Multiobjective Lagrangian duality for portfolio optimization with risk measures

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Abstract. In this paper we present an application for a multiobjective optimization problem. The objective functions of the primal problem are the risk and the expected pain associated to a portfolio vector. Then, we present a Lagrangian dual problem for it.

In order to formulate this problem, we introduce the theory about risk measures for a vector of random variables. The definition of this kind of measures is a very evolving topic; moreover, we want to measure the risk in the multidimensional case without exploiting any scalarization technique of the random vector.

We refer to the approach of the image space analysis in order to recall weak and strong Lagrangian duality results obtained through separation arguments.

Finally, we interpret the shadow prices of the dual problem providing new definitions for risk aversion and non-satiability in the linear case.

Keywords: Multivariate risk measures, Vector Optimization, Lagrangian Duality, Shadow prices, Image Space Analysis.

AMS Classification: 90C, 49K, 91B30, 91G10

JEL Classification: C61, D81, G11.

1   Introduction and preliminaries

The first measure of the riskiness of an investment was the variance, or the standard deviation. Markowitz [18] proposed an optimization problem for minimizing the risk and maximizing the return of a portfolio, where the risk was given by the standard deviation. Now, there are indeed two approaches: the "old" or classic one, which exploits the variance or standard deviation of the portfolio, and the more recent one, which introduces a set of axioms and properties for a function to quantify the risk of an investment. The former gives only positive values; it does not account for fat tails of the underlying distribution and treats the positive and negative deviation from the mean in the same way, and therefore, it is inappropriate to describe the risk of low probability events; whereas, the latter gives also negative amounts and thinks to a risk measure as to the necessary capital for covering of losses.

One of the critical steps for financial institutions is to construct a proper risk measure. Risk measurement is a very evolving topic in theoretical and practical fields in recent years. We are interested in giving a characterization of risk measures in order to formulate a multicriteria optimization problem, where the objective functions are the risk and the expected pain of a portfolio vector.

The paper is organized as follows: in the remaining part of this section, we give some introductive notions about the theory of risk measures for a random variable; we introduce the properties of these measures and the definitions of a monetary, convex and coherent measure of risk. In Sect.2, we discuss the recent developments about this theory in order to introduce the definition and the properties of a risk measure for a vector of random variables. There is not a unique approach about it, hence, we propose the concepts which will be exploited later in the paper. In Sect.3, it is described the vector optimization problem that we want to deal with and the formulation of the Lagrangian dual problem. In order to do this, we introduce the so called generalized Pareto problem and the Lagrangian duality results are obtained through separation arguments and the image space analysis. Sect.4 is devoted to describe an application of the optimization problem recalled in the previous section. We propose a minimization problem of two objective functions: the risk and the expected pain associated to a portfolio vector under some constraints. The risk is represented by a risk measure, instead of by the standard deviation, or the variance and the expected pain is the opposite of the expected utility. The interpretation of the shadow prices of this problem allows us to provide new definitions for risk aversion and non-satiability in a

multicriteria setting.

We consider risk measures in a static situation without reference to the hedging strategies that could minimize or eliminate risk, and without discounted net worths.

A random variable X is a measurable function mapping Ω to the real numbers, that is, $X : \Omega \to \mathbb{R}$, and such that $X^{-1}((-\infty, x]) \in \mathcal{F}$, $\forall x \in \mathbb{R}$, where $X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\}$. We recall that $(\Omega, \mathcal{F}, \mathcal{P})$ is a measurable space equipped with the set of outcomes Ω, the σ-algebra $\mathcal{F}$ and with the probability measure $\mathcal{P}$, which is considered known.

We call $\mathcal{X}$ the space of all random functions $X(\omega)$ and we suppose that it is a Banach space. The article that lays the foundations of the axiomatic approach in defining a risk measure is due to Artzner, Delbaen, Eber, Heath [1], followed by many others, we cite only [9] which generalizes the previous approach to the space $\mathcal{L}^\infty$.

