



Working Paper Series
Department of Economics
University of Verona

Estimation and pricing with the Cairns-Blake-Dowd model of mortality

Edmund Cannon

WP Number: 65

December 2009

ISSN: 2036-2919 (paper), 2036-4679 (online)

Estimation and pricing with the Cairns-Blake-Dowd model of mortality

Edmund Cannon

edmund.cannon@bristol.ac.uk

Abstract

Parametric forecasts of future mortality improvements can be based on models with a small number of factors which summarise both the improvement in mortality and changes in the relationship between mortality and age. I extend the analysis of the two-factor model of Cairns, Blake and Dowd (2006) to a more general dynamic process for the factors and also consider the problems arising from modelling estimated rather than observed factors. The methods are applied to mortality data for sixteen countries and are used to estimate the value of an annuity and measures of risk. The consequences for the money's worth of an annuity and reserving are also considered.

Keywords

stochastic mortality, mortality projections, annuity, money's worth

This research was undertaken during a visit to the Department of Economics at University of Verona, whose hospitality and financial support are gratefully acknowledged. I have also benefited from discussions with Ian Tonks. Any remaining errors are my own.

1. Introduction

Recent advances in actuarial practice have resulted in a variety of models for describing and projecting mortality: a convenient survey and exposition is provided by Pitacco *et al* (2009).¹ One of the important features of the more recent models is that mortality projections are stochastic rather than deterministic. This is important for two reasons. First, the value of an annuity or any similar pension-type product is a non-linear function of future mortality and hence calculations of annuity values should be based upon the entire distribution rather than just the expected future mortality. Secondly, risk management requires knowledge of the distribution of the annuity and this can only be calculated with knowledge of the mortality distribution. This paper describes several important modelling, estimation and forecast issues within the context of the model proposed by Cairns, Blake and Dowd (2006) (which I shall refer to as the “CBD model”). Most of the results have wider applicability.

The CBD model is a “two-factor” model and is one among a large number of contenders for projecting mortality. The underlying idea is that there is a (downward) trend in mortality, which is presumably either a stochastic trend or a deterministic trend with some variation about the trend. If improvements in mortality were perfectly correlated at all ages then it would be possible to project mortality using a “one-factor” model such as the simple Lee-Carter (1992) model. However, improvements in mortality do not just consist of downward shifts in the functional relationship between mortality and age but also changes in the “shape” of the relationship. If this relationship were sufficiently complicated, or the changes were sufficiently complicated, then this might need to be modelled non-parametrically. This is the P-spline type approach of Eilers and Marx (1996) or CMI (2006). However, empirically it is possible to approximate well the relationship between mortality and age by fairly simple functional forms involving relatively few parameters (Cairns *et al*, 2009). In the simplest case only one parameter is needed to describe the relationship and hence mortality projection

¹ Of course, this does not mean that practising actuaries actually use these new models. CMI (2009, p.6) reports that 83% of life insurers and 82% of pension funds still used a particular version of the deterministic projections made by the UK’s Institute of Actuaries in 1999 and updated in CMI (2002).

requires two factors, which jointly provide a description of the relationship between mortality and age and the trend in mortality over time.

This results in a wide variety of modelling strategies: should the model predict log mortality or the log-odds of mortality (e.g., Cairns *et al*, 2009); should there be two or more factors (e.g., Plat, 2009, suggests four); should there be additional cohort effects (Renshaw and Haberman, 2006)? Merely surveying a sub-set of these possibilities takes up a substantial number of pages in Pitacco *et al* (2009). However, any n-factor model (with n equal to a small number greater than one, such as two) will face the question of how the factors evolve over time and it is this question that I consider here. In the simplest case of two factors, there are three possibilities: both factors are stochastic trends (the original CBD model); both factors vary stochastically about deterministic trends (Sweeting, 2009); or the factors are stochastic trends but share a unit root so that they are cointegrated.

In section 2 of this paper I shall provide an exposition of the model under all three cases in Section 2. A consequence of this class of model which has not received much attention in the literature is that there is ambiguous relationship between the variance of mortality forecasts and the expected value of an annuity: I prove this in section 3. In section 4 I discuss the application of the two factor model to mortality data for sixteen countries taken from the *Human Mortality Database*. In section 5 I discuss the problems that arise from the fact that the CBD methodology first estimates the factors and then analyses their dynamic properties: in the light of this I report tests for distinguishing the models and quantify the importance of measurement error. The resulting analysis provides estimates of annuity values, measures of risk and measures of the consequences for the money's worth of annuities actually sold. Section 6 discusses my results and concludes.

2. An outline of the two-factor model

The CBD model works with a logistic transform of death probabilities or death rates. Given constraints on data availability it is often necessary to work in a discrete-time model with such variables and accordingly in this paper I work consistently with one-year death probabilities, where $q_{x,t}$ is the probability of

dying within one year for someone aged x in year t and $p_{x,t} \equiv 1 - q_{x,t}$.² The original CBD model uses period rather than cohort life tables and can be written

$$(1) \quad \ln\left(\frac{q_{x,t}}{p_{x,t}}\right) = A_t^1 + A_t^2 x \quad x \in \{60, \dots, 95\}, t \in \{0, \dots, T-1\}$$

where the choice of ages 60 to 95 is driven mainly by considerations of data availability and partially to reduce problems of heteroskedasticity which occur when very high ages are included.

Equation (1) says that the log-odds of death probabilities for different ages in year t is a linear function of age, where both the intercept A_t^1 and the slope A_t^2 of the linear function can vary through time. At the risk of stating the obvious, because this is a model of different ages at the same point in time, it is using data from cohorts born in different years. A_t^1 and A_t^2 are the two “factors”. In practice it is impossible to observe either factor, so they must be estimated: CBD do this using OLS although alternative procedures are available.³ In this section I ignore the problem of estimation. In a generalisation of the CBD model, the evolution of the two factors through time is modelled by

$$(2) \quad \begin{pmatrix} A_t^1 \\ A_t^2 \end{pmatrix} = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} A_{t-1}^1 \\ A_{t-1}^2 \end{pmatrix} + \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \delta^1 \\ \delta^2 \end{pmatrix} t + \begin{pmatrix} \zeta_t^1 \\ \zeta_t^2 \end{pmatrix} \quad \begin{pmatrix} \zeta_t^1 \\ \zeta_t^2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}\right)$$

To simplify notation I re-write equations (1) and (2) respectively as

$$(3) \quad Q_t = XA_t$$

$$(4) \quad A_t = \pi A_{t-1} + \mu + \delta t + \zeta_t, \quad \zeta_t \sim N(0, V)$$

where

² The relationship between one-year death probabilities, death rates and mortality, are discussed in actuarial texts such as Bowers *et al* (1997) or Pitacco *et al* (2009).

³ A simple alternative to control for heteroskedasticity would be GLS; when information on the exposed-to-risk is also available a more efficient estimator would explicitly model q with the binomial distribution.

$$(5) \quad \mathbf{Q}_t \equiv \left(\ln \left(\frac{q_{60,t}}{1 - q_{60,t}} \right) \quad \dots \quad \ln \left(\frac{q_{99,t}}{1 - q_{99,t}} \right) \right)', \quad \mathbf{X} \equiv \begin{pmatrix} 1 & 1 & \dots & 1 \\ 60 & 61 & \dots & 99 \end{pmatrix}', \quad \mathbf{x} = \begin{pmatrix} 1 \\ x \end{pmatrix}'$$

$$\mathbf{A}_t \equiv (\mathbf{A}_t^1 \quad \mathbf{A}_t^2)', \quad \boldsymbol{\mu} \equiv (\mu^1 \quad \mu^2)', \quad \text{etc}, \quad \boldsymbol{\pi} \equiv \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}, \quad \text{etc},$$

\mathbf{I} is the identity matrix and $\mathbf{0}$ a vector of zeros. With appropriate estimators of the parameters, this model can be used to predict numerically the density function of the factors using equation (4).

The crux of the matter is how to model equation (4), the interpretation of which depends crucially upon the rank of the matrix $\boldsymbol{\pi} - \mathbf{I}$. CBD assume

$$(6) \quad \mathbf{A}_t = \mathbf{A}_{t-1} + \boldsymbol{\mu} + \boldsymbol{\zeta}_t, \quad \text{rank}(\boldsymbol{\pi} - \mathbf{I}) = 0$$

i.e. the two factors are independent random walks with drift (stochastic trends),⁴ whose only possible relationship is contemporaneous correlation in the shock terms through non-zero off-diagonal components of \mathbf{V} .⁵ Two obvious alternative parameterisations are

$$(7) \quad \Delta \mathbf{A}_t = (\boldsymbol{\pi} - \mathbf{I}) \mathbf{A}_{t-1} + \boldsymbol{\mu} + \boldsymbol{\zeta}_t, \quad \text{rank}(\boldsymbol{\pi} - \mathbf{I}) = 1$$

$$\boldsymbol{\delta} = \mathbf{0} \quad \boldsymbol{\pi} - \mathbf{I} \equiv \boldsymbol{\gamma} \boldsymbol{\beta}, \quad \boldsymbol{\gamma} \equiv (\gamma_1 \quad \gamma_2)', \quad \boldsymbol{\beta} \equiv (\mathbf{1} \quad \beta)$$

$$(8) \quad \mathbf{A}_t = \boldsymbol{\pi} \mathbf{A}_t + \boldsymbol{\mu} + \boldsymbol{\delta} t + \boldsymbol{\zeta}_t, \quad \text{rank}(\boldsymbol{\pi} - \mathbf{I}) = 2$$

The second of these assumes that both factors are deterministic trends (rather than stochastic trends), whereas the first assumes that there is one stochastic trend and the two factors are cointegrated.

