac{p_n T_n}{T_k}} = \frac{T_k}{\sum_{j=1}^{n} p_j T_j} \tag{35}$$

We now prove the lemma, focusing on the outlay functions $p_i x_i^*$.

i) In the case with n attributes, the Moving Away condition applied to the outlay in the less consumed good means:

$\frac{p_1 \partial x_1^*}{\partial y}(y,\cdot) \leq \frac{1}{n}$. This is an immediate consequence of increasing tolerance, and of the assumptions formulated on prices and weigths.

ii) Requiring progressivity means $\frac{p_1 \partial x_1^*}{\partial y}(y,\cdot) \leq \frac{p_1 x_1^*}{y}$. From (35) we get:

$\frac{p_1 T_1}{\sum_{i=1}^{n} p_i T_i} \leq \frac{p_1 x_1^*}{\sum_{i=1}^{n} p_i x_i^*}$. It follows

$p_1 T_1 \sum_{i=2}^{n} p_i x_i^* \leq p_1 x_1^* \sum_{i=2}^{n} p_i T_i$, which can be rewritten as:

$$\frac{T_1}{x_1^*} \leq \frac{\sum_{i=2}^{n} p_i T_i}{\sum_{i=2}^{n} p_i x_i^*}.$$

Observing that $\frac{\sum_{i=2}^{n} p_i T_i}{\sum_{i=2}^{n} p_i x_i^*} = \frac{p_1 x_1^*}{\sum_{i=2}^{n} p_i x_i^*} \frac{p_1 T_1}{p_1 x_1^*} + \frac{p_2 x_2^*}{\sum_{i=2}^{n} p_i x_i^*} \frac{p_2 T_2}{p_2 x_2^*} + ... + \frac{p_n x_n^*}{\sum_{i=2}^{n} p_i x_i^*} \frac{p_n T_n}{p_n x_n^*}$, the conclusion follows from the assumptions on the weigths and relative increasing tolerance.

iii) is proved by Gollier (2001).Q.E.D

Proof of Proposition 6

Consider the last period T, when $c_T = y_T = y_{T-1}(1+r) - c_{T-1}(1+r)$, or equivalently:

$$c_{T-1} + \frac{c_T}{(1+r)} = y_{T-1} \tag{36}$$

Replacing $y_{T-1} = y_{T-2}(1+r) - c_{T-2}(1+r)$ into (36) and dividing again for $(1+r)$, we get:

$$c_{T-2} + \frac{c_{T-1}}{(1+r)} + \frac{c_T}{(1+r)2} = y_{T-2}.$$

By iteration, we get for a generic period t:

$$c_t + \frac{c_{t+1}}{(1+r)} + ... + \frac{c_T}{(1+r)^{T-t}} = y_t. \quad (37)$$

Therefore, the problem of maximizing the stream of consumption from period t to T disposing of an initial wealth y_1 presents the same structure as that analyzed in Lemma 1. We can conclude that the link between the function $c_t(y_t)$ and the risk attitude of the agent under exam is fully described by Lemma 1.Q.E.D.