Consider the following positions:

$$\mathcal{X}_+ = \{X \in \mathcal{X} : X(\omega) \geq 0 \ \ \forall \omega \in \Omega\} \text{ and } \mathcal{X}_- = \{X \in \mathcal{X} : X(\omega) \leq 0 \ \ \forall \omega \in \Omega\}.$$

Now we can give the definition of acceptance set of future net worths, or investments.

Definition 1 *Given a set $\mathcal{A} \subset \mathcal{X}$ it is acceptable if and only if it satisfies all these axioms:*

1. $\mathcal{X}_+ \subset \mathcal{A}$;

2. $\mathcal{A} \cap \mathcal{X}_- = \{0\}$;

3. $\mathcal{A}$ is a cone;

4. $\mathcal{A} + \mathcal{A} \subset \mathcal{A}$.

Axioms 1 and 2 say that a nonnegative outcome does not need that extra capital is to be added to regulate the risky position. Axioms 3 and 4 lead to positive homogeneity of the cone $\mathcal{A}$: this means that every nonnegative multiple of an acceptable investment is again acceptable and that the combination of two acceptable investments is again acceptable. Furthermore, we can introduce, instead of these axioms, a convexity requirement for those investors which do not want loose too much, or more than a threshold amount.

If a set $\mathcal{A}$ satisfies all the axioms, it is a pointed, convex cone and thus $\mathcal{A}$ defines an ordering relation in $\mathcal{X}$. A random variable (an investment or a risk position) is preferred to another one, say Y, if $X \geq_\mathcal{A} Y$ and thus $X - Y \in \mathcal{A}$.

Let us make the point that, contrary to a convention often adopted in the literature, we choose

to account positively for net losses, hence, X is an effective loss.

Once defined which kind of investment we want to deal with, we can proceed in measuring the risk of this investment.

Definition 2 *A risk measure is a functional ρ mapping a loss (an investment, or a risky position) X to the real space $\mathbb{R}$.*

If $\rho(X)$ is negative, it represents the minimal extra cash which if added to X (the initial position), and invested in a risk free manner, makes the total investment acceptable, otherwise if it is positive, it gives the maximal amount of cash that could be withdrawn such that the reduced result remains acceptable (this is a consequence of the translation property requirement).

We consider now some desirable properties for univariate risk measures:

- Expectation boundedness: $\rho(X) \geq E(X)$, $\forall X \in \mathcal{X}$;

- Non-excessive loading: $\rho(X) \leq \max(X)$, $\forall X \in \mathcal{X}$;

- Monotonicity: $Pr(X \leq Y) = 1 \Rightarrow \rho(X) \leq \rho(Y)$, $\quad \forall X, Y \in \mathcal{X}$;

- Translation invariance: $\rho(X + b) = \rho(X) + b$, $\forall X \in \mathcal{X}, \forall b \in \mathbb{R}$, where b is a sure initial amount. We have in particular the equality $\rho(X - \rho(X)) = 0$, that is a way to obtain a neutral position and to define the acceptance set associated to a risk measure as $\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \geq 0\}$, that is the set of those positions that do not require extra capital;

- Positive homogeneity: $\rho(aX) = a\rho(X)$, $\quad \forall X \in \mathcal{X}, \forall a \geq 0$;

- Constancy: $\rho(c) = c$, $\forall c \in \mathbb{R}$. A special case is $\rho(0) = 0$, that is called normalization property;

- Convexity: $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$, $\quad \forall X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$. This property implies diversification effects;

- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y$ that reflects again the idea that risk can be reduced by diversification. The diversification effect is thus $\sum_{i=1}^{n} \rho(X_i) - \rho(\sum_{i=1}^{n} X_i)$ and it is positive for subadditive risk measures.

We can make a discrimination among all these properties; this will influence our choice of a risk measure. The interested reader may find some reference about these definitions and other properties in [11], [12] and [13].