A special case of (8) is that $\boldsymbol{\pi} = \mathbf{0}$, which is considered by Sweeting (2009) in the context of a model where the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are subject to infrequent and stochastic shifts.

⁴ When the system in equation (4) has $\text{rank}(\boldsymbol{\pi} - \mathbf{I}) = 0, 1$ it is assumed that $\boldsymbol{\delta} = \mathbf{0}$ and the stochastic trend is modelled by the parameter $\boldsymbol{\mu}$; when $\text{rank}(\boldsymbol{\pi} - \mathbf{I}) = 2$ the deterministic trend is modelled by the parameter $\boldsymbol{\delta}$.

⁵ In my analyses I find that the correlation between $\hat{\zeta}_t^1$ and $\hat{\zeta}_t^2$ from model (5) is almost unity (as do CBD), which may suggest that the model be under-parameterised.

3. Pricing an annuity

In this section I discuss the use of the two-factor model for valuing a financial product, specifically an annuity (although it could also be used to value products such as a mortality bond as in the original CBD article).

Consider a simple life annuity paying an annual income of one unit per period in arrears without proportion. The general formula for the expected value of an annuity sold at time T is

$$(9) \quad a_{x,T} = \sum_{i=1}^{\infty} R_{T+i} \prod_{j=0}^{i-1} \tilde{p}_{x+j,T+j}$$

where R is the discount factor for term-to-maturity i . In actuarial textbooks this is often assumed to be the same for all terms but in economists' analysis of the money's worth it is usually taken from estimates of the yield curve. The future probabilities are unknown and have to be projected, denoted by $\tilde{p}_{x+j,T+j}$. To simplify notation, and without loss of generality, I assume that the maximum number of periods that the annuitant will live is two so equation (9) can be written

$$(10) \quad a_{x,T} = R_1 p_{x,t} + R_1 p_{x,t} p_{x+1,t+1}$$

Note that when the annuity is sold the most recent data available will be for period $T - 1$ (in practice the most recent data available will be older than this).

3.1 Deterministic projection

The simplest projection methodology would be to ignore all of the uncertainty. Then

$$(11) \quad \begin{aligned} \tilde{A}_T^{\{1\}} &= \hat{\pi} A_{T-1} + \hat{\mu} + \hat{\delta} T \\ \tilde{A}_{T+1}^{\{1\}} &= \hat{\pi} \tilde{A}_T^{\{1\}} + \hat{\mu} + \hat{\delta} (T + 1) \\ \tilde{p}_{x,T}^{\{1\}} &= \left(1 + \exp \left\{ \left(1 - x \right) \tilde{A}_T^{\{1\}} \right\} \right)^{-1} \\ \tilde{p}_{x+1,T+1}^{\{1\}} &= \left(1 + \exp \left\{ \left(1 - x + 1 \right) \tilde{A}_{T+1}^{\{1\}} \right\} \right)^{-1} \\ a_{x,T}^{\{1\}} &= R_1 \tilde{p}_{x,T}^{\{1\}} + R_1 \tilde{p}_{x,T}^{\{1\}} \tilde{p}_{x+1,T+1}^{\{1\}} \end{aligned}$$

where the superscript {1} denotes the method of calculating the annuity.⁶ This method of calculation is similar to the projections methods used in the UK in the construction of the “80”, “92” or “00” tables, where a statistical estimation method was used to estimate a relationship between mortality and age and then projected forward deterministically using a trend.⁷

3.2 Stochastic projection taking parameters as certain

The uncertain nature of the factors can be modelled using

$$(12) \quad \begin{aligned} \tilde{A}_T^{\{2\}} &= \hat{\pi}A_{T-1} + \hat{\mu} + \hat{\delta}T + \tilde{\zeta}_T \\ \tilde{A}_{T+1}^{\{2\}} &= \hat{\pi}\tilde{A}_T^{\{2\}} + \hat{\mu} + \hat{\delta}(T+1) + \tilde{\zeta}_{T+1} \end{aligned}$$

where the density function of the factors can be simulated by generating pseudo-random values for the shock terms from the estimated distribution $\tilde{\zeta}_T, \tilde{\zeta}_{T+1} \sim N(0, \hat{V})$. In the simulations below I generate 100,000 simulations allowing me to calculate 100,000 values of the annuity using

$$(13) \quad \begin{aligned} \tilde{p}_{x,T}^{\{2\}} &= \left(1 + \exp\left\{\left(1-x\right)\tilde{A}_T^{\{2\}}\right\}\right)^{-1} \\ \overline{p_{x,T}p_{x+1,T+1}}^{\{2\}} &= \left(1 + \exp\left\{\left(1-x\right)\tilde{A}_T^{\{2\}} + \varepsilon_T\right\}\right)^{-1} \\ &\quad \times \left(1 + \exp\left\{\left(1-x+1\right)\tilde{A}_{T+1}^{\{2\}} + \varepsilon_{T+1}\right\}\right)^{-1} \\ a_{x,T}^{\{2\}} &= R_1\tilde{p}_{x,T}^{\{1\}} + R_1\overline{p_{x,T}p_{x+1,T+1}}^{\{2\}} \end{aligned}$$

Note that this method involves generating the survival probabilities to each age rather than each annual survival probability: the latter would substitute (13) by

$$(14) \quad \begin{aligned} \tilde{p}_{x+1,T+1}^{\{2B\}} &= \left(1 + \exp\left\{\left(1-x+1\right)\tilde{A}_{T+1}^{\{2\}} + \varepsilon_{T+1}\right\}\right)^{-1} \\ a_{x,T}^{\{2B\}} &= R_1\tilde{p}_{x,T}^{\{2\}} + R_1\tilde{p}_{x,T}^{\{2\}}\tilde{p}_{x+1,T+1}^{\{2B\}} \end{aligned}$$

⁶ Note that in models (6) and (7) there is no δ term.

⁷ The trend was based on data supplemented by judgements about ceilings or floors on mortality or mortality improvements. One of the most recent publications by the UK actuarial profession advocates this approach (CMI, 2009).

and hence ignore the correlation between the survival probabilities in different periods.⁸

3.3 Stochastic projection taking parameters as uncertain

The final possibility is to acknowledge that the parameter values are themselves estimates and hence uncertain. Then

$$(15) \quad \begin{aligned} \tilde{A}_T^{\{3\}} &= \tilde{\pi}A_{T-1} + \tilde{\mu} + \tilde{\delta}T + \tilde{\zeta}_T \\ \tilde{A}_{T+1}^{\{3\}} &= \tilde{\pi}\tilde{A}_T^{\{3\}} + \tilde{\mu} + \tilde{\delta}(T+1) + \tilde{\zeta}_{T+1} \end{aligned}$$

In each of my 100,000 simulations I draw a set of parameters from their assumed distribution and then add the pseudo-random shock terms. Generation of the parameter values is slightly different in each model (6), (7) and (8) and is detailed in the appendix. Finally

$$(16) \quad \begin{aligned} \tilde{p}_{xT}^{\{3\}} &= \left(1 + \exp\left\{\left(1-x\right)\tilde{A}_T^{\{3\}} + \varepsilon_T\right\}\right)^{-1} \\ \overline{p_{xT}p_{x+1,T+1}}^{\{3\}} &= \left(1 + \exp\left\{\left(1-x\right)\tilde{A}_T^{\{3\}} + \varepsilon_T\right\}\right)^{-1} \\ &\quad \times \left(1 + \exp\left\{\left(1-x+1\right)\tilde{A}_{T+1}^{\{3\}} + \varepsilon_{T+1}\right\}\right)^{-1} \\ a_{xT}^{\{3\}} &= R_1\tilde{p}_{xT}^{\{1\}} + R_1\overline{p_{xT}p_{x+1,T+1}}^{\{3\}} \end{aligned}$$

3.4 Relationship between the different valuations

Perhaps surprisingly there is no reason to believe that incorporating risk has an unambiguous effect on the value of the annuity. The reason for this is that the annuity formulae in equations (13), (14) and (16) may be concave functions of the risk in mortality improvement: the concavity or otherwise of the function depends upon the actual survival probabilities and interest rates. By a standard application of Jensen's inequality, a value function which is a concave function of a stochastic variable will have a negative relationship to the variance of the variable. To see this, rewrite equation (10) as

$$(17) \quad a_{xT} = \frac{1}{1+e^{F_1}} \left(R_1 + \frac{R_2}{1+e^{F_2}} \right)$$

⁸ From the equation on page 694 of CBD it appears that they used the method in equation (14) rather than that in equation (13).

where the stochastic component is $F_i = A_{T-1+i}^1 + A_{T-1+i}^2 x$. The relevant derivatives are

$$\begin{aligned}
\frac{\partial a}{\partial F_1} &= \frac{-e^{F_1}}{(1+e^{F_1})^2} \left(R_1 + \frac{R_2}{1+e^{F_2}} \right) < 0 \\
\frac{\partial a}{\partial F_2} &= \frac{-R_2 e^{F_2}}{(1+e^{F_1})(1+e^{F_2})^2} < 0 \\
\frac{\partial^2 a}{\partial F_1^2} &= \frac{e^{F_1}(e^{F_1}-1)}{(1+e^{F_1})^3} \left(R_1 + \frac{R_2}{1+e^{F_2}} \right) \quad \begin{cases} < 0 & \Leftrightarrow q_1 < \frac{1}{2} \\ > 0 & \Leftrightarrow q_1 > \frac{1}{2} \end{cases} \\
\frac{\partial^2 a}{\partial F_2^2} &= \frac{R_2(e^{F_2}-1)}{(1+e^{F_1})(1+e^{F_2})^3} \quad \begin{cases} < 0 & \Leftrightarrow q_2 < \frac{1}{2} \\ > 0 & \Leftrightarrow q_2 > \frac{1}{2} \end{cases} \\
\frac{\partial^2 a}{\partial F_1 \partial F_2} &= \frac{R_2 e^{F_1} e^{F_2}}{(1+e^{F_1})^2 (1+e^{F_2})^2} > 0
\end{aligned}
\tag{18}$$

Thus both of the diagonal elements of the Hessian matrix will be negative if the probability of dying in the first period is sufficiently low, namely less than one-half. From 1970 onwards in the UK such high death probabilities were only found among men aged 96 or more and this is also true for most other developed countries. For the function to be concave function, the Hessian would need to be negative semi-definite. From the derivatives in (27), the determinant of the Hessian is

$$\begin{aligned}
\det \begin{pmatrix} \frac{\partial^2 a}{\partial F_1^2} & \frac{\partial^2 a}{\partial F_1 \partial F_2} \\ \frac{\partial^2 a}{\partial F_1 \partial F_2} & \frac{\partial^2 a}{\partial F_2^2} \end{pmatrix} &\propto R_1 e^{F_1} (1-e^{2F_1})(1-e^{F_2}) \\
&+ R_2 \left\{ (1-e^{F_1})(1-e^{F_2}) - e^{2F_1+2F_2} \right\} \leq 0
\end{aligned}
\tag{19}$$

so it is quite possible that the function will be concave, confirming that the effect of the variance on the value of the annuity is ambiguous.

4. Data and preliminary discussion of the model

In the rest of the paper I apply the BCD model and its extensions to male population mortality data taken from the *Human Mortality Database* for sixteen

countries for which good data are available and which are representative of most developed countries. In this section I briefly introduce the data.

The England & Wales log-odds data is plotted for the whole period in Figure 1, which confirms that the log-odds is an approximately planar surface in age-year space. There is sufficient detail in the graph to see an oblique kink running through the data for the cohort aged 60 in 1985 (born 1925), suggesting that there may be a significant cohort effect as noted in Willets (2004). An obvious extension to this paper would be to replicate the analysis using cohort data.⁹

Figure 1 about here

Figures 2 and 3 illustrate my calculated values of the two factors for all sixteen countries: the first figure shows how the factors evolve over time and second is a scatter plot joining temporally consecutive points. Visual inspection shows that there is a large structural break in the trend for most countries in the middle of the post-war period. For nearly all countries the time-series plot for the second factor is almost a mirror image of that for the first factor. This is confirmed in the scatter plots where the points lie close to a straight line, although in many countries the scatter plot appears to be in the shape of a letter “V” lying on its side.

Certainly over the period 1980 onwards and possibly for the whole period there appears to be a fairly tight relationship between the two factors. This could be either because models (7) or (8) fit the data better than model (6), or because model (6) is correct but the magnitude of the drift terms, μ , is large relative to the variance of the shocks in the random walk process.

Figures 2 and 3 about here

The data for the USA are particularly problematic: while sharing many of the features present in the data for other countries, there is an additional change in behaviour of the factors after 2000. Having been in decline from about 1975 onwards the first factor A_t^1 starts to increase and the opposite happens for the second factor. At the same time the relationship between these two factors changes as can be seen from the cross-plot in Figure 3. Such a change in

⁹ This would introduce further complications since the factors for the youngest cohorts would have to be estimated from fewer data observations.

behaviour would be difficult to reconcile with any model and it is unsurprising that the BCD model is unable to fit these data.

5. Analysis of the BCD model and extensions

5.1 Measurement error in the factors

The analysis so far in both this paper and BCD has assumed that the factors are perfectly observed, but in fact they have to be estimated. Consider replacing equations (1) and (3) with

$$(20) \quad Q_t = XA_t + \varepsilon_t \quad \varepsilon_t \sim N(0, \omega^2 I)$$

Using a “hat” to denote the fitted values of the factors,

$$(21) \quad \hat{A}_t \equiv X_t (X_t' X_t)^{-1} X_t' Q_t = A_t + \eta_t \quad \eta_t \sim N(0, H)$$

where the vector η_t can be interpreted as a form of measurement error.¹⁰

Substituting (10) into (4) one obtains

$$(22) \quad \hat{A}_t = \pi \hat{A}_{t-1} + \mu + \delta t + \zeta_t + \eta_t - \pi \eta_{t-1} = \pi \hat{A}_{t-1} + \mu + \delta t + \xi_t$$

leading to the standard result that the OLS estimator is inconsistent, since

$$(23) \quad \text{plim}[\hat{\pi}^{\text{OLS}}] = \pi \left(I - \text{HE} \left[R_{t-1} R_{t-1}' \right]^{-1} \right)$$

where R_t are the residuals obtained from regressing \hat{A}_t on a constant and a trend.¹¹

There are only two cases in which the OLS estimator will be consistent. If $\pi = 0$ there is no problem, but this is implausible and rejected by the data. If the assumption of the original CBD model, namely $\pi = I$, be true it is unnecessary to

¹⁰ Visual examination of the estimated errors suggests mild heteroskedasticity, but I ignore that here to save space. Measurement errors are usually assumed to be serially uncorrelated.

¹¹ Since ξ_t is orthogonal to the constant and the trend they can be “partialled out”. Notice that $E[R_{t-1} R_{t-1}']$ only has a limiting distribution if the data generating process is that of equation (8), but $\left(E[R_{t-1} R_{t-1}'] \right)^{-1}$ has a limiting distribution regardless of the DGP.

estimate π and it is possible to obtain unbiased estimates of μ .¹² Continuing with the general case it follows that

$$(24) \quad E[\hat{\xi}_t \hat{\xi}_t'] = E\left[\left(\zeta_t + \eta_t - \pi \eta_{t-1}\right)\left(\zeta_t + \eta_t - \pi \eta_{t-1}\right)'\right] = V + H + \pi H \pi'$$

showing that the true residuals from the second-stage regression are a combination of the stochastic evolution of the factors and the measurement error. In the special case of the BCD model (equation 6) the OLS estimates are unbiased and the right hand side of equation (24) simplifies to $V + 2H$, so a possible estimator for the relevant matrix would be

$$(25) \quad \hat{V} = T^{-1} \sum_{t=0}^{T-1} \hat{\xi}_t \hat{\xi}_t' - 2\hat{H}$$

where an estimator for H could be obtained from the first-stage regressions

$$(26) \quad \hat{H} = \text{var}[\hat{A}_t] = \hat{\omega}^2 (\mathbf{X}'\mathbf{X})^{-1}, \quad \hat{\omega}^2 = \text{vec}(\hat{\hat{\epsilon}}_{x,t})' \text{vec}(\hat{\hat{\epsilon}}_{x,t}) / (34T)$$

Unfortunately there is no guarantee that the expression in equation (25) will be positive definite and using the data discussed above (from 1980 onwards) it is so for only the Netherlands and Norway. In fact for some countries even the diagonal elements of equation (25) – that is the variances – are negative. This problem could arise either through sampling error or because the simple BCD model is incorrect: regardless of the cause of the problem, the consequence is that it is impossible to implement a logically consistent version of the BCD model for most of the countries in my sample using OLS alone.¹³

For models (7) or (8) projecting mortality is more complicated. Using equation (23) a possible estimator for π would be

¹² The estimates are unbiased (rather than just consistent) in this case because the only regressor is a constant, which is obviously fixed in repeated samples.