A risk measure is *monetary* if it is monotone and translation invariant. If a monetary measure of risk is positively homogeneous, then it is normalized. A risk measure is *convex* if it is monetary and also fulfills the convexity property, and it is *coherent* if it is translation invariant, subadditive, positively homogeneous and monotone. A coherent risk measure is a convex one which satisfies also positive homogeneity. Some properties are not universally accepted, thus one can modify them to obtain different approaches and different coherent risk measures.

We note that if a "large position" in the market is doubled in its size, the risk of the final position can be more than doubled, because bid prices will depend on the position size. This fact could allow us to prefer convex risk measures rather than coherent ones.

As noted before, through translation property, a monetary risk measure ρ induces the acceptance set $\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \geq 0\}$ and thus a position is acceptable if it does not require extra capital.

In recent years the researchers address themselves toward a very interesting aspect of the theory of measuring risk; i.e., risk measures for a vector $\mathbf{X} = (X_1, \ldots, X_n)$ of random variables. We give some introductive notions in the subsequent section.

2   Risk measures for portfolio vectors

A portfolio is made of components, which correspond to a specific security market and thus we can consider the loss or the return of these investments all together and not one by one, even because investors are in general not able to aggregate their portfolio because of liquidity problems or transaction costs. Besides, to aggregate the risk measured on the marginals X_h, $h = 1, \ldots, n$, could give a different result than measuring the joint risk of all components, caused by their variation or by the dependence structure among the assets.

Some works that treat this argument are: [16], where Jouini, Meddeb and Touzi defined coherent risk measures as set valued maps from $\mathcal{L}_d^\infty$ into $I\!\!R^n$. They provided an aggregation function to pass from a $I\!\!R^d$-valued random portfolio to a $I\!\!R^n$-valued measure of risk and gave necessary and sufficient conditions of coherent aggregation. In [6], Burgert and Rüschendorf studied the consistency of risk measures, axiomatically defined, for portfolio vectors with respect to various classes of orderings and they introduced some ways of aggregating the different components of the portfolio; in [21], Rüschendorf deepened this study and in [22] he introduced the multivariate distorted measures of risk, proposed again later by [7], where Cardin and Pagani

gave a mathematical formalization of this kind of measures, also through some results about the representation of subadditive distorted risk measures as combination of vector CVaRs, and introduced a new multivariate measure called Product Stop-loss Premium. In [3], Bentahar constructed quantile functions for vector portfolios with the aim of proposing the Tail Conditional Expectation for random vectors and in [2], Balbás and Guerra generalized the theory of vector risk measures for a map with values on a Banach lattice. Deviations and Expectation bounded risk measures are analyzed and it is proposed a vector CVaR.

We consider now $\bar{\mathcal{X}} = \mathcal{X}^n$, a collection of non-negative random vectors (effective losses or debts) $\mathbf{X} = (X_1, \ldots, X_n)$, such that $X_h : \Omega \to I\!\!R_+, \ h = 1, \ldots, n$.

A risk measure for portfolio vectors is the function $R : \bar{\mathcal{X}} \to I\!\!R$, mapping $\mathbf{X} \in \bar{\mathcal{X}}$ to the real space.

We underline that this setting could have also an actuarial interpretation, since the measures of risk are also considered as premium principles. Moreover, these risk measures may be interpreted as functions that evaluate the additional risk deriving from dependent events.

We want to extend some known results of the one-dimensional case to the multidimensional one.