¹³ The possibility of this problem arises due to estimating the parameters in a two-stage procedure: first estimating the A parameters and only secondly estimating their dynamic properties. This might be avoided if both were estimated simultaneously (perhaps through Maximum Likelihood), but such an extension is beyond the scope of the current paper.

$$(27) \quad \hat{\pi} = \hat{\pi} \left(\mathbf{I} - \hat{\mathbf{H}} \left(T^{-1} \sum_{t=1}^T \hat{\mathbf{R}}_{t-1} \hat{\mathbf{R}}'_{t-1} \right)^{-1} \right)^{-1}$$

where for model (8) $\hat{\pi}$ would be the OLS estimate from the VAR and for model (7) $\hat{\pi}$ could be the estimate with $\text{rank}(\pi - \mathbf{I}) = 1$ imposed. The variance might then be estimated using

$$(28) \quad \hat{\mathbf{V}} = T^{-1} \sum_{t=0}^{T-1} \hat{\xi}_t \hat{\xi}'_t - \hat{\mathbf{H}} - \hat{\pi} \hat{\mathbf{H}} \hat{\pi}'$$

where the estimated residuals are those obtained using $\hat{\pi}$ rather than $\hat{\pi}$. This formula is only positive definite for Denmark and Norway. Again the problem arises partly from sampling error; at best the expressions in equations (27) and (28) would be consistent and the sample size available is relatively small.

5.2 Distinguishing the models

My analysis so far has considered three versions of the BCD model, based on the three possibilities for the rank of the matrix $\pi - \mathbf{I}$. If these models were to result in very similar valuations of pension or life products then it would not matter which were used: but for some data sets the models give very different answers, so some guidance is needed on which model to use.

An obvious first step is to test the estimated factors for a unit root individually. Table 1 reports Dickey-Fuller statistics for these series for each country, accompanied by conventional p-values for the test under the null hypothesis of a unit root. For six countries (Australia, Belgium, France, Germany, Spain and Sweden) the null of unit roots is comprehensively rejected for both factors and it is marginal for a further two (Poland and Switzerland). This result appears similar to that of Sweeting (2009) although he obtained different results for England & Wales.

However, all of this analysis assumes that there is no measurement error in the factors. I re-calculate the p-values under the null hypothesis assuming that there is measurement error, where the variances of the shocks driving the unit root and the measurement error are estimated using equations (25) and (26): this procedure is only possible where the resulting estimated variance of the shocks is positive. Of the countries for which the exercise is possible, only Australia and Belgium

appear unambiguously not to have unit roots, with mixed evidence for the Netherlands and Sweden. Failure to reject the null of a unit root does not appear to be due to unduly low power: the final column calculates the power of the test under the alternative hypothesis that the auto-regressive parameter is 0.9 and these figures seem acceptable given the small sample sizes.

If there be a unit root it is now necessary to distinguish model (6) from model (7): are the two factors cointegrated? Johansen (1988, 1995) provides a ML procedure to distinguish the models in equations (6), (7) and (8) using the estimated eigenvalues of $\hat{\pi} - I$ to construct the trace statistic:¹⁴ this test is just the multivariate extension of the Dickey-Fuller test. The null hypothesis is that equation (6) is correct: the alternative hypothesis is that either one or both eigenvalues are non-zero.¹⁵ The univariate analysis in Table 1 suggests that measurement error has a big effect on the correct size of tests and this will presumably be true for multivariate tests also.

The asymptotic 5 per cent critical value when there is no measurement error is 25.32 and the trace statistics for the Netherlands and Norway respectively are 29.88 and 14.86, which would suggest rejecting the null for the Netherlands. This is *prima facie* consistent with the result that the Dutch factor \hat{A}_t^2 does not have a unit root when tested in isolation. However, the critical value is too small when there is measurement error, since this biases the test towards rejecting the null. Using the estimated parameter values for these two countries a Monte Carlo experiment suggests that the correct critical values should be 39.18 and 32.10 respectively, so it is impossible to reject the null hypothesis of the model in equation (6).

As noted in the previous section, the Netherlands and Norway are the only two countries for which equation (25) can be used to obtain a positive definite \hat{V} , and hence the only two countries for which I can construct confidence intervals under the assumption of measurement error. Thus for Norway and (to a lesser extent)

¹⁴ Note that, due to non-linearity, the ML estimates of the eigenvalues are not the same as the eigenvalues of the ML estimator $\hat{\pi} - I$.

¹⁵ The Johansen VAR regression includes a trend restricted to lie in the cointegrating space since, under one of the alternative hypotheses (model 8), there must be such a trend for the model to fit the data.

the Netherlands, the mortality data for these countries appears consistent with the original BCD model. For the other countries the problem remains that the procedures used here are insufficient to obtain a satisfactory estimator of the variance using either \hat{V} or $\hat{\hat{V}}$ so it is impossible either to distinguish the models or use them for valuing an annuity.

5.3 Annuity valuation using different versions of the two-factor model

Given the problems in operationalising the model when there is measurement error in the factors, I start by ignoring the problem (ie imposing the assumption that $\mathbf{H} = \mathbf{0}$). While this is not ideal it does allow me to make some comparisons of the different models under discussion. So I use the models in equations (6), (7) and (8) with post-1980 data for all sixteen countries to generate the annuity values $a^{\{2\}}$ and $a^{\{3\}}$. The annuities are valued assuming a constant interest rate of 3 per cent. As has been discussed above, there are particular problems with the data from the USA, but I continue to include that country for purposes of comparison. The simulated expected values, together with the upper and lower deciles and the 90:10 spread, are reported in Table 2 and the density plots are illustrated in Figures 4, 5 and 6. Perhaps surprisingly, the median value is usually close to the mean value and the distributions are approximately symmetric although there is some strange behaviour in the tails (of course these are the parts of the distribution which are modelled least well).

Figures 4, 5 and 6 about here

Table 21 about here

The expected value of the annuity tends to be highest using the model with $\text{rank}(\pi - I) = 2$ and lowest using the model with $\text{rank}(\pi - I) = 1$, but there are many exceptions to this generalisation. To emphasise the very different annuity prices of the three models I look at the “spread”, ie the difference between the largest and smallest expected values from the three models divided by the average expected value. So for England and Wales the annuity is valued (with parameter uncertainty) as either 15.68, 15.35 or 15.55 depending on which model is used: a difference of 0.34 between the highest and lowest price, equal to 2.2 per cent of the annuity value, clearly a large discrepancy. Where the “spread” is greater than two per cent it is shaded in the table, which occurs for seven countries other than

the USA. The latter is notable in that the model with $\text{rank}(\pi - I) = 2$ fits so badly as to be nonsense, unsurprisingly given the behaviour of the A factors for that country shown in Figures 2 and 3: interestingly despite the strange behaviour of the estimated factors model (6) still provides “plausible” densities, suggesting that analysis of this model in isolation may prompt an inappropriate reliance on the results of the model.

For all countries, the result of modelling parameter uncertainty is fairly small: what changes when parameter uncertainty is introduced is the 90:10 spread. In section 3 I established that the effect of greater variance in the mortality has an ambiguous effect on the annuity price. From the table it can be seen that greater variance tends to increase the price slightly under model (6) but reduce the price slightly under model (8): ie the effects are opposite for the two models which I tended to find difficult to choose from the unit root tests.

For either prudential or regulatory reasons a life insurer might sell an annuity not at the actuarially fair price but at a higher price which would limit the probability of a policy making a loss: for example, the price might be set to ensure that the policy be expected to make a loss only 10 per cent of the time. This is one of the possible reasons why the “money’s worth” which is observed for annuity price quotes is less than one (Cannon and Tonks, 2008). Note that the money’s worth is conventionally calculated assuming the expected value of the annuity, whereas the prudential pricing I have described would result in prices based on the upper decile. So to calculate the resulting money’s worth, I simply calculate the ratio of the expected value to the upper decile, using the numbers in Table 2 with parameter uncertainty. This assumes that there no other transactions costs or reasons for unfair pricing such as adverse selection. The results are reported in Table 3. Excluding the special case of the USA, it can be seen that the money’s worth would be in the range 93 to 97 per cent if life insurers were using model (6). This is an upper bound to the money’s worth since there may be additional mortality variance (which I have not modelled) arising from the possibility of future structural breaks such as that which appeared to occur in the late 1970s for many countries. Cannon and Tonks (2008, chapter 6) survey money’s worths for different countries and time periods and find that the money’s worth is typically in the range 80 to 100 per cent and a rough average would be 90 per cent.

Comparing the figures for model (6) in Table 3 to a money's worth of 90 per cent would therefore suggest that reserving against unexpectedly high mortality is playing a relatively large rôle in low money's worths. Consequently problems such as transactions costs, adverse selection or other market failures may be less important than assumed by economists until now.

[Table 3 about here](#)

I now turn to the issue of measurement error and calculate the annuity values for the Netherlands and Norway (ie the two countries for which I can estimate the matrix V using equation 25). Results for these countries under model (6) are reported in Table 4, where I calculate annuity values on the assumption that parameters of the underlying processes are uncertain. The differences between the calculations that assume measurement error and those that explicitly model it are surprisingly small. This provides some evidence that measurement error in the factors is quantitatively unimportant.

[Table 4 about here](#)

6. Conclusion

In this paper I have extended the Cairns, Blake and Dowd (2006) two-factor model in two important ways: first, to generalise the dynamic processes underlying the modelling of the factors; and secondly, to account for the measurement error arising from using estimated rather than observed factors. The two-step procedure used by CBD means that there is no guarantee that estimators of the relevant covariance matrices will be logically admissible (i.e., positive definite) and thus it is impossible to separate the errors in the dynamic process from the measurement error. This is most likely due to be sampling error arising from the short time-series data available. Where it is possible to estimate the covariances, the effect on annuity valuation appears to be minimal, so this problem does not appear to be a major one.

However, the effect of measurement error does have important implications for tests to distinguish the models. Contrary to the results of Sweeting (2009) I find adequate evidence that the factors do follow unit root processes for most countries and my results differ due to the measurement error issue. However, I agree with

Sweeting's analysis that there appears to be a structural break in about 1980 and incorporating this into a unit root framework is clearly a job for future research.

The choice between models can have significant differences in the estimated value of an annuity with differences of up to 2 per cent arising purely from model uncertainty (and this is model uncertainty within the class of log-odds-mortality two-factor models).

Increased uncertainty within a model does not necessarily mean that an annuity will be more costly (in the sense that its expected value is higher), since the effect of the variance is ambiguous. My simulations in Table 2 demonstrate that this is a practical possibility since greater uncertainty, arising from explicit modelling of parameter uncertainty, can increase or decrease expected annuity values. This has important implications for pricing of annuities by life insurers and monitoring of annuity prices by government regulators.¹⁶

International evidence suggests that the money's worth of annuities (the price at which they are actually sold compared to the expected actuarial value) is less than one. Although there are other reasons for low money's worths, this could arise from purely prudential motives of the life insurer or be required by a regulator who was concerned about the solvency of life insurers if realised mortality was less than expected when the annuities were sold. Cannon and Tonks (2009) quote a letter from the UK regulator (the Financial Services Authority) explicitly asking life insurers to build in adequate safeguards against insolvency. Although widely recognised as a possible contributory reason for low money's worths of annuities, I know of no attempt to quantify this hitherto. Using the models estimated in this paper, I calculate how much the money's worth of annuities might be reduced if life insurers priced not on the expected value but on the upper decile. The resulting money's worth under the CBD model would be about 93 to 97 per cent, suggesting a relatively important rôle for reserving in money's worths observed in markets around the world.

¹⁶ The issue is particularly acute in the UK where, in exchange for tax privileges, it is compulsory to annuitise part of a personal pension fund. Government studies of the annuity market include Cannon and Tonks (2009) for the Dept of Work and Pensions and HM Treasury (200###).

Bibliography

- Bowers, Newton L., Gerber, Hans U., Hickman, James C., Jones, Donald A., and Nesbitt, Cecil J. (1997) *Actuarial Mathematics* (Illinois: The Society of Actuaries).
- Cairns, A.J.G., Blake, D. and Dowd, K. (2006) "A two-factor model for stochastic mortality with uncertainty: theory and calibration." *Journal of Risk and Insurance*, 73(4), pp.687-718.
- Cairns, A.J.G., Blake, D., Dowd, K., Coughlan, G.D., Epstein, D., Ong, A., and Balevich, I. (2009) "A quantitative comparison of stochastic mortality models using data from England & Wales and the United States." *North American Actuarial Journal*, 13(1), pp.1-35
- Cannon, E., and Tonks, I. (2008) *Annuity Markets* (Oxford: Oxford University Press).
- Cannon, E., and Tonks, I. (2009) "Money's worth of pension annuities." Dept for Work and Pensions Research Report No 563.
- CMI (2002) "An interim basis for adjusting the '92' series mortality projections for cohort effects." Continuous Mortality Investigation Working Paper 1.
- CMI (2006) "Stochastic projection methodologies: further progress and P-spline model features, example results and implications." Continuous Mortality Investigation Working Paper 20.
- CMI (2009) "A prototype mortality projections model: part one – an outline of the proposed approach." Continuous Mortality Investigation Working Paper 38.
- Dickey, D.A., and Fuller, W.A. (1979) "Distribution of the estimators for autoregressive time series with a unit root." *Journal of the American Statistical Association*, 74, pp.427-431.
- Eilers, P.H.C., and Marx, B.D. (1996) "Flexible smoothing with B-splines and penalties." *Statistical Sciences*, 11, pp.89-121.
- H.M. Treasury (2006) *The Annuities Market* (London: The Stationary Office)
- Human Mortality Database. University of California, Berkeley (USA) and MaxPlanck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 5 August 2008).

Johansen, S. (1988) "Statistical analysis of cointegrating vectors." *Journal of Economic Dynamics and Control*, 12, pp.231-254.

Johansen, S. (1995) *Likelihood-based Inference in Cointegrated Vector Autoregressive Models* (Oxford: Oxford University Press).

Lee, R. and Carter, L.R. (1992) "Modelling and forecasting U.S. mortality." *Journal of the American Statistical Association*, 87(14), pp.659-675.

Pitacco, E., Denuit, M., Haberman, S., and Olivieri, A. (2009) *Modelling Longevity Dynamics for Pensions and Annuity Business* (Oxford: Oxford University Press).

Plat, R. (2009) "On stochastic mortality modeling." Pensions Institute Discussion Paper PI-0908.

Renshaw, A.E., and Haberman, S. (2006) "A cohort-based extension to the Lee-Carter model for mortality reduction factors." *Insurance: Mathematics and Economics*, 38(3), pp.556-570.

Sweeting, P. (2009) "A trend-change extension of the Cairns-Blake-Dowd model." Pensions Institute Discussion Paper PI-0904.

Willets, R. (2004) "The cohort effect: insights and explanations." *British Actuarial Journal*, 10(4), pp.833-877.

Appendix: Simulation procedures with uncertain parameters

For the model in equation (6) we are assuming that both factors are random walks and imposing $\pi = I$. Therefore we only need to estimate the drift term and the variance of the shocks. I simulate the drift term from

$$(A.1) \quad \tilde{\mu} \sim N\left(\hat{\mu}, (T-1)^{-1} \hat{V}\right)$$

Simulating the variance term is slightly more complicated and I use the procedure suggested by BCD: first, generate pseudo-random vectors y from the distribution

$N(\mathbf{0}, T^{-1} \hat{V}^{-1})$ and then use $\tilde{V} = \left(\sum_{s=1}^{T-1} y_s y_s'\right)^{-1}$. I use this method to calculate the

variance matrix for all three models

The model in equation (7) is also fairly straightforward: the OLS standard errors of the parameter estimates are used so that

$$(A.2) \quad \text{vec}\left(\tilde{\pi} \quad \tilde{\mu} \quad \tilde{\delta}\right) \sim N\left(\text{vec}\left(\hat{\pi} \quad \hat{\mu} \quad \hat{\delta}\right), \hat{V} \otimes (Z'Z)^{-1}\right)$$

where Z is the stacked vector of explanatory variables in the VAR. This procedure sometimes results in a value of $\tilde{\pi}$ with an eigenvalue numerically close to zero (in which case the model resembles that of equation 8) and I discard all such simulations. This happened with France, Italy, Norway, Spain and the USA (the maximum number of simulations discarded was sixteen out of 100,000).