We can consider a generalization of the properties of risk measures, as proposed in [7]:

- Non-negative loading: $R(\mathbf{X}) \geq E(X_1 \ldots X_n), \forall \mathbf{X} \in \bar{\mathcal{X}}$;

- Non-excessive loading: $R(\mathbf{X}) \leq \sup_{\omega \in \Omega} \{|X_1(\omega)|, \ldots, |X_n(\omega)|\}, \ \forall \mathbf{X} \in \bar{\mathcal{X}}$;

- Monotonicity: $R(\mathbf{X}) \leq R(\mathbf{Y}), \ \forall \mathbf{X}, \mathbf{Y} \in \bar{\mathcal{X}}$ such that $\mathbf{X} \preceq \mathbf{Y}$ in some stochastic sense;

- Translation invariance: $R(\mathbf{X} + \mathbf{a}) = R(\mathbf{X}) + \bar{a}, \ \forall \mathbf{X} \in \bar{\mathcal{X}}, \forall \mathbf{a} \in I\!\!R^n$, where $\mathbf{a}$ is a vector of sure initial amounts and $\bar{a}$ is the componentwise product of the elements of the vector $\mathbf{a}$;

- Positive homogeneity of order n: $R(c\mathbf{X}) = c^n R(\mathbf{X}), \ \forall \mathbf{X} \in \bar{\mathcal{X}}, \forall c \geq 0$;

- Constancy: $R(\mathbf{b}) = \bar{b}, \ \forall \mathbf{b} \in I\!\!R^n$. A special case is $R(\mathbf{0}) = 0$, that is called normalization property;

- Convexity: $R(\lambda \mathbf{X} + (1 - \lambda)\mathbf{Y}) \leq \lambda R(\mathbf{X}) + (1 - \lambda) R(\mathbf{Y}), \quad \forall \mathbf{X}, \mathbf{Y} \in \bar{\mathcal{X}}$ and $\lambda \in [0, 1]$; this property implies diversification effects as subadditivity does;

- Subadditivity: $R(\mathbf{X} + \mathbf{Y}) \leq R(\mathbf{X}) + R(\mathbf{Y}), \forall \mathbf{X}, \mathbf{Y} \in \bar{\mathcal{X}}$ that reflects the idea that risk can be reduced by diversification.

3 Lagrangian duality

In this work we want to deal with an optimization problem with two objectives under some constraints. Following this aim, we introduce this *generalized Pareto problem*:

$$\min_{C\backslash\{O\}} f(\mathbf{X}) \text{ subject to } \mathbf{X} \in S := \{\mathbf{X} \in \bar{\mathcal{X}} : g(\mathbf{X}) \geq O\}, \tag{1}$$

where C is a convex, closed and pointed cone with apex at the origin, and with int$C \neq \emptyset$, namely with nonempty interior; $f : \bar{\mathcal{X}} \to \mathbb{R}^\ell$ and $g : \bar{\mathcal{X}} \to \mathbb{R}^m$, moreover, O_n denotes the n-tuple, whose entries are zero, when there is no fear of confusion the subfix is omitted; for $n = 1$, the 1-tuple is identified with its element, namely, we set $O_1 = 0$.

The sentence $\min_{C\backslash\{O\}}$ denotes vector minimum with respect to the cone $C\backslash\{O\}$: $\mathbf{Y} \in S$ is a (global) vector minimum point (for short, v.m.p.) of (1), if and only if (iff)

$$f(\mathbf{Y}) \not\geq_{C\backslash\{O\}} f(\mathbf{X}), \ \forall \mathbf{X} \in S, \tag{2}$$

where the inequality means $f(\mathbf{Y}) - f(\mathbf{X}) \notin C\backslash\{O\}$. We will assume that v.m.p. exist.

We comply with the approach of [14] and we recall some introductive notions useful to understand what follows.

Let $\mathbf{Y} \in S$ be a v.m.p. of (1), $\mathcal{H} := (C\backslash\{O\}) \times \mathbb{R}^m_+$ and $\mathcal{K}(\mathbf{Y}) := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u = f(\mathbf{Y}) - f(\mathbf{X}), \ v = g(\mathbf{X}), \ \mathbf{X} \in \bar{\mathcal{X}}\}$. $\mathcal{H}$ and $\mathcal{K}(\mathbf{Y})$ are subsets of $\mathbb{R}^\ell \times \mathbb{R}^m$, that is called *image space*; $\mathcal{K}(\mathbf{Y})$ is called *image* of problem (1).