For model (8), the variance of the parameters except β is conditional on the value of β itself. In the simulations I took the value of β as given and then used the OLS standard errors so that

$$(A.3) \quad \text{vec}\left(\tilde{\gamma} \quad \tilde{\mu}\right) \sim N\left(\text{vec}\left(\hat{\gamma} \quad \hat{\mu}\right), \hat{V} \otimes (Z'Z)^{-1}\right)$$

Figures and Tables

Figure 1: Mortality by age and year for England and Wales

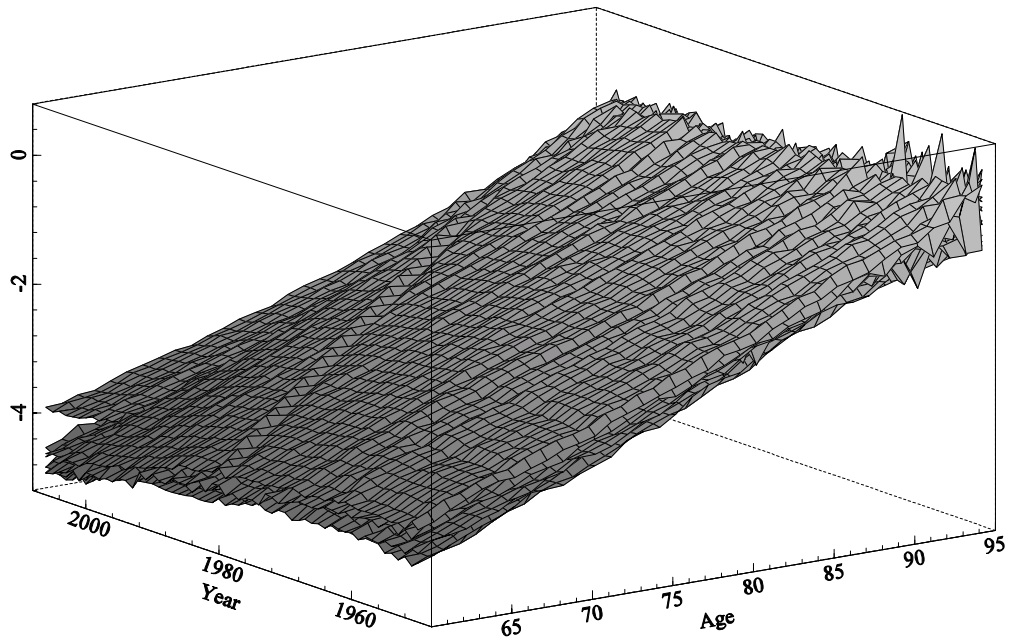


Figure 2, part 1: Time series of A factors

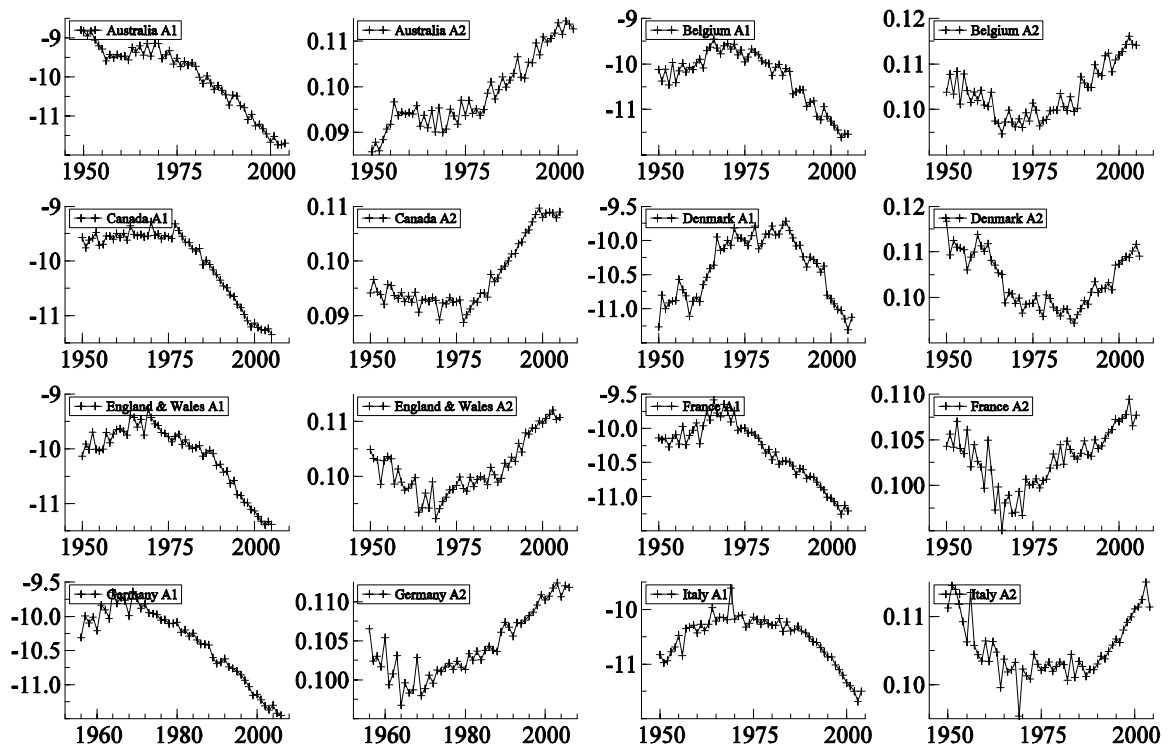


Figure 2, part 2: Time series of A factors