We note that $\mathbf{Y} \in \bar{\mathcal{X}}$ is a v.m.p. of (1), iff the system

$$f(\mathbf{Y}) - f(\mathbf{X}) \in C, \ f(\mathbf{Y}) - f(\mathbf{X}) \neq O, \ g(\mathbf{X}) \geq O, \ \mathbf{X} \in \bar{\mathcal{X}}$$

is impossible. This is equivalent to state that $\mathcal{H} \cap \mathcal{K}(\mathbf{Y}) = \emptyset$. This disjunction can be proved by means of a sufficient condition, that consists in providing a function, such that two of its disjoint level sets [1] contain $\mathcal{H}$ and $\mathcal{K}(\mathbf{Y})$, respectively.

This class of linear functions, called *separation functions*, is

$$w = w(u, v, \Theta, \Lambda) = \Theta u + \Lambda v, \ \ \Theta \in U^*_{C\backslash\{O\}}, \Lambda \in V^*_C,$$

where Θ, Λ are parameters; $U^*_{C\backslash\{O\}}$ and V^*_C are the positive vector polar sets [2] of $U = C\backslash\{O\}$ and $V = \mathbb{R}^m_+$.

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[1] The positive and non positive level sets of w are defined as follows: $W_{C\backslash\{O\}} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u, v, \Theta, \Lambda) \geq C\backslash\{O\}\}$; $\overline{W}_{C\backslash\{O\}} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u, v, \Theta, \Lambda) \not\geq C\backslash\{O\}\}$.

[2] The positive vector polar set of a set $A \subseteq \mathbb{R}^p$ with respect to a cone $C \subseteq \mathbb{R}^q$ is defined as: $A^*_C := \{M \in \mathbb{R}^{q\times p} : Ma \geq_C O, \ \forall a \in A\}$.

Let us now define the following vector optimization problem of set-valued functions:

$$\text{Max}_{C\backslash\{O\}} \quad \text{min}_{C\backslash\{O\}} L_V(\mathbf{X}, \tilde{\Lambda}) \tag{3}$$
$$\tilde{\Lambda} \in V_C^* \qquad \mathbf{X} \in \bar{\mathcal{X}}$$

where $L_V(\mathbf{X}, \tilde{\Lambda}) = f(\mathbf{X}) - \tilde{\Lambda}g(\mathbf{X})$ is the *vector Lagrangian function*. Max denotes the vector Maximum of a set-valued map. In what follows, problem (3) will be called the *vector dual problem* of (1).

Moreover, we want to observe that it is possible to formulate a generalized vector dual problem as follows:

$$\text{Max}_{C\backslash\{O\}} \qquad \text{min}_{C\backslash\{O\}} L_V(\mathbf{X}, \Theta, \Lambda)$$
$$\Theta \in U_{C\backslash\{O\}}^* \quad \mathbf{X} \in \bar{\mathcal{X}} \tag{4}$$
$$\Lambda \in V_C^*$$

where $L_V(\mathbf{X}, \Theta, \Lambda) = \Theta f(\mathbf{X}) - \Lambda g(\mathbf{X})$ is the *generalized vector Lagrangian function* and (4) can be replaced by problem (3), if $rg(\Theta) = \ell$, and then $\tilde{\Lambda} = \Theta^{-1}\Lambda$.

Theorem 1 *For any* $\mathbf{Y} \in S$ *and* $\Lambda \in V_C^*$ *it results:*

$$f(\mathbf{Y}) \nleq_{C\backslash\{O\}} z, \ \forall z \in \min_{\mathbf{X}\in\bar{\mathcal{X}}} {}_{C\backslash\{O\}}[f(\mathbf{X}) - \tilde{\Lambda}g(\mathbf{X})]. \tag{5}$$

PROOF: It may be obtained applying the same arguments of those presented in Theorem 9 of [14], where the functions take values in $I\!\!R^n$. □

Theorem 1 states that the vector of the objectives of the primal (1), evaluated at any feasible solution $\mathbf{Y}$, is not less than or equal to the vector of the objectives of the dual (3), calculated at any $\Lambda \in V_C^*$; hence, Theorem 1 is a weak Duality Theorem, in the vector case.

Lemma 1 *There exist* $\mathbf{Y} \in S$ *and* $\bar{\Lambda} \in V_C^*$ *such that* $[f(\mathbf{Y}) - f(\mathbf{X})] + \bar{\Lambda}g(\mathbf{X}) \ngeq_{C\backslash\{O\}} O, \ \forall \mathbf{X} \in \bar{\mathcal{X}}$ *iff* $O \in \Delta$.

PROOF: It may be obtained trough the same passages of Lemma 4 of [14], where the functions take values in $I\!\!R^n$. □

Definition 3 *Let* Z *be a nonempty set,* A *be a convex cone in* $I\!\!R^p$ *with* $intA \neq \emptyset$ *and* $F : Z \to I\!\!R^p$. *F is said A-subconvexlike on* Z *iff there exists* $\alpha_0 \in intA$ *such that for all* $\varepsilon > 0$

$$(1 - \alpha)F(Z) + \alpha F(Z) + \varepsilon\alpha_0 \subseteq F(Z) + A, \ \forall\alpha \in (0, 1).$$

Now define the following sets:

$$\Delta_1 := \min_{\substack{C \setminus \{O\} \\ \mathbf{X} \in S}} f(\mathbf{X}) \quad \text{and} \quad \Delta_2 := \max_{\substack{C \setminus \{O\} \\ \tilde{\Lambda} \in V_C^*}} \ [f(\mathbf{X}) - \tilde{\Lambda} g(\mathbf{X})].$$

Observe that if $\Delta_1 \cap \Delta_2 \neq \emptyset$, or equivalently $O \in \Delta := \Delta_1 - \Delta_2$, then there exists an optimal solution of the primal problem and an optimal solution of the dual such that the corresponding optimal vector values are equal; i.e., the two problems possess at least a common optimal value.

Definition 4 *If $M \neq \emptyset$ and $\bar{x} \in clM$, then the set of $\bar{x} + x \in \mathbb{R}^n$ for which $\exists \{x^i\} \subseteq clM$, with $\lim_{i \to +\infty} x^i = \bar{x}$, and $\{a_i\} \subset \mathbb{R}_+ \setminus \{O\}$ such that $\lim_{i \to +\infty} a_i(x^i - \bar{x}) = x$ is called tangent cone (or Boulingand tangent cone) to M at $\bar{x}$ and denoted by $T(\bar{x}, M)$.*

We stipulate that $T(\bar{x}, \emptyset) = \emptyset$. If $\bar{x} = 0$, then $\bar{x}$ is omitted from the notation of the cone.

Definition 5 *Let $C = \mathbb{R}_+^\ell$. If $U_i \not\subseteq T(\mathcal{K} - cl\mathcal{H}) \ \forall i = 1, \dots, \ell$, where $U_i := \{(u, v) \in \mathbb{R}^{\ell+m} : u_i \geq 0; u_j = 0, \ j = 1, \dots, \ell \text{ and } j \neq i; v_k = 0, \ k = 1, \dots, m\}$, and $\mathbf{Y} \in S$ is any v.m.p. of (1), then we say that the regularity condition (RC) holds.*

Theorem 2 *Consider problem (1) with $C = \mathbb{R}_+^\ell$; let $\mathbf{Y} \in S$ a v.m.p. of (1). If $(-f, g)$ is $(cl\mathcal{H})$-subconvexlike on $\bar{\mathcal{X}}$ and the regularity condition (RC) holds, then $O \in \Delta$.*

PROOF: The result implies the same concepts of Theorem 10 of [14], where the functions take values in $\mathbb{R}^n$. □

In the following section, we exploit weak and strong Lagrangian duality results for a multicriteria optimization problem in order to interpret the shadow prices of our problem.