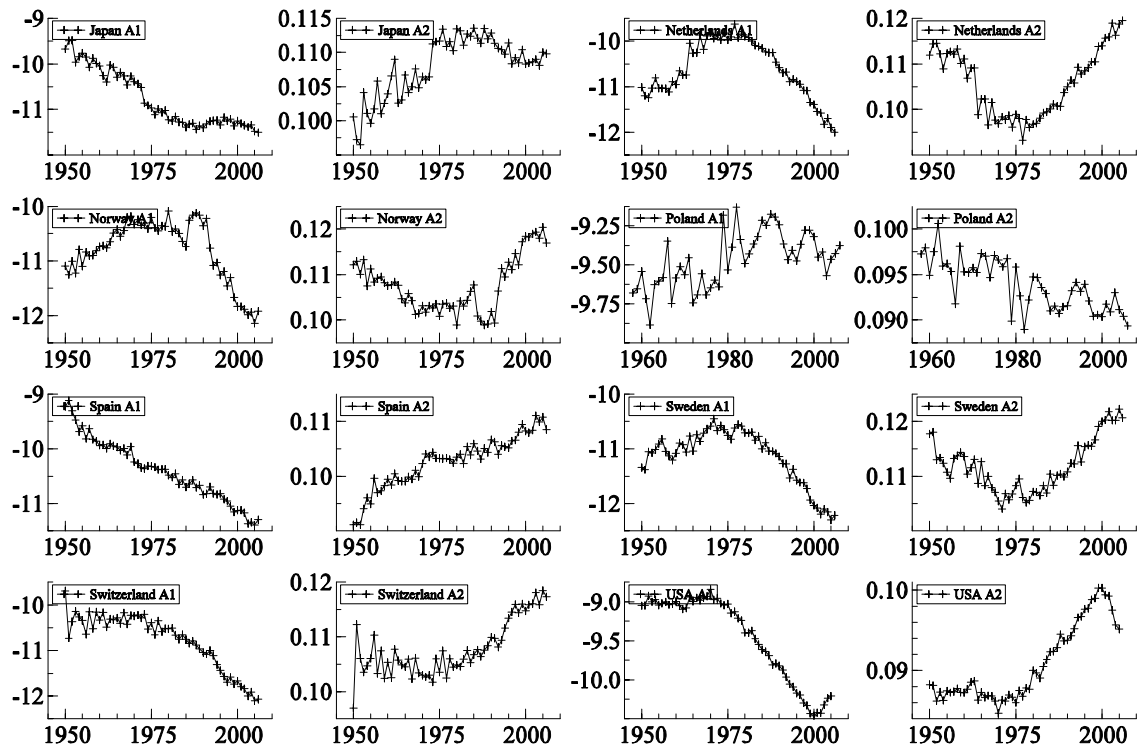


Figure 3: Cross plots of A factors

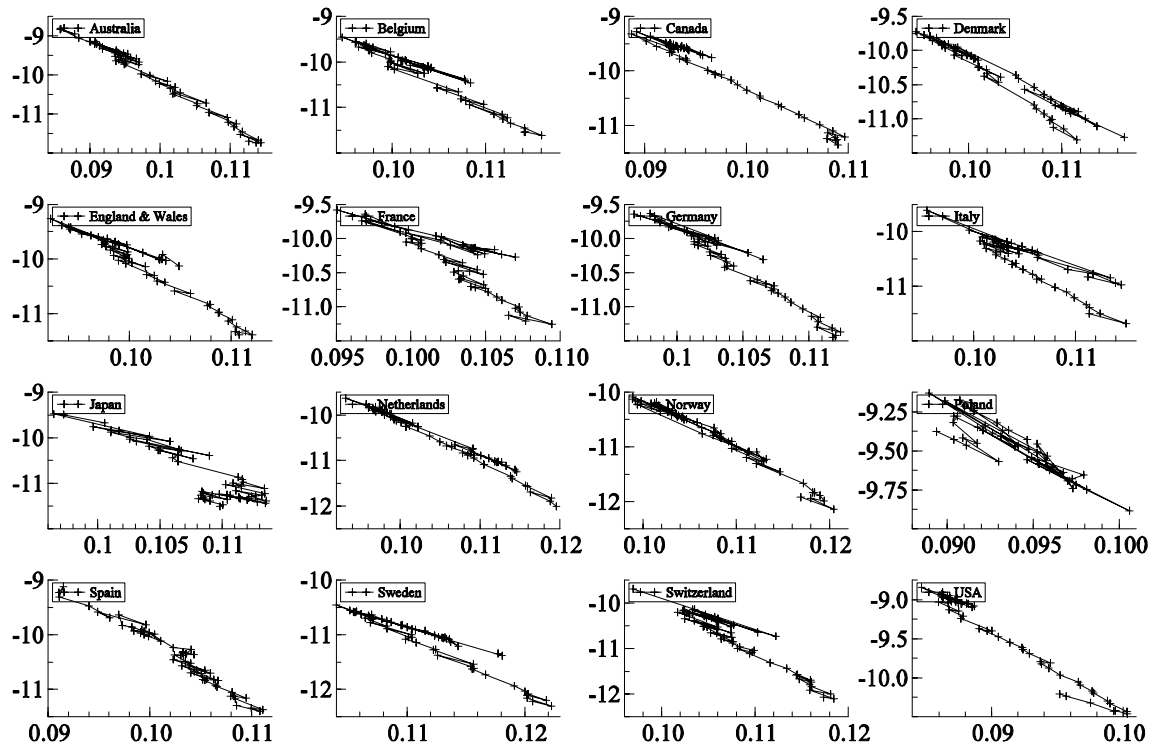


Figure 4: Density Plots of Annuity Valuation with $\text{rank}(\pi - I) = 0$, parameters certain (solid line) and uncertain (dotted line).

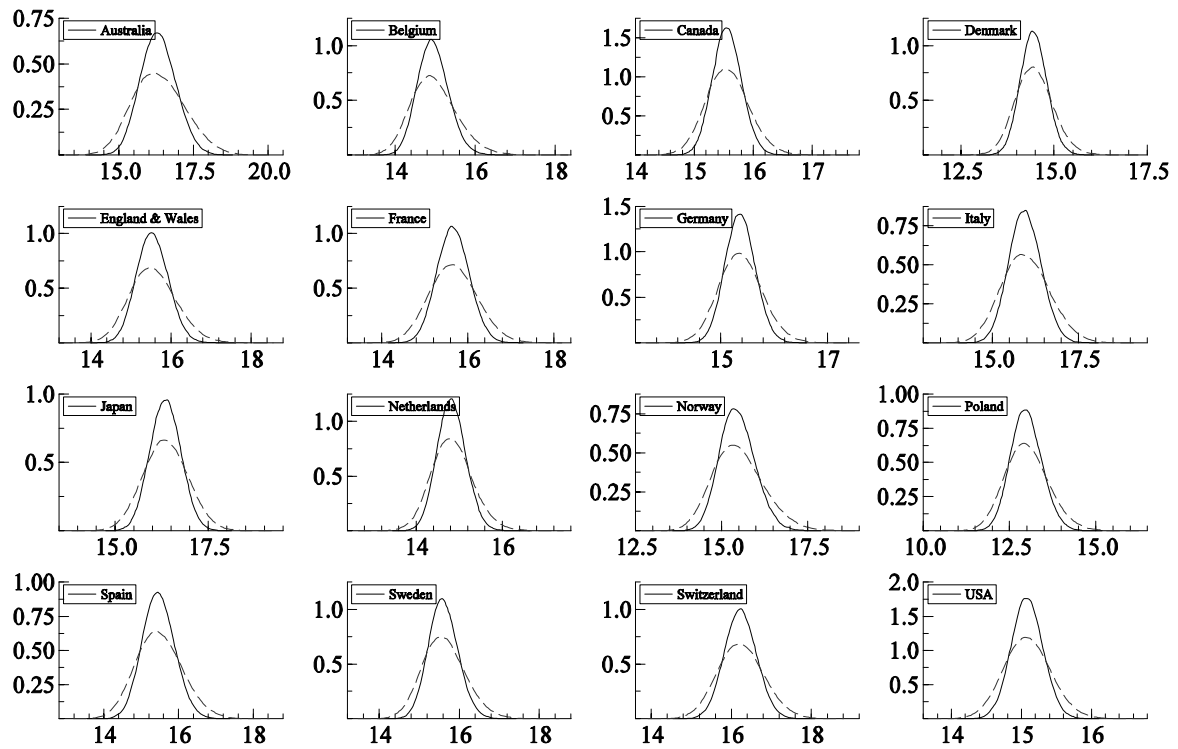


Figure 5: Density Plots of Annuity Valuation with $\text{rank}(\pi - I) = 1$, parameters certain (solid line) and uncertain (dotted line).

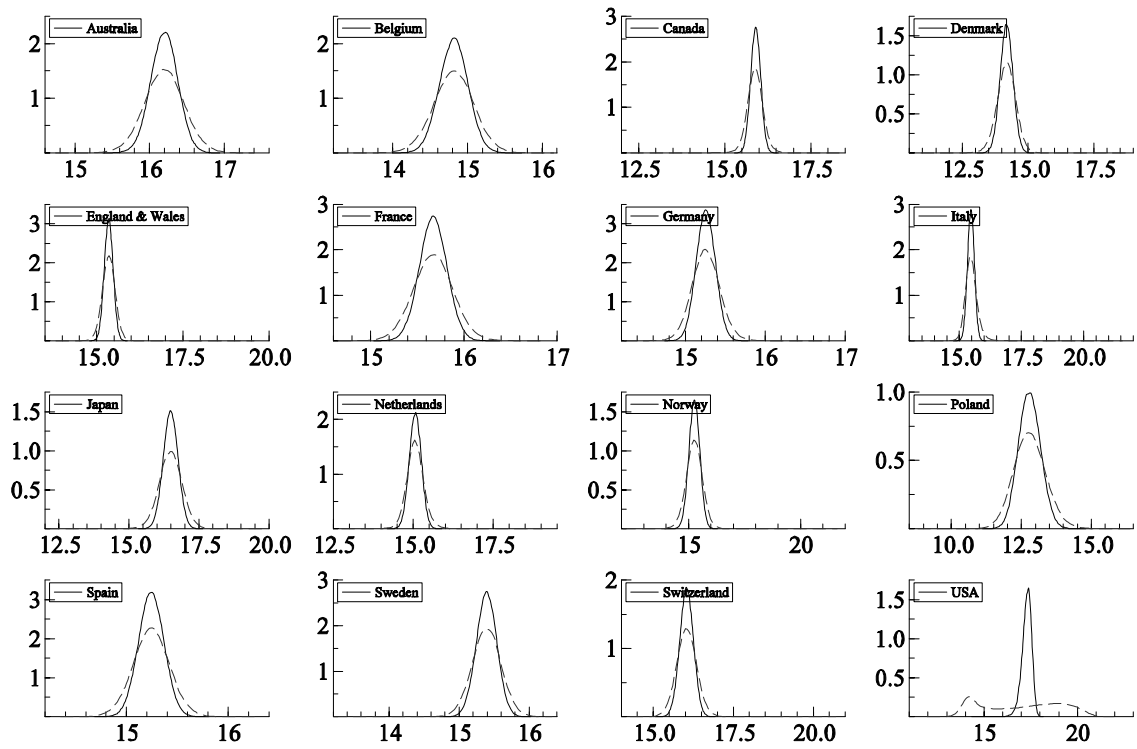


Figure 6: Density Plots of Annuity Valuation with $\text{rank}(\pi - I) = 2$, parameters certain (solid line) and uncertain (dotted line).