4 Vector optimization with risk measures

There exists several approaches to model decision making under risk. For instance, the expected utility theory of von Neumann and Morgenstern [24], which specifies a utility function, concave and increasing, for a maximization problem, or the worst case scenarios. A recent approach, that takes origin from the mean-variance theory of Markowitz [18], is proposed, for instance, in [23], [4] and [5], where necessary and sufficient optimality conditions are given for convex optimization problems, where the objective function is a scalar convex risk function, or a scalar convex deviation measure, and duality results are provided by means of conjugate duality.

Finally, we want also cite [25] and [26], where the authors proposed a multiobjective duality problem for the classical Markowitz portfolio problem, by means of a scalarization technique, but they did not interpret the mathematical investigations.

To describe our primal problem, let us recall equation (1) and make the following positions. The objective functions are given by: $f(\mathbf{X}) = (R, E(v))(\mathbf{X})$, where $R(\mathbf{X})$ is a multivariate risk measure, and $E(v(\mathbf{X}))$ is the mathematical expectation of a function $v(\mathbf{X})$ that describes the pain associated with a vector of losses (debts) of amount $\mathbf{X} = (X_1, \ldots, X_n)$. Let us consider $u : \bar{\mathcal{X}} \to I\!R$ as the utility function of the decision maker. It holds the relation: $v(\mathbf{X}) = -u(-\mathbf{X})$; hence, if u is a non-decreasing concave utility function related to a vector of goods or profits, v is a non-decreasing convex pain function related to a vector of losses. We note that, in one dimension, saying that a decision maker, with utility function u, is risk averse is equivalent to say that the pain function v is convex.

In the classical portfolio optimization models, we are used to minimize the opposite of the expected return of an investment; in this case, we could minimize the expected losses of the n investments or the expected loss of the sum of the n investments. We observe that the former case implies the minimization of $(n + 1)$ functions, i.e., the risk measure and the n expected values of the vector $\mathbf{X}$; while, the latter is a particular case of our pain function, because the sum of the components of the vector $\mathbf{X}$ is embedded in the function $v(\mathbf{X})$, if it is linear.

Now, in order to understand the meaning of the dual variables, and hence, of the dual problem (as formulated in (3) or (4)), we dwell on the linear case.

We will apply a known method, called ε-constraint method (see, for example, [10], Sect. 4.2.). The following theorem holds:

Theorem 3 *The solution $\mathbf{Y}$ is a v.m.p. of (1), if and only if, it is a global minimum point of the following scalar problem:*

$$P_k(\mathbf{Y}) : \quad \min f_k(\mathbf{X}) \ s.t.$$

$$\mathbf{X} \in S, f_i(\mathbf{X}) \le f_i(\mathbf{Y}), i \in I \setminus \{k\}, \forall k \in I, \tag{6}$$

where $I = 1, \ldots, \ell$.

Define the perturbed problem:

$$\min f_k(\mathbf{X}), \ \text{s.t.}$$

$$\mathbf{X} \in R_k(\eta; \xi) := \{\mathbf{X} \in \bar{\mathcal{X}} : g_j(\mathbf{X}) \geq \eta_j, -f_i(\mathbf{X}) \geq -f_i(\mathbf{Y}) + \xi_i, j \in J, i \in I_{n_2^k}\}, \qquad (7)$$

where J is the set of binding constraints at $\mathbf{Y}$, n_1 is the number of binding constraints, $\eta \in I\!R^{n_1}$ and $\xi \in I\!R^{n_2}$, $I_{n_2^k}$ is the subset of indexes of $I \setminus \{k\}$ of cardinality n_2, with $n_1 + n_2 = n$ and $\mathbf{Y}$ is a global vector minimum point.

Now, we call $\lambda^k \in I\!R_+^{n_1}$, $\vartheta_i^k \in I\!R_+$, $i \in I_{n_2^k}$, the dual variables of problem (7); they are the derivatives of the optimal value of the k-th objective function with respect to the variable representing a change in the level of the constraints; thus, we obtain:

$$\lambda_i^k = \frac{\partial}{\partial \eta_i} \left(\min_{\mathbf{X} \in R_k(\eta; \xi)} f_k(\mathbf{X}) \right), \quad i = 1, \ldots, n_1, \qquad (8)$$

$$\vartheta_i^k = \frac{\partial}{\partial \xi_i} \left(\min_{\mathbf{X} \in R_k(\eta; \xi)} f_k(\mathbf{X}) \right), \quad i \in I_{n_2^h}. \qquad (9)$$

Hence, λ_i^k, $k = 1, 2$, $i = 1, \ldots, n_1$, describes an information about the change of the k-th objective function with respect to a variation in the level of the i-th constraint, while ϑ_i^k, $k = 1, 2$, $i \in I_{n_2^k}$, represents a change of the k-th objective function, when it occurs a variation in the level of the i-th objective.

We suppose that the first objective function is given by the risk of our investment and hence, if, for instance, the first constraint is a budgetary constraint, we may say that λ_1^1 can be interpret as the personal feeling of the decision maker toward a change in her budget and the consequences that this variation has upon her attitude in investing in that portfolio. In other words, λ_i^1 may be viewed as a risk aversion index concerning the i-th constraint.

The second objective function is the expected value of a pain function related to the losses of our investment, hence, in this case, λ_1^2 may give some information about her non-satiability, when it occurs a variation in her budget. If we consider, for instance, the expected loss of the sum of the X_h, $h = 1, \ldots, n$, as second objective function (i.e., $E(X_1 + \cdots + X_n)$), we may interpret λ_1^2 as the propensity to gain of the decision maker. Moreover, we may suppose that if there is an increase in the budget of the decision maker, there may be a decrease in the index of risk aversion, since we have a larger cover of losses, but also, she may have an increase in the non-satiability feeling since, with this added amount of money, she could face more purchases; but the money spent in one direction, obviously, cannot be spent also in the opposite one.

Finally, the shadow price ϑ_2^1 may describe what happens to the risk measure when there is a change in the level of the expected pain (the expected utility).

As stated in [15], from these shadow prices, λ_i^k, $k = 1, 2$, $i = 1, \ldots, n_1$ and ϑ_i^k, $k = 1, 2$, $i \in I_{n_2^k}$, we can construct the shadow prices matrices $\Lambda \in I\!\!R^{2 \times m}$ and $\Theta \in I\!\!R^{2 \times 2}$. The matrix Λ is composed by the shadow prices corresponding to the binding constraints and its k-th row is the vector λ^{kT}. The matrix Θ is given by the entries $\vartheta_{ki} = -\vartheta_i^k$ (because of the change in the sign of the objective functions in (7)), and its diagonal entries are $\vartheta_{ii} = 1$, since they represent the variation of a function with respect to a change in the level of itself.

5 Further developments

It is important to note that the definition of risk aversion in the multivariate case is an open problem, we can cite many different approaches, for example, [8], [17], [20] and [19].

We can consider the meaning of risk aversion, that is the particular attention (subjective probability) attributed to negative events (or dependent events), or the attention given to events that could have disastrous consequences.

This approach may be viewed as an alternative way of defining risk aversion and non-satiability with respect to a particular target.

In the linear case, we can consider two kind of multipliers (shadow prices). It could be interesting to have some results of this kind also in the nonlinear case in order to interpret these kind of multipliers.

Finally, we note that the duality results exploited in this paper (see, for instance, Theorem 2) refer to a non-convex primal problem; hence, it is possible to weaken the usual hypothesis of convexity about the risk function and the pain function, if it could be necessary.

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