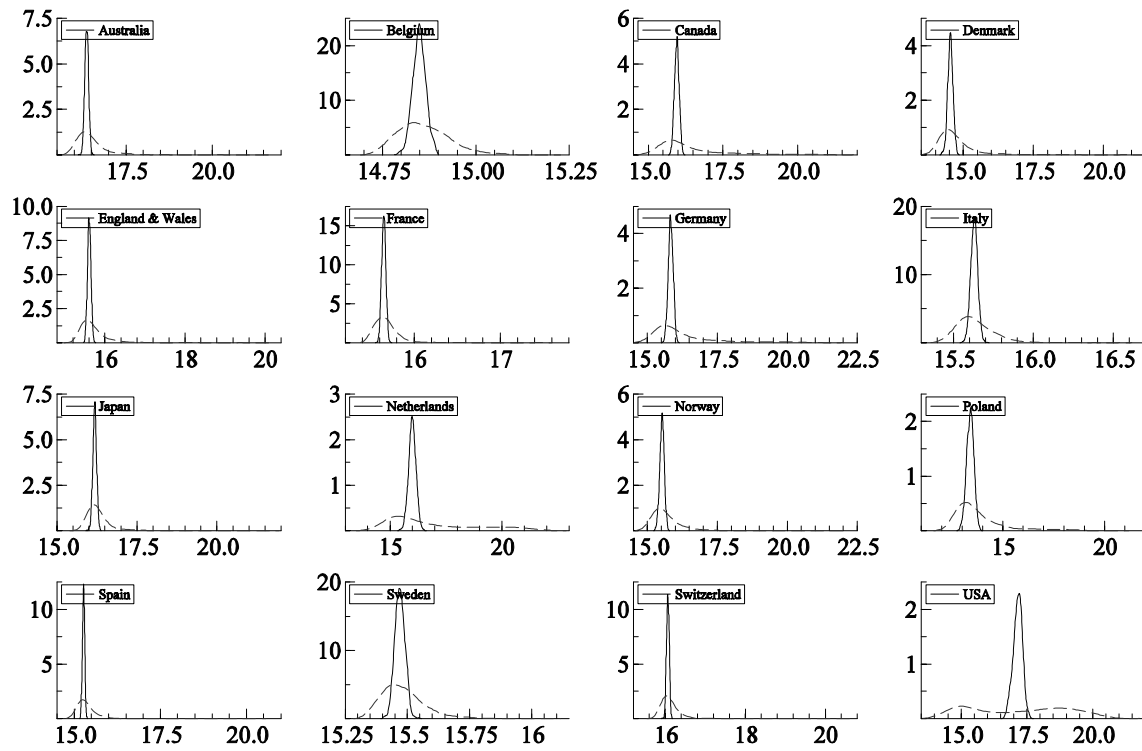


Table 1: Unit Root Tests on Estimated Factors

	Analysis of \hat{A}_t^1			Analysis of \hat{A}_t^2		
	DF test	Measurement error		DF test	Measurement error	
	[conventional p-value]	p-value	power	[conventional p-value]	p-value	power
Australia	-4.62 [0.01]	0.02	0.95	-5.36 [0.00]	0.00	0.00
Belgium	-4.49 [0.01]	0.03	0.91	-4.65 [0.01]	0.02	0.00
Canada	-2.42 [0.36]	0.99	1.00	-1.88 [0.63]	0.94	0.00
Denmark	-3.06 [0.14]	0.87	1.00	-3.42 [0.07]	0.54	1.00
England & Wales	-2.74 [0.24]	0.56	0.32	-3.25 [0.10]	0.20	0.00
France	-3.95 [0.03]			-4.35 [0.01]		
Germany	-3.57 [0.05]			-4.46 [0.01]		
Italy	-2.70 [0.25]	0.95	1.00	-3.16 [0.12]	0.37	0.00
Japan	-2.73 [0.24]			-4.29 [0.01]	0.44	1.00
Netherlands	-3.17 [0.11]	0.47	0.27	-4.72 [0.01]	0.03	0.00
Norway	-1.88 [0.63]	0.76	0.19	-2.09 [0.52]	0.64	0.00
Poland	-3.37 [0.08]			-3.43 [0.07]	0.88	1.00
Spain	-3.68 [0.04]			-4.36 [0.01]	0.16	1.00
Sweden	-4.35 [0.01]	0.80	1.00	-5.11 [0.00]	0.05	1.00
Switzerland	-3.30 [0.09]			-3.60 [0.05]		
USA	0.99 [1.00]			0.79 [1.00]		

Table 2: Annuity valuation under different models of the factors

	Australia		Belgium		Canada		Denmark		Eng & Wales		France		Ge
	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert
Rank 2 (equation 7)													
Expected value	16.36	16.49	14.85	14.85	15.96	16.44	14.53	14.76	15.61	15.68	15.65	15.68	15.8
Lower decile	16.29	16.04	14.83	14.76	15.86	15.43	14.42	14.11	15.55	15.36	15.62	15.51	15.7
Upper decile	16.43	17.00	14.87	14.94	16.06	18.48	14.65	15.54	15.66	16.07	15.68	15.85	15.9
90:10 range	0.14	0.96	0.05	0.18	0.19	3.05	0.23	1.43	0.11	0.71	0.06	0.34	0.2
Rank 1 (equation 8)													
Expected value	16.20	16.20	14.81	14.81	15.90	15.88	14.18	14.18	15.35	15.35	15.67	15.67	15.2
Lower decile	15.97	15.86	14.57	14.47	15.72	15.60	13.87	13.73	15.18	15.11	15.49	15.40	15.1
Upper decile	16.43	16.53	15.06	15.15	16.08	16.16	14.49	14.62	15.51	15.58	15.86	15.94	15.4
90:10 range	0.46	0.67	0.49	0.68	0.37	0.56	0.62	0.89	0.33	0.47	0.37	0.54	0.3
Rank 0 (equation 6)													
Expected value	16.32	16.35	14.95	14.97	15.56	15.56	14.46	14.47	15.54	15.55	15.66	15.67	15.3
Lower decile	15.58	15.26	14.47	14.28	15.25	15.11	14.01	13.84	15.04	14.83	15.19	14.97	15.0
Upper decile	17.09	17.49	15.44	15.72	15.87	16.03	14.91	15.11	16.04	16.31	16.14	16.38	15.7
90:10 range	1.51	2.23	0.98	1.44	0.62	0.92	0.89	1.27	1.00	1.48	0.96	1.41	0.7
"Spread"	1.0%	1.8%	0.9%	1.1%	2.5%	5.5%	2.4%	4.0%	1.7%	2.2%	0.1%	0.1%	3.8%

Table 2 (continued)

	Japan		Netherlands		Norway		Poland		Spain		Sweden		Swit
	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert	Unc	Cert
Rank 2 (equation 7)													
Expected value	16.18	16.28	16.00	16.81	15.53	15.67	13.40	13.75	15.27	15.40	15.47	15.48	16.0
Lower decile	16.10	15.91	15.80	14.89	15.42	15.14	13.18	12.63	15.23	15.10	15.44	15.37	16.0
Upper decile	16.25	16.68	16.20	20.34	15.63	16.28	13.62	15.43	15.31	15.71	15.50	15.59	16.0
90:10 range	0.14	0.77	0.40	5.45	0.21	1.14	0.44	2.80	0.08	0.61	0.05	0.23	0.0
Rank 1 (equation 8)													
Expected value	16.49	16.48	15.07	15.07	15.26	15.26	12.79	12.81	15.24	15.24	15.39	15.39	16.0
Lower decile	16.15	15.96	14.83	14.74	14.95	14.80	12.29	12.09	15.08	15.02	15.20	15.13	15.7
Upper decile	16.82	17.00	15.30	15.40	15.57	15.71	13.30	13.54	15.40	15.47	15.58	15.65	16.3
90:10 range	0.67	1.03	0.48	0.66	0.62	0.91	1.01	1.45	0.32	0.45	0.37	0.52	0.5
Rank 0 (equation 6)													
Expected value	16.35	16.35	14.82	14.84	15.47	15.50	12.99	13.01	15.47	15.49	15.60	15.62	16.2
Lower decile	15.83	15.60	14.40	14.25	14.84	14.60	12.43	12.23	14.93	14.71	15.14	14.96	15.7
Upper decile	16.88	17.12	15.25	15.46	16.13	16.48	13.58	13.83	16.02	16.30	16.08	16.32	16.7
90:10 range	1.05	1.52	0.85	1.21	1.30	1.88	1.15	1.61	1.10	1.59	0.93	1.36	1.0
"Spread"	1.9%	1.2%	7.7%	12.7%	1.7%	2.6%	4.7%	7.2%	1.5%	1.6%	1.4%	1.5%	1.2

Table 3: Consequences for the money's worth (parameters uncertain)

	Rank 2 (equation 7)	Rank 1 (equation 8)	Rank 0 (equation 6)
Australia	0.970	0.980	0.935
Belgium	0.994	0.978	0.953
Canada	0.890	0.982	0.971
Denmark	0.950	0.970	0.957
Eng & Wales	0.976	0.985	0.954
France	0.989	0.983	0.956
Germany	0.894	0.986	0.967
Italy	0.990	0.981	0.946
Japan	0.976	0.970	0.955
Netherlands	0.827	0.978	0.960
Norway	0.962	0.971	0.941
Poland	0.892	0.946	0.940
Spain	0.980	0.985	0.950
Sweden	0.993	0.983	0.957
Switzerland	0.984	0.976	0.956
USA	0.867	0.860	0.972

Table 4: The effect of incorporating factor measurement error into projections

	Ignoring measurement error in factors	Modelling measurement error in factors
Netherlands		
Mean	14.838	14.836
Lower decile	14.257	14.276
Upper decile	15.450	15.414
90:10	1.192	1.138
Money's worth	0.960	0.962
Norway		
Mean	15.505	15.509
Lower decile	14.611	14.648
Upper decile	16.462	16.446
90:10	1.851	1.798
Money's worth	0.942	0